Journal of the Egyptian Mathematical Society (2016) **24**, 337–347

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Fundamental solutions of the fractional diffusion and the fractional Fokker–Planck equations

E.A. Abdel-Rehim*

Department of Mathematics and Computer Science, Faculty of Science, Suez Canal University, Ismailia, Egypt

Received 17 November 2014; revised 6 June 2015; accepted 20 August 2015
Available online 16 October 2015

Keywords

α-Stable distribution;
Green function;
Similarity variable;
Feller operator;
Scale invariant method;
Fractional diffusion equations

Abstract The solutions of the space–time fractional diffusion equations and that of the space–time fractional Fokker–Planck equation are probabilities evolving in time and stable in the sense of Lévy. The fundamental solution, Green function, of the space–time fractional diffusion equation, is early obtained by using the scale invariant method. In this paper, I use this reduced Green functions and the scale invariant method to obtain the fundamental solution, Green function, of the fractional diffusion equation and henceforth I obtain the solution of the space–time fractional Fokker–Planck equation, by applying the Billerś transformation between the independent spatial coordinates of these fractional differential equations. Henceforth, I simulate these solutions in the 3D for all the possible values of the space and time fractional orders and also for different values of the skewness.

MSC: 26A33; 45K05; 60J60; 60G50; 60G51; 65N06; 80-99; 42A38

1. Introduction

The aim of this paper is to produce the simulation of the fundamental solution of the fractional diffusion and the fractional Fokker–Planck equations. For this purpose I use both the concept of the similarity variable and the Billeŕ s transformation to transform the fundamental solution of both models. The main job of the similarity variable is to reduce the partial differential equations to first order differential equations. Then the functional form of the partial differential is obtained by solving the first order differential equations. Blumen and Cole [1] initiated using the similarity variable to obtain a solution to the heat equation,

$$\frac{\partial}{\partial t}u(x,t) = a\frac{\partial^2}{\partial x^2}u(x,t), \quad u(x,0) = \delta(x), \quad -\infty < x < \infty, \tag{1.1}$$

* Tel.: +20 1221554921.
E-mail address: entsarabdelrehim@yahoo.com, entsarabdelrehim@gmail.com
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.08.006

here $a > 0$ and is called the diffusion constant. $u(x, t)$ is a probability density function and the conditions imposed on it are $u(x,t) \geq 0, u(\pm\infty,t) = 0$ and $\int_{\infty}^{\infty} u(x,t)dx = 1$. This equation describes the diffusion of the free particle, i.e. the particle diffuses under the action of no force other than those due to the molecules of the surrounding medium are acting and its solution is well known as the corresponding Green function or the fundamental solution.

The same authors in [2] applied the similarity scaling methods for other differential equations. Gorenflo and Luchko [3] applied the scale invariant method to find the fundamental solutions to differential equations having fractional orders. In their paper the authors firstly introduced the scaling variable $z = xt^{-\frac{\alpha}{\beta}}$ as an independent variable used to find a special types of solutions being invariant under a subgroup of the fully symmetry group of the space–time fractional differential equations, *STFDE*,

$$\underset{t\ *}{D^{\beta}} u(x,t) = a \underset{x\ \theta}{D^{\alpha}} u(x,t), \quad u(x,0) = \delta(x), \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1, \tag{1.2}$$

here θ is the skewness parameter having the restricted values $|\theta| \leq \min(\alpha, 2-\alpha)$. Later, Mainardi et al. [4] obtained the fundamental solution (Green function), $G^{\theta}_{\alpha,\beta}(x,t)$ of the space–time fractional diffusion Eq. (1.2) using the similarity variable z. They discussed many particular values of the fractional orders α, β and the skewness θ and gave convergent series for each case. They discussed in their long paper the scaling property of the Green function and its relation to the Lévy strictly stable densities, for $\beta = 1$, and the reduced Green function, for $\beta \neq 1$, and gave convergent expansions for each case.

Recently, Rodringues [5] followed Gorenflo and Luchko [3]. He used the Banach fixed point theorem and the independent variable $z = xt^{-\frac{\alpha}{\beta}}$ to study the existence and uniqueness of solution to the partial differential equation of the fractional orders. I wish in this paper to complete my previous works on studying the Fokker–Planck equation

$$v(\zeta,\tau) - v(\zeta,0) = L_{FP} v(\zeta,\tau), \tag{1.3}$$

with $L_{FP} = \frac{\partial^2}{\partial \zeta^2} v(\zeta,\tau) - \frac{d(F(\zeta)v(\zeta,\tau))}{d\zeta}$. $F(\zeta)$ is an external force. I am interested here in using $F(\zeta) = -b\zeta$ as a linear attraction force. So far it reads

$$\frac{\partial}{\partial \tau} v(\zeta,\tau) = a \frac{\partial^2}{\partial \zeta^2} v(\zeta,\tau) + b \frac{d(\zeta v(\zeta,\tau))}{d\zeta}, \quad v(\zeta,0) = \delta(\zeta). \tag{1.4}$$

If $b = 0$, then one has Eq. (1.1) having the solution $u(x,t) = \frac{1}{2\sqrt{\pi a t}} e^{-x^2/(4at)}$. Consequently by using the Galilei transformation of the independent variables (ζ, t) to $(x - bt, t)$, then the solution of (1.4) is $v(\zeta,\tau) = u(x - bt, t) = \frac{1}{2\sqrt{\pi a t}} e^{-(x-bt)^2/(4at)}$ with $t = \tau$. I have studied Eq. (1.4) as the space has fractional order only or the has time fractional order only so I am going to complete my results obtained in [6] and [7] to obtain the fundamental solution with both the time and the space have fractional orders. Therefore, the space–time fractional Fokker–Planck equation, *STFFPE*, or as I prefer to call it the son model, is explicitly written as

$$\underset{t\ *}{D^{\beta}} v(\zeta,\tau) = a \underset{\zeta\ \theta}{D^{\alpha}} v(\zeta,\tau) + b \frac{d(\zeta v(\zeta,\tau))}{d\zeta}, \quad v(\zeta,0) = \delta(\zeta), \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1. \tag{1.5}$$

The procedure begins by using the fundamental solution $G^{\theta}_{\alpha,\beta}(x,t)$ of Eq. (1.2), *(the mother model)*, obtained by Mainardi et al. [4]. The next step to obtain the fundamental solution of the *son model* is to use the transformation between the spatial coordinates (x, t) of Eq. (1.2) and (ζ, τ) of Eq. (1.5), introduced by Biller et al. [8] for $\beta = 1$. The new point is that according to the inversion of Fourier transformation of the characteristic function of the son model, Eq. (1.5), one can deduce that the transformation between the two models is also possible for $0 < \beta < 1$.

This paper is organized as follows: In Section 2, I provide the reader with the needed notations and used symbols. In Section 3, the fundamental solutions $G^{\theta}_{\alpha,\beta}$, the Green function, the similarity variable, and the Bileŕ s transformation between the both studied models are given. In Section 4, I give the particular formulae of the reduced Green functions and the Lévy stable functions. Finally, Section 5 is devoted to the simulation of both models for all the different values of α, β and θ with the interpretation of the results.

2. Definition of used symbols

The used $\underset{x\ \theta}{D^{\alpha}}$ is called the Riesz–Feller pseudo fractional differential operator which in Fourier domain reads

$$\mathcal{F}\left\{\underset{x\ \theta}{D^{\alpha}} f(x);\kappa\right\} = -|\kappa|^{\alpha} e^{isig(\kappa)\theta\pi/2} \widehat{f}(\kappa) = -\psi^{\theta}_{\alpha}(\kappa)\widehat{f}(\kappa), \quad 0 < \alpha \leq 2, \quad \kappa \in \mathbb{R}, \tag{2.1}$$

where $0 < \alpha \leq 2, |\theta| \leq \min\{\alpha, 2-\alpha\}$. $-\psi^{\theta}_{\alpha}(\kappa)$ is its symbol with skewness θ being restricted to the domain $|\theta| \leq \min(\alpha, 2-\alpha)$, see [15,16]. In the case $\alpha = 1$

$$\mathcal{F}\left\{\underset{0}{D^{1}}\phi(x);\kappa\right\} \neq \mathcal{F}\left\{\frac{d\phi(x)}{dx};\kappa\right\}.$$

Since $-|\kappa|^{\alpha} = -(\kappa^2)^{\alpha/2}$, one can set $\underset{x0}{D^{\alpha}} = -(\frac{d^2}{dx^2})^{\alpha/2}$, see [10]. This symbol is the logarithm of the characteristic function of α-stable probability density. The used $\underset{t\ *}{D^{\beta}}$ is the Caputo fractional operator which is defined through its image in the Laplace transform domain as

$$\mathcal{L}\left\{\underset{*}{D^{\beta}} f(t);s\right\} = s^{\beta}\widetilde{f}(s) - s^{\beta-1}f(0) - \dot{f}(0)s^{\beta-2} - \cdots - f^{(m-1)}(0)s^{\beta-m}, \quad s > 0.$$

It is obvious that the Caputo differential operator depends directly on the initial condition $f(0)$. This is the reason that makes it suitable for dealing with time differentiation. The properties of many distributions are easily investigated in terms of their characteristic functions. In this paper, one needs to use the characteristic function which is a variant of the Fourier transform of the probability density function $p(x)$ and is defined as

$$\mathcal{F}\{p(x);\kappa\} = \widehat{p}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x} p(x)dx.$$

This characteristic function belongs to an α-stable distribution, $\alpha \in (0, 2]$, see [9], if and only if it has the form

$$\log\widehat{p}(\kappa) = \begin{cases} i\mu'\kappa - c\,|\kappa|^{\alpha}\{1 + i\beta'\frac{\kappa}{|\kappa|}w(|\kappa|,\alpha)\} & \text{if } \alpha \neq 1, \\ i\mu'\kappa - c\,|\kappa|\{1 + i\beta'\frac{\kappa}{|\kappa|}w(|\kappa|,\alpha)\} & \text{if } \alpha = 1. \end{cases} \tag{2.2}$$

Here $\kappa \in \mathbb{R}$, $c \geq 0$, $\mu' > 0$ and $|\beta'| \leq 1$ being the symmetry parameter. It determines the skewness of the distribution. $\beta' = 0$ corresponds to a symmetric distribution. c is the scale parameter. It measures the spread of the samples from a distribution around the mean. μ' is the location parameter and $\exp[i\mu'\kappa]$ basically corresponds to a shift in the x-axis of the probability density function. For $1 < \alpha \leq 2$, μ' represents the mean and for $0 < \alpha \leq 1$, it represents the median. A stable distribution is said to be standard if $\mu' = 0$ and $c = 1$. The used function $w(\kappa, \alpha)$ is defined as

$$w(|\kappa|,\alpha) = \begin{cases} \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1, \\ (2/\pi)\log|\kappa| & \text{if } \alpha = 1, \end{cases} \tag{2.3}$$

For $\alpha = 2$, $w(|\kappa|, \alpha) = 0$ is the special case of the normal distribution. According to Feller another definition for the characteristic function of Lévy strictly stable densities is denoted by $\widehat{p}_{\alpha}(\kappa;\theta)$ and is defined as

$$\widehat{p}_{\alpha}(\kappa;\theta) = \exp\left[-|\kappa|^{\alpha}e^{\frac{i\theta\pi}{2}sig(\kappa)}\right]. \tag{2.4}$$

The range of the parameters is given as: $0 < \alpha \leq 2$, $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and is visualized by the Feller–Takaysau diamond. The relation between the Gnedebko–Kolmogrov and Feller in the other side is related to the skewness θ of $\widehat{p}_{\alpha}(\kappa;\theta)$ at Eq. (2.4) and the skewness β' of $\widehat{p}_{\alpha}(\kappa;\theta)$ at Eq. (2.3) as follows:

$$\beta' = \frac{\tan\left(\theta\frac{\pi}{2}sig(\kappa)\right)}{\tan\left(\alpha\frac{pi}{2}\right)}, \quad \alpha \neq 1, \tag{2.5}$$

with $\mu' = 0$, $c = 1$. Using these definitions, one can write the characteristic function of Eq. (1.2) as

$$\widehat{u}(\kappa,t) = E_{\beta}(-|\kappa|^{\alpha}t^{\beta}\,e^{isig(\kappa)\theta\pi/2}), \tag{2.6}$$

where $E_{\beta}(z)$ is the Mittag–Leffler function, see [11,12]. So far, the Fourier–Laplace transformation of Eq. (1.2) reads

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^{\beta} + \psi_{\alpha}^{\theta}(\kappa)}, \quad s > 0, \quad \kappa \in \mathbb{R}, \tag{2.7}$$

and its inverse Laplace transformation gives the characteristic function of the mother model, Eq. (2.6). Hence fourth in the Fourier–Laplace domain, see [13], Eq. (1.5), reads

$$\widehat{\widetilde{v}}(\kappa,s) = \frac{s^{\beta-1}}{s^{\beta} + ib\kappa + a\psi_{\alpha}^{\theta}(\kappa)}.$$

Therefore, the solution of Eq. (1.5) with the space fractional derivatives $\alpha \in (0, 2]$ and the time fractional derivative $\beta \in (0, 1]$ is related to the solution of Eq. (1.2), in the Fourier domain, by the relation

$$\widehat{v}(\kappa,t) = E_{\beta}\left[\left(-a\psi_{\alpha}^{\theta}(\kappa) + ib\kappa\right)t^{\beta}\right], \tag{2.8}$$

and for $\beta = 1$, it reads

$$\widehat{v}(\kappa,\tau) = \exp\left[-|\kappa|^{\alpha}\frac{a}{b\alpha}(1 - e^{-b\alpha\tau})e^{i\theta\frac{\pi}{2}sig(\kappa)}\right],$$

where $sig(\kappa)$ is the $sign(\kappa)$. As we see the characteristic functions of the mother and son models are closely related. Therefore, one can detect that also the Green functions are related and can be transformed to each others. In the following section I will explain this idea in full details.

3. The concept of the similarity properties and Biler's transformation

In this section, a survey about the Green function, the similarity variable, and the Biler's transformation is given. These information must be in hand before extracting the solutions of the both studied models. The Green function $G(x, s)$ is the solution of the partial differential equation written in the form $LG(x, s) = \delta(x - s)$ where L is a linear differential operator and $\delta(x)$ is the Dirac delta function. Therefore, using $u(x, 0) = \delta(x)$ makes it possible to write the solution of Eq. (1.2) in the following integral form:

$$u(x,t) = \int_{-\infty}^{\infty} G_{\alpha,\beta}^{\theta}(y,t)\delta(x-y)dy, \tag{3.1}$$

where $G_{\alpha,\beta}^{\theta}(y,t)$ is the Green function or the fundamental solution. Use Eq. (3.1) and apply the Laplace–Fourier transformation on Eq. (1.2), you get $\widehat{\widetilde{G}}_{\alpha,\beta}^{\theta}(\kappa,s) = \widehat{\widetilde{u}}(\kappa,s)$. Using the scaling properties of the Fourier and Laplace transformations

$$\mathcal{F}\{f(ax);\kappa\} = a^{-1}\widehat{f}(\kappa/a), \quad a > 0,$$
$$\mathcal{L}\{f(bt);s\} = b^{-1}\widetilde{f}(s/b), \quad b > 0. \tag{3.2}$$

Using Eqs. (3.1) and (3.2), one can deduce the Green function as,

$$G_{\alpha,\beta}^{\theta}(x,t) = t^{-\frac{\beta}{\alpha}}K_{\alpha,\beta}^{\theta}\left(xt^{-\frac{\beta}{\alpha}}\right), \tag{3.3}$$

where $z = xt^{-\frac{\beta}{\alpha}}$ is the similarity variable and $K_{\alpha,\beta}^{\theta}(xt^{-\frac{\beta}{\alpha}})$ is a function to be detected later. I am using here the same notation of [4] to avoid confusion. Using Eq. (2.1) and its properties, one can prove that

$$G_{\alpha,\beta}^{\theta}(\kappa,t) = \overline{G_{\alpha,\beta}^{\theta}(-\kappa,t)} = G_{\alpha,\beta}^{-\theta}(-\kappa,t), \tag{3.4}$$

and $G_{\alpha,\beta}^{\theta}(0,t) = E_{\beta}(0) = 1$. Biller et al. [8] proved in their papers that the following transformations occur between the two independent pairs (x, t) and (ζ, τ) of Eqs. (1.2) and (1.5) for $\beta = 1$, see also [14],

$$\xi = x(\alpha t + 1)^{-1/\alpha}, \quad \tau = \alpha^{-1}log(\alpha t + 1), \tag{3.5}$$

or vice versa

$$x = \xi e^{\tau}, \quad t = \frac{1}{\alpha}(e^{\alpha\tau} - 1). \tag{3.6}$$

Using Eqs. (3.5) and (3.6), you can write

$$v(\xi,\tau) = (\alpha t + 1)^{1/\alpha}u(x,t), \tag{3.7}$$

or using (3.7), you can write

$$u(x,t) = e^{-\tau}\, v(\xi, \tau).$$

Using these transformations, one deduces that every solution of Eq. (1.2) is a solution to Eq. (1.5). According to Eq. (2.8), this is truth also in the case $\beta < 1$ as the solution is expressed in terms of the Mittag–Leffler function, $E_\beta(-t^\beta)$ which has the following properties. It exhibits a behavior similar to that of a stretched exponential for $0 < \beta < 1$ and for small values of t

$$E_\beta(-t^\beta) \approxeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \approxeq \exp\{-t^\beta/\Gamma(\beta+1)\}, \quad 0 \ll t \ll 1. \tag{3.8}$$

Whereas for large t, it has the asymptotic representation

$$E_\beta(-t^\beta) \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, \qquad t \to \infty. \tag{3.9}$$

4. Particular formulae of the Green functions

The particular solutions of the mother model are given according to the special formulae of $K^\theta_{\alpha,\beta}(xt^{-\frac{\beta}{\alpha}})$ and the similarity variable $z = xt^{-\frac{\beta}{\alpha}}$ which in accordance are related to the values of α, β and θ, see [15] and the appendices of [14]. These special formulae, characteristic functions, are a class of Lévy strictly stable densities. For ease of writing as $\beta = 1$, we can replace the notation $K^\theta_{\alpha,1}(z)$ by the Lévy notation $L^\theta_\alpha(z)$ which with the similarity variable is satisfying

$$G^\theta_{\alpha,1}(x,t) = t^{-1/\alpha} L^\theta_\alpha(x/t^{1/\alpha}). \tag{4.1}$$

The procedure is to use $L^\theta_\alpha(x)$, then apply Eq. (4.1), you get the Green function, fundamental solution, of Eq. (1.2). After that apply the Biler's transformations, Eqs. (3.5) and (3.6), you get the solution of the son model at Eq. (1.5).

The Gaussian case, the standard diffusion, is for $\alpha = 2$ and $\theta = 0$, and is defined as

$$L^0_2(z) = \frac{1}{2\sqrt{\pi}} exp(-z^2/4), \qquad z = xt^{-\frac{1}{2}}. \tag{4.2}$$

The Cauchy case is in accordance to $\alpha = 1$ and $\theta = 0$ and is defined as

$$L^0_1(z) = \frac{1}{\pi}\frac{1}{z^2+1}, \qquad z = x/t, \tag{4.3}$$

then apply the previous steps. It is worth to define the singular Dirac function $L^\pm_1(x) = \delta(x \pm 1)$.

The Lévy–Smirnov case is in accordance to $\alpha = 1/2$ and $\theta = -1/2$, and is written as

$$L^{-1/2}_{1/2}(z) = \frac{z^{-3/2}}{2\sqrt{\pi}} e^{-1/4z}, \qquad z = x/t^2. \tag{4.4}$$

For $\alpha = 1$ and $0 < |\theta| < 1$, one has to use the following special spectral function:

$$L^\theta_1 = \frac{1}{\pi}\frac{\cos(\theta\pi/2)}{[z+\sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2} \quad -\infty < z < \infty. \tag{4.5}$$

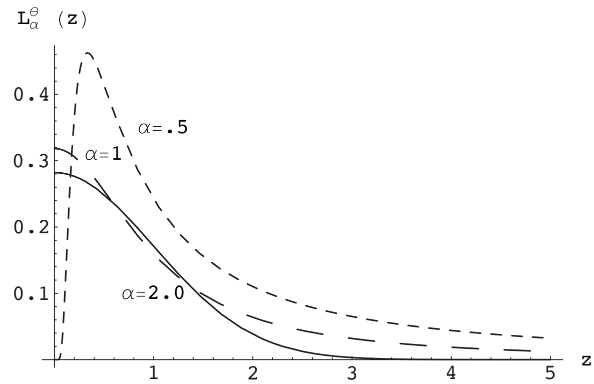

Fig. 1 Special formulae of $L^\theta_\alpha(z)$.

These special cases of the spectral functions, $L^\theta_\alpha(z)$, Eqs. (4.2) and (4.4), are plotted in Fig. 1. Unfortunately, no closed form expressions exist for general α-stable distributions other than Cauchy, Gaussian and Lévy–Smirnov. However, power series expansions can be derived for $L^\theta_\alpha(z)$. Feller [16] has given convergent and asymptotic power series of $L^\theta_\alpha(z)$, where z is the similarity variable. Later, Schneider [17] has corrected these formulae. The series are given for $z > 0$ and you can use the symmetry relation $L^\theta_\alpha(-z) = L^{-\theta}_\alpha(z)$ to determine the behavior as $z < 0$ while the behavior near zero and infinity is given separately.

First the case $0 < \alpha < 1$: this case has three convergent series. The formulae for $-\alpha \le \theta \le \alpha$,

$$L^\theta_\alpha(z) = \frac{1}{\pi z}\sum_{n=1}^{\infty}(-z^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!}\sin\Big(\frac{n\pi}{2}(\theta-\alpha)\Big), \quad z > 0, \quad z = x/t^{1/\alpha}, \tag{4.6}$$

its formulae near the origin but for $-\alpha < \theta \le \alpha$

$$L^\theta_\alpha(z) \sim \frac{1}{\pi z}\sum_{n=1}^{\infty}(-z)^n \frac{\Gamma(1+n/\alpha)}{n!}\sin\Big(\frac{n\pi}{2\alpha}(\theta-\alpha)\Big), \quad z \to 0^+, \tag{4.7}$$

finally near the origin and for $\theta = -\alpha$

$$L^{-\alpha}_\alpha(z) \sim \left\{\frac{1}{2\pi(1-\alpha)}\alpha^{1/(1-\alpha)}\right\}^{1/2} z^{-a_1} e^{-b_1 z^{c_1}}, \tag{4.8}$$

with

$$a_1 = \frac{2-\alpha}{2(1-\alpha)}, \quad b_1 = (1-\alpha)\alpha^{\alpha/(1-\alpha)}, \quad c_1 = \frac{\alpha}{1-\alpha}.$$

Second the case $1 < \alpha < 2$: this case has a convergent series for any $z \in \mathbb{R}$, where $z = x/t^{1/\alpha}$ and two asymptotic representations corresponding to the value of θ.

$$L^\theta_\alpha(z) = \frac{1}{\pi z}\sum_{n=1}^{\infty}(-z)^n \frac{\Gamma(1+n/\alpha)}{n!}\sin\Big(\frac{n\pi}{2\alpha}(\theta-\alpha)\Big), \quad z > 0, \quad |\theta| \le 2-\alpha, \tag{4.9}$$

for $\alpha - 2 < \theta \le 2 - \alpha$ and $z \to \infty$

$$L^\theta_\alpha(z) \sim \frac{1}{\pi z}\sum_{n=1}^{\infty}(-z^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!}\sin\Big(\frac{n\pi}{2}(\theta-\alpha)\Big), \tag{4.10}$$

Table 1 Used parameters in the calculations.

Case	Figure	α	β	θ
$L_1^0(z), L_2^0(z), L_{1/2}^{-1/2}(z)$	Fig. 1	$\alpha = 1, \alpha = 2, \alpha = .5$	$\beta = 1$	$\theta = 0, \theta = 0, \theta = 0.5$
Mother model $u(x, t)$	Fig. 2	$\alpha = 2$	$\beta = 1$	$\theta = 0$
Son model $v(\zeta, \tau)$	Fig. 3	$\alpha = 2$	$\beta = 1$	$\theta = 0$
Mother model $u(x, t)$	Fig. 4	$\alpha = 1.0$	$\beta = 1$	$\theta = -0.5$
Son model $v(\zeta, \tau)$	Fig. 5	$\alpha = 1$	$\beta = 1$	$\theta = -0.5$
Mother model $u(x, t)$	Fig. 6	$\alpha = 0.5$	$\beta = 1$	$\theta = -0.5$
Son model $v(\zeta, \tau)$	Fig. 7	$\alpha = 0.5$	$\beta = 1$	$\theta = -0.5$
Mother model $u(x, t)$	Fig. 8	$\alpha = 1$	$\beta = 1$	$\theta = 0.75$
Son model $v(\zeta, \tau)$	Fig. 9	$\alpha = 1$	$\beta = 1$	$\theta = 0.75$
Spectral $L_{0.8}^{0.5}(z)$	Fig. 10	$\alpha = 0.8$	$\beta = 1$	$\theta = 0$
Spectral $L_{0.9}^{0.4}(z)$	Fig. 11	$\alpha = 0.9$	$\beta = 1$	$\theta = 0$
Mother model $u(x, t)$	Fig. 12	$\alpha = 0.75$	$\beta = 1$	$\theta = 0.5$
Son model $v(\zeta, \tau)$	Fig. 13	$\alpha = 0.75$	$\beta = 1$	$\theta = 0.5$
Spectral $L_{1.5}^{0.4}(z)$	Fig. 14	$\alpha = 1.5$	$\beta = 1$	$\theta = 0.4$
Spectral $L_{1.75}^{0.1}(z)$	Fig. 15	$\alpha = 1.75$	$\beta = 1$	$\theta = 0.1$
Mother model $u(x, t)$	Fig. 16	$\alpha = 1.75$	$\beta = 1$	$\theta = 0.25$
Son model $v(\zeta, \tau)$	Fig. 17	$\alpha = 1.75$	$\beta = 1$	$\theta = 0.25$
Wright function M	Fig. 18	$\alpha = 2$	$\beta = 0.75$	
Wright function M	Fig. 19	$\alpha = 2$	$\beta = 0.5$	
Mother model $u(x, t)$	Fig. 20	$\alpha = 2$	$\beta = 0.75$	$\theta = 0$
Son model $v(\zeta, \beta)$	Fig. 21	$\alpha = 2$	$\beta = 0.75$	$\theta = 0$
$K_{\alpha,\alpha}^{\theta}$	Fig. 22	$\alpha = 0.75$	$\beta = 0.75$	$\theta = 0.5$
$K_{\alpha,\alpha}^{\theta}$	Fig. 23	$\alpha = 1.5$	$\beta = 1.5$	$\theta = 0.4$
Mother model $u(x, t)$	Fig. 24	$\alpha = 0.75$	$\beta = 0.75$	$\theta = 0.6$
Son model $v(\zeta, \tau)$	Fig. 25	$\alpha = 0.75$	$\beta = 0.75$	$\theta = 0.6$
Mother model $u(x, t)$	Fig. 26	$\alpha = 1.5$	$\beta = 1.5$	$\theta = 0.4$
Son model $v(\zeta, \tau)$	Fig. 27	$\alpha = 1.5$	$\beta = 1.5$	$\theta = 0.4$
Mother model $u(x)$	Fig. 28	$\alpha = 2$	$\beta = 0.75$	$\theta = 0$
Son model $v(\zeta)$	Fig. 29	$\alpha = 2$	$\beta = 0.75$	$\theta = 0$
Mother model $u(x)$	Fig. 30	$\alpha = 0.75$	$\beta = 0.75$	$\theta = 0.4$
Son model $v(\zeta)$	Fig. 31	$\alpha = 0.75$	$\beta = 0.75$	$\theta = 0.4$
Mother model $u(t)$	Fig. 32	$\alpha = 1.25$	$\beta = 1$	$\theta = 0.5$
Mother model $u(x)$	Fig. 33	$\alpha = 1.4, \alpha = 1.6, \alpha = 1.8$	$\beta = 1$	$\theta = 0.5$

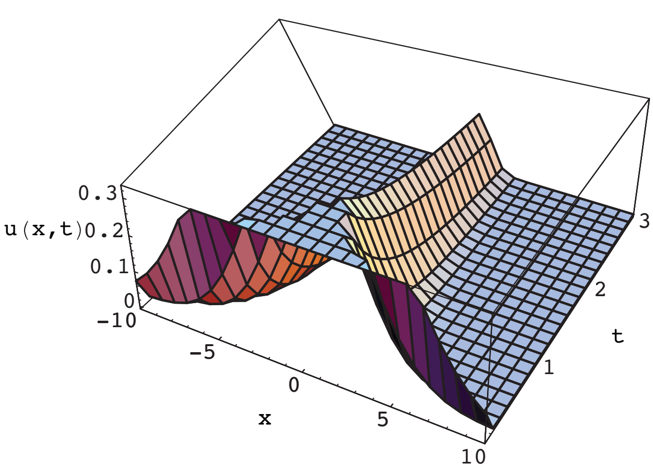

Fig. 2 Gaussian mother model.

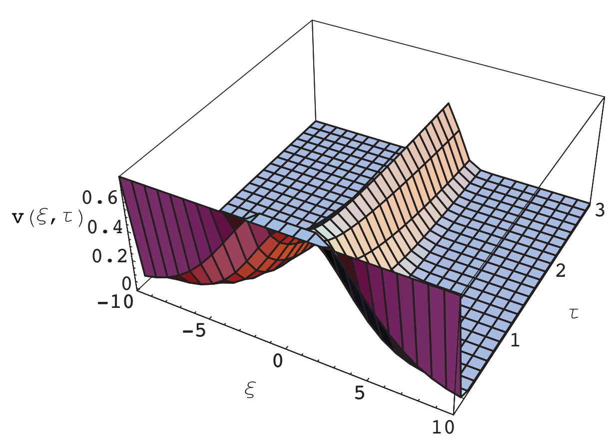

Fig. 3 Gaussian son model.

and finally for $\theta = \alpha - 2$ and $z \to \infty$

$$L_\alpha^{\alpha-2}(z) \sim (2\pi(\alpha-1)\alpha^{1/(\alpha-1)})^{-1/2\, a_2} e^{-b_2 z^{c_2}}, \tag{4.11}$$

where

$$a_2 = \frac{2-\alpha}{2(\alpha-1)}, \quad b_2 = (\alpha-1)\alpha^{\alpha/(\alpha-1)}, \quad c_2 = \frac{\alpha}{\alpha-1}.$$

I have listed all the needed convergent series for $z > 0$ and the asymptotic formulae but to plot these relations we still need their values at $z = 0$ which for $\beta = 1$ is calculated from the following equations:

$$L_\alpha^\theta(0) = K_{\alpha,1}^\theta(0) = \frac{1}{\pi\alpha}\Gamma(1/\alpha)\, cos\left(\frac{\theta\pi}{2\alpha}\right), \quad 0 < \alpha \le 2. \tag{4.12}$$

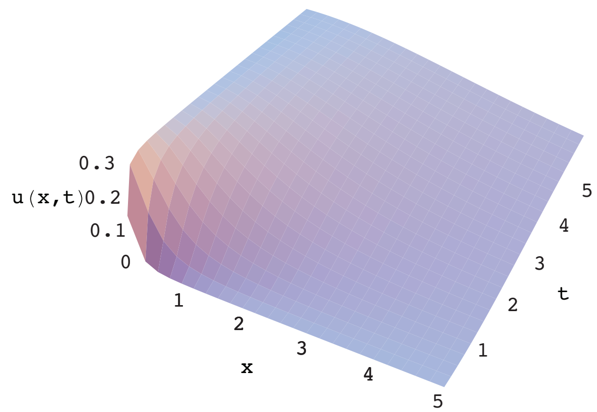

Fig. 4 Cauchy mother model.

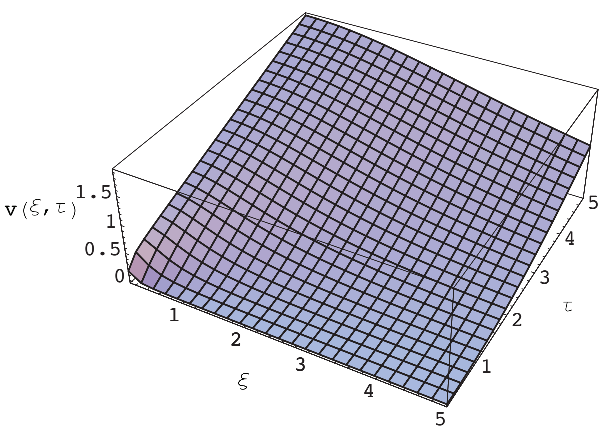

Fig. 5 Cauchy son model.

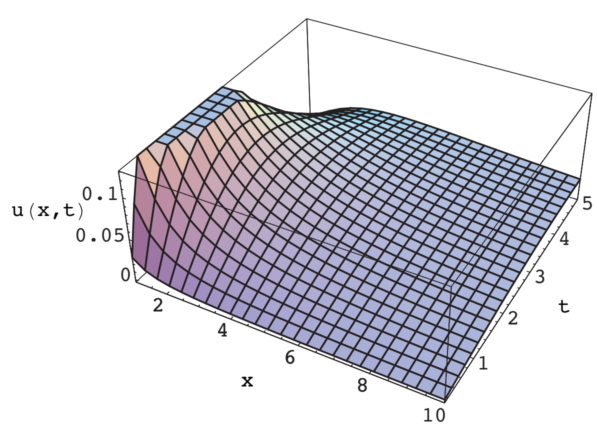

Fig. 6 Mother, $\alpha < 1$.

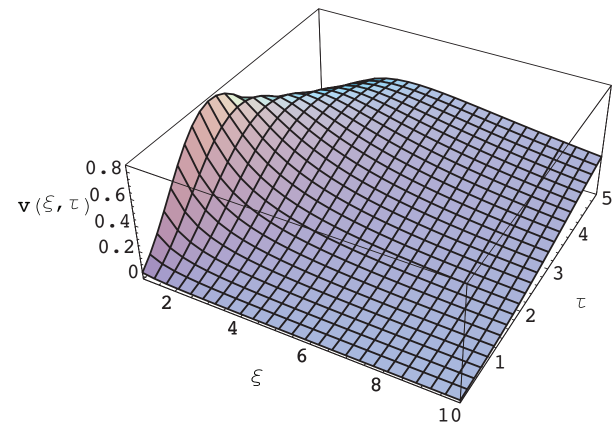

Fig. 7 Son, $\alpha < 1$.

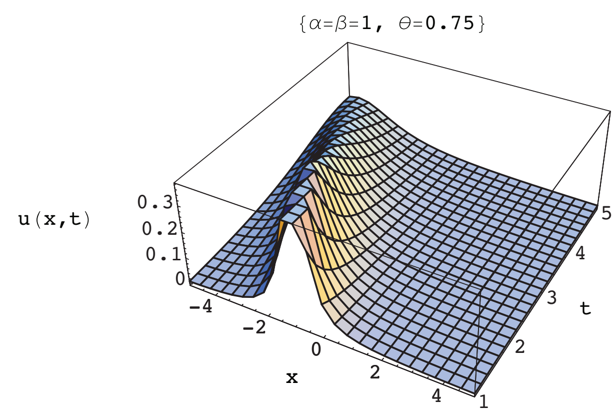

Fig. 8 Mother, $\alpha = \beta = 1$.

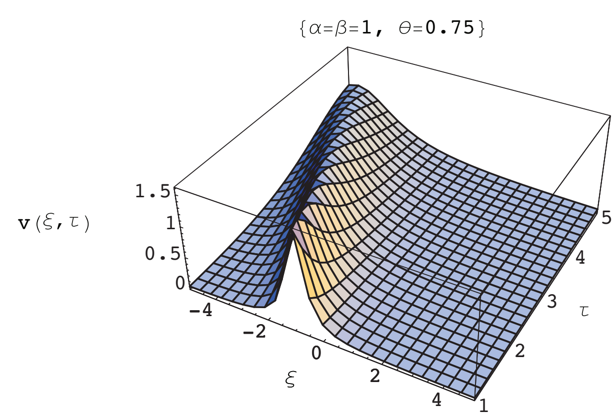

Fig. 9 Son, $\alpha = \beta = 1$.

That means, Eq. (4.12) is valid for all values of $\alpha \in (0, 2]$.

The matter with the time fractional is slightly different. In order to get the solution of Eq. (2.6) as a function of (x, t), one needs to use the so called M-function which is a special form of the so called Wright function, see [18–21]. In [21], Mainardi has presented the Wright function and the Mittag–Leffler functions and the relation between them in full details. M-function is defined as

$$\begin{aligned} M_\beta(z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\beta n + (1-\beta)]} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\beta n) \sin(\pi \beta n) \; z \geq 0, \end{aligned} \tag{4.13}$$

where $0 < \beta < 1$. The first special case is the time-fractional diffusion equation having the Green function

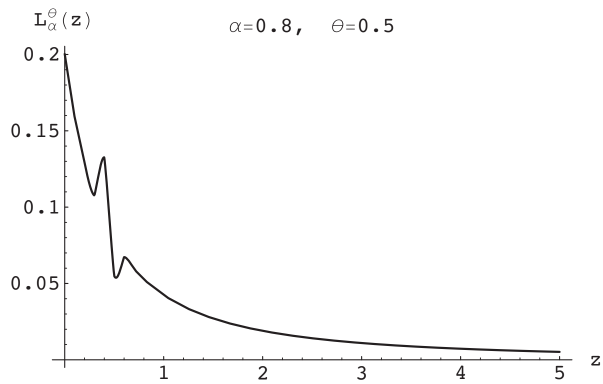

Fig. 10 Spectral, $\alpha < 1$.

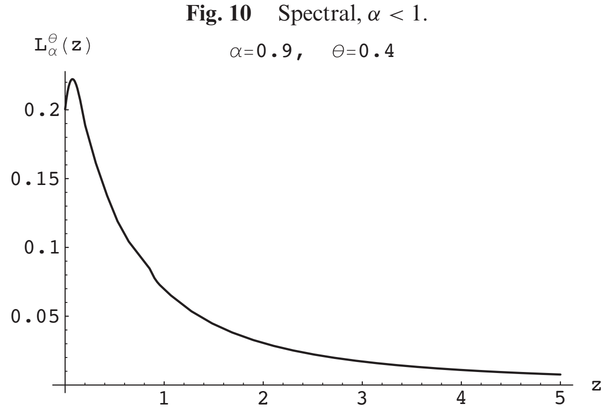

Fig. 11 Spectral, $\alpha < 1$.

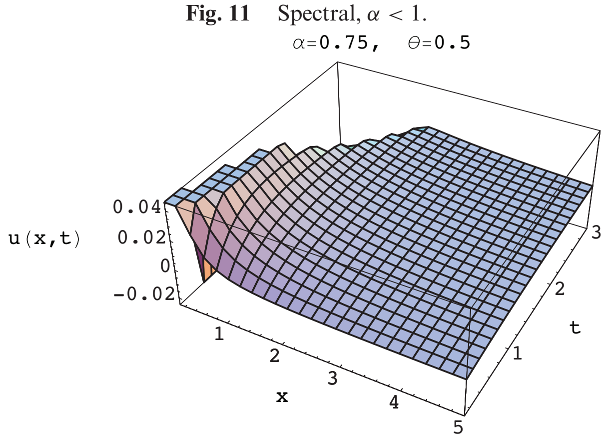

Fig. 12 Mother, $\alpha < 1$.

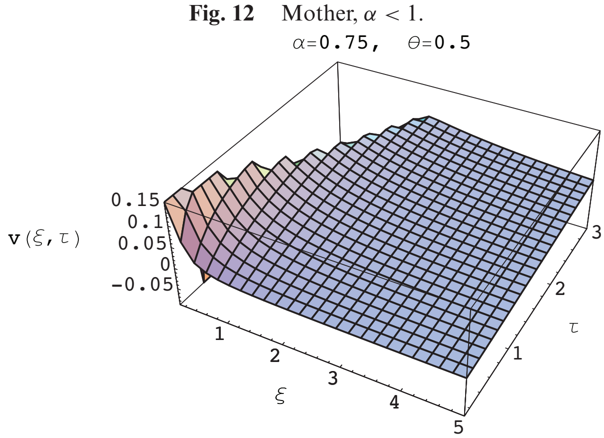

Fig. 13 Son, $\alpha < 1$.

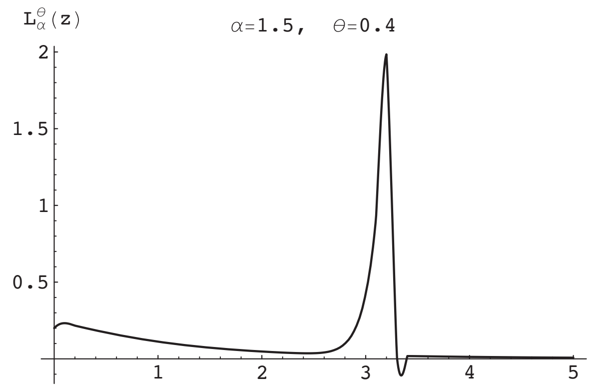

Fig. 14 Spectral, $\alpha > 1$.

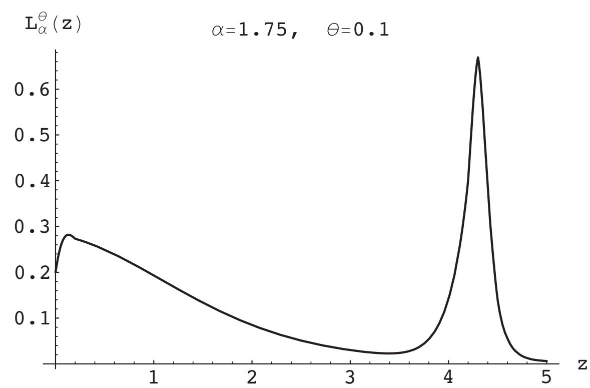

Fig. 15 Spectral, $\alpha > 1$.

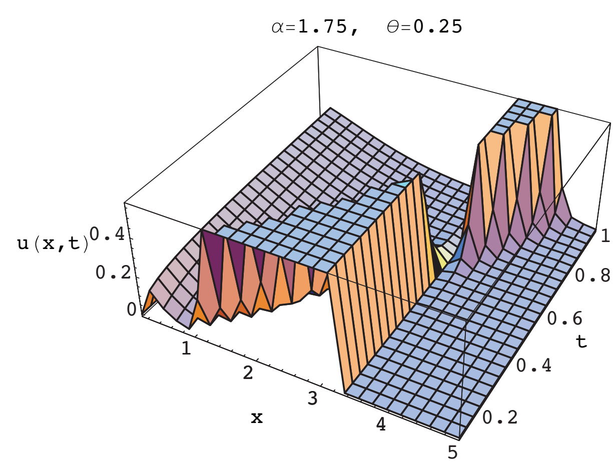

Fig. 16 Mother, $\alpha > 1$.

$$G^0_{2,\beta}(x,t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2}(|x|/t^{\beta/2}), \quad -\infty < x < \infty, \quad t \geq 0. \tag{4.14}$$

For $z \to \infty$, $G^0_{2,\beta}(z)$ is interpreted as a symmetric probability density function evolving in time with a stretched exponential decay as

$$G^0_{2,\beta}(z) = \frac{1}{2} M_{\beta/2}(z) \sim a z^{(\beta-1/2)/(1-\beta)}\, exp[-b z^{1/(1-\beta)}],$$

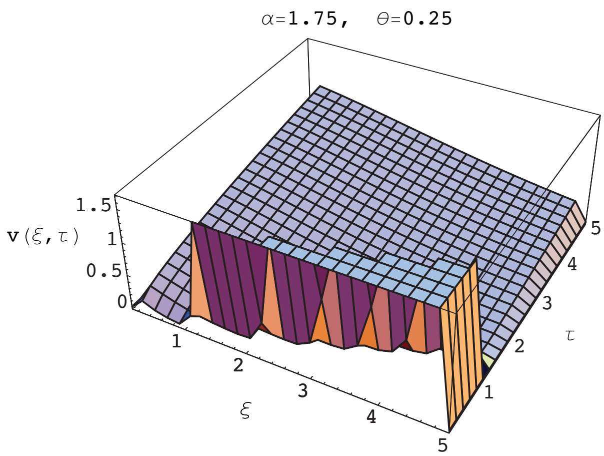

Fig. 17 Son, $\alpha > 1$.

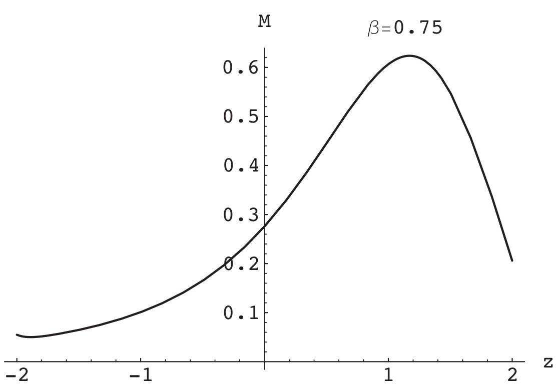

Fig. 18 Wright fun.

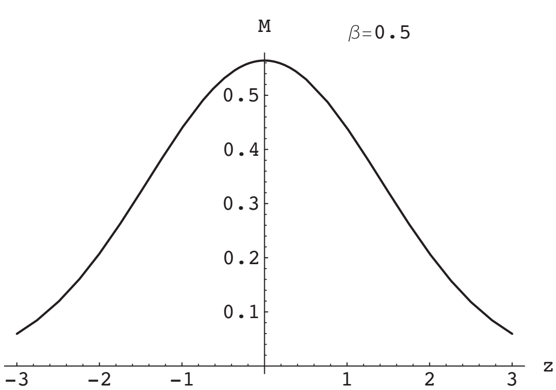

Fig. 19 Wright fun.

where

$$a = \frac{1}{\sqrt{2\pi(1-\beta)}} > 0, \quad b = \frac{1-\beta}{\beta} > 0$$

Dealing with both the fractional orders α and β is more complicated. You have to distinguish the cases $\alpha = \beta, \alpha < \beta$, and $\alpha > \beta$. First the case $\alpha = \beta$, to calculate the spectral functions and hence forth the fundamental solutions by using the similarity variable $z = (x/t^{\beta/\alpha})$, you have to use the following formula:

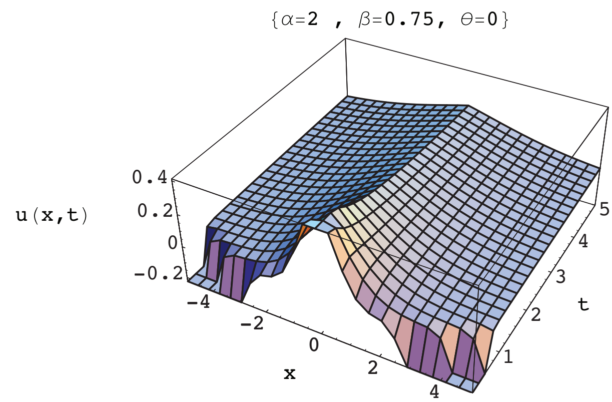

Fig. 20 Mother, $\beta < 1$.

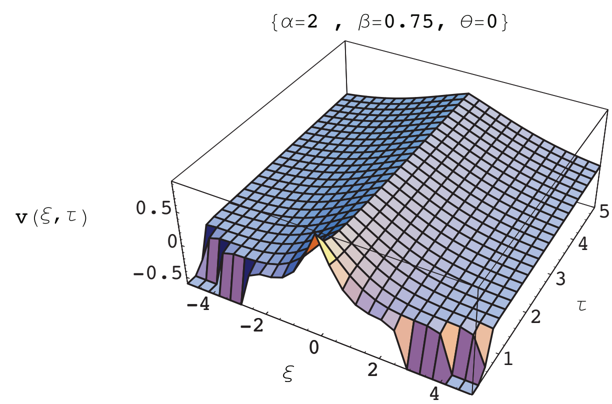

Fig. 21 Son, $\beta < 1$.

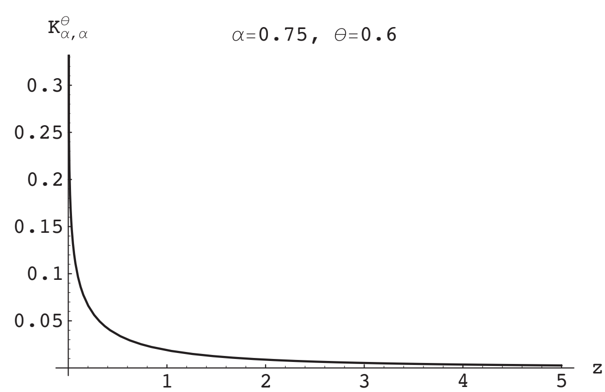

Fig. 22 $K^{\theta}_{\alpha,\alpha}$, $\alpha < 1$.

$$K^{\theta}_{\alpha,\alpha}(z) = \frac{1}{\pi} \frac{z^{\alpha-1}\sin\left[\frac{\pi}{2}(\alpha-\theta)\right]}{1 + 2z^{\alpha}\cos\left[\frac{\pi}{2}(\alpha-\theta)\right] + z^{2\alpha}},$$
$$0 < \alpha < 2, \quad 0 < z < \infty. \tag{4.15}$$

Second the case as $\alpha < \beta$, $\beta \neq 1$ with $|\theta| \leq \alpha$, one has to use the formula

$$K^{\theta}_{\alpha,\beta}(z) = \frac{1}{\pi z} \sum_{n=1}^{\infty} \frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)} \sin\left[\frac{n\pi}{2}(\theta-\alpha)\right](-z^{-\alpha})^n,$$
$$0 < z < \infty. \tag{4.16}$$

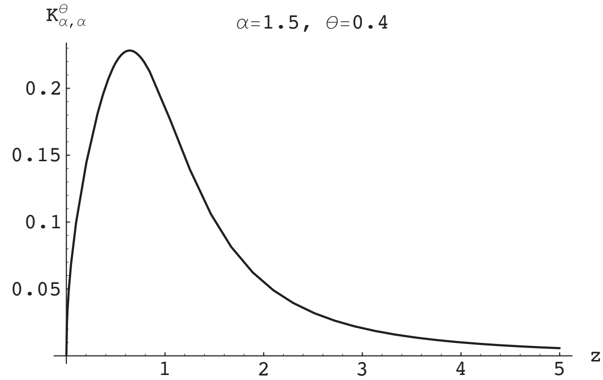

Fig. 23 $K^{\theta}_{\alpha,\alpha}$, $\alpha > 1$.

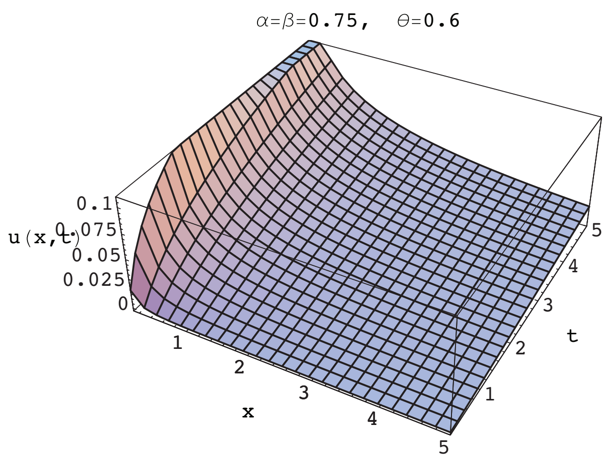

Fig. 24 Mother, $\alpha = \beta < 1$.

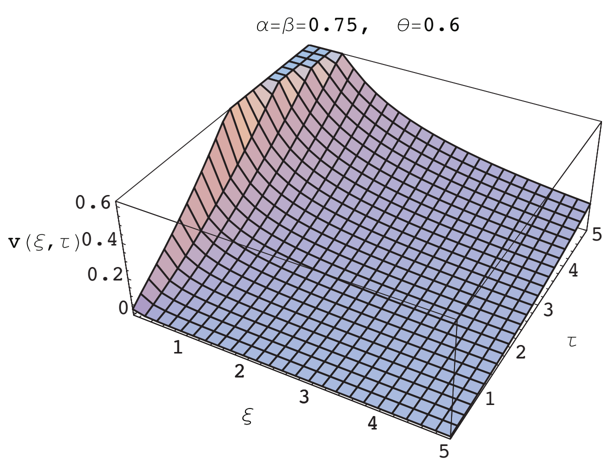

Fig. 25 Son, $\alpha = \beta < 1$.

Third the case as $\alpha > \beta$, i.e. $1 < \alpha < 2$ with $|\theta| \le 2 - \alpha$, you have to use the formula

$$K^{\theta}_{\alpha,\beta}(z) = \frac{1}{\pi z} \sum_{k=0}^{\infty} \frac{\Gamma(1-\alpha k)}{\Gamma(1-\beta k)} \sin\left[\frac{k\pi}{2}(\theta - \alpha)\right](-z^{\alpha})^{k}$$
$$+ \frac{1}{\pi z} \sum_{k=0}^{\infty} \frac{\Gamma(1-k/\alpha)\Gamma(1+k/\alpha)}{k!\Gamma(1-\frac{\beta}{\alpha}k)} \sin$$
$$\times \left[\frac{k\pi}{2\alpha}(\theta - \alpha)\right](-z)^{k} \; z > 0 \qquad (4.17)$$

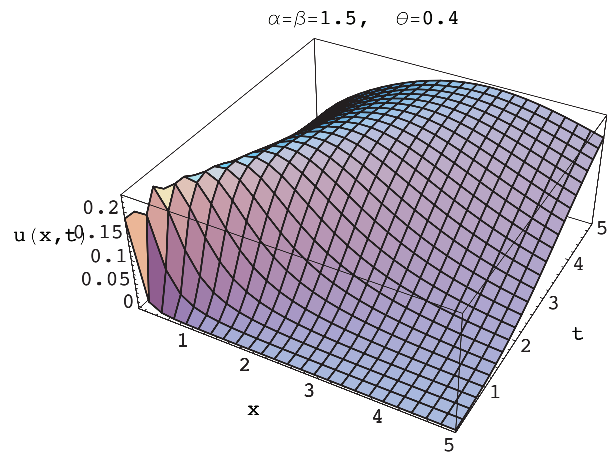

Fig. 26 Mother, $\alpha > 1$.

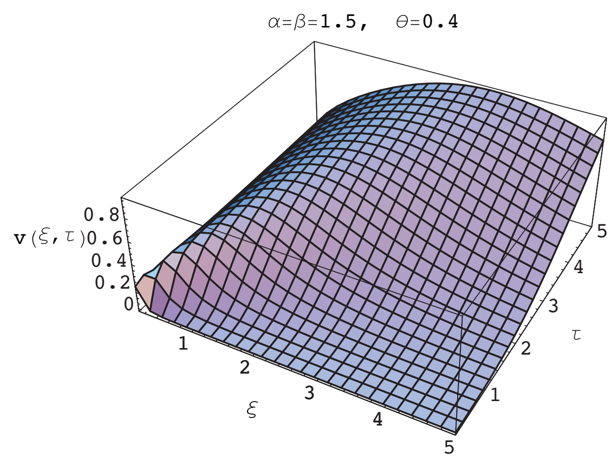

Fig. 27 Son, $\alpha > 1$.

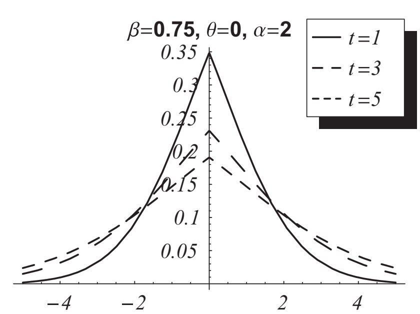

Fig. 28 Mother, t varies.

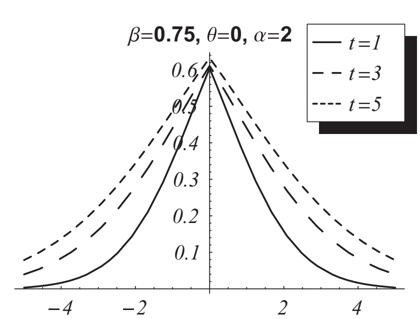

Fig. 29 Son, t varies.

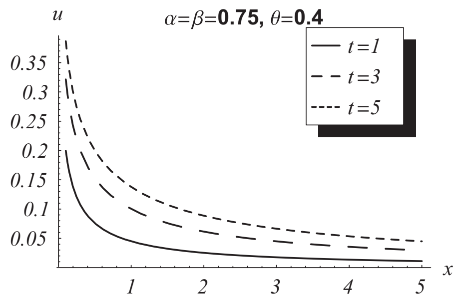

Fig. 30 Mother, t varies.

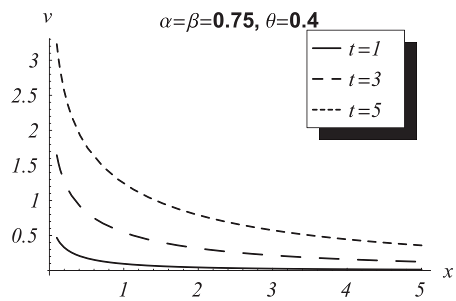

Fig. 31 Son, t varies.

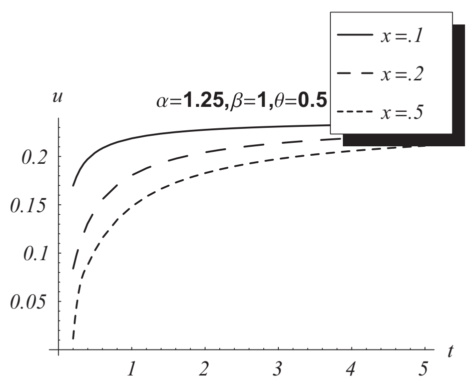

Fig. 32 Mother, x varies.

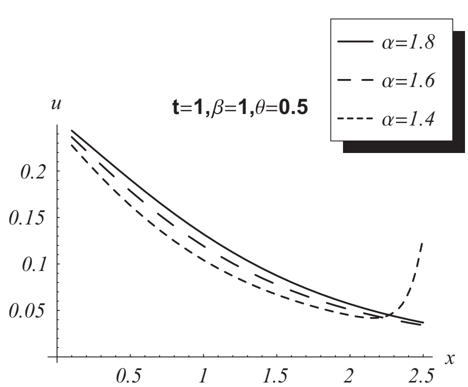

Fig. 33 Son, x varies.

When computing this equation you find that the factor $\Gamma(1-k/\alpha)\Gamma(1+k/\alpha)$ gives infinity of some values of k, and so you cannot plot this equation. The alternative way is to compute only its asymptotic representation as $z \to \infty$, with taking into consideration that θ has not any extremal values i.e. for $1 < \alpha < 2$, then $\theta < 2 - \alpha$, namely

$$K^{\theta}_{\alpha,\beta}(z) \sim \frac{1}{\pi z}\sum_{n=1}^{\infty}\frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)}\sin\left[\frac{n\pi}{2}(\theta-\alpha)\right](-z^{-\alpha})^{n},\quad z \to \infty. \tag{4.18}$$

If θ has extremal value, i.e. $\theta = -\alpha$, for the case $0 < \alpha < \beta$, then one has to use for z: $0 \to \infty$ the following formula:

$$K^{\theta}_{\alpha,\beta}(z) \sim \frac{1}{\pi z}\sum_{n=1}^{\infty}\frac{\Gamma(1+\alpha n)}{\Gamma(1+\beta n)}\sin\left[\frac{n\pi}{2}(\theta-\alpha)\right](-z^{-\alpha})^{n},\quad z: 0 \to \infty. \tag{4.19}$$

As in the case $\beta = 1$, you have to compute this case at $z = 0$ from a separate formula, namely

$$K^{\theta}_{\alpha,\beta}(0) = \frac{1}{\pi\alpha}\frac{\Gamma(1/\alpha)\Gamma(1-1/\alpha)}{\Gamma(1-\beta/\alpha)}\cos\left[\frac{\theta\pi}{2\alpha}\right]. \tag{4.20}$$

Now one has all the needed formulae to simulate the fundamental solutions of Eq. (1.2) and of Eq. (1.5) for all values of α, β and θ. One has to be aware of the value of the spatial coordinates in computing the numerical results.

5. Numerical results

In the following Table 1 lists all the numerical results which I have calculated to simulate the previous formulae for different values of the variables α, β, θ, x, ζ and τ. To plot the spectral function, you have to divide the interval into three intervals the first is at $z = 0$, the second is for $z \to 0$, and the third for $z \to \infty$. As you see in the case $1 < \alpha < 2$, with $|\theta| < 2 - \alpha$, the interval z: $0 \to 1$ is more stable than in the case $0 < \alpha < 1$, with $|\theta| \le \alpha$. The results obtained are consistent with all my previous works. For example in [6,7], I used the finite difference scheme to find the approximation solution of the son model. In this paper I used the Green function to simulate the analytical solution of the mother and son models in 2D and 3D. There are other authors who also used other methods to simulate the space time fractional diffusion processes. For example, El Danaf [22] studied the numerical solution of the s space–time fractional diffusion equation by using the spline function. The author analyzed the stability of the derived difference scheme. Liu et al. [23] used the method of lines to solve a special form of Fokker–Planck equation that describe the concentration of large number of independent solute particles.

Acknowledgment

I really want to thank the referees for their fruitful advices. By the aid of their comments and advices the paper has been appeared in this good form.

References

[1] G.W. Blumen, J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.
[2] G.W. Blumen, J.D. Cole, Similarity Methods for Differential Equations, Applied Mathematical Sciences No. 13, Springer Verlag, New York, 1974.

[3] Y. Luchko, R. Gorenflo, Scale-invariant Solutions of a partial differential equation of fractional order, Fract. Calc. Appl. Anal. 1 (1) (1998) 63–78.

[4] F. Mainardi, Yu. Luchko, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation, Fract. Calc. Appl. Anal. 4 (2) (2001) 153–192. Dedicated to R. Gorenflo on the occasion of his 70th birthday. Available on: www.fracalmo.org, preprint 0101.

[5] M.M. Rodringues, Study of solutions of a nonlinear fractional partial differential equation, in: Proceedings of the World Congress on Engineering, vol. 1, London, U.K., 2011.

[6] R. Gorenflo, E.A. Abdel-Rehim, Simulation of continuous time random walk of the space-fractional diffusion equations, J. Comput. Appl. Math. 222 (2008) 274–285.

[7] E.A. Abdel-Rehim, From the Eehrenfest model to time-fractional stochastic processes, J. Comput. Appl. Math. 233 (2009) 197–207.

[8] P. Biler, G. Karch, W.A. Woyczyn'ski, Fokker–Planck equations and conservation laws involving diffusion generators, in: Miscellanea No. 22 August 2002, Mini Proceedings on Lévy Processes, Theory and Applications, 2002, pp. 64–68. Available at: http://www.maphysto.dk.

[9] B.V. Gnedenko, A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley Publishing Company, Cambridge, MA, 1954. (translated from Russian by K.L. Chung).

[10] A.I. Saichev, G.M. Zaslavsky, Fractional kinetic equations: solutions and applications, Chaos 7 (1997) 753–764.

[11] M.M. Djrbashian, Integral Transforms and Representations of Functions in the Complex Plane, Nauka, Moscow, 1966. (in Russian).

[12] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. 1–2, McGraw-Hill, New York, 1953–1955.

[13] R. Metzler, J. Klafter, I.M. Sokolov, Anomalous transported in external fields: continous time random walks and fractional diffusion equations extended, Phys. Rev. E 48 (2) (1998) 1621–1633.

[14] E.A. Abdel-Rehim, Modelling and Simulating of Classical and Non-classical Diffusion Processes by Random Walks, Mensch & Buch Verlag, 2004 ISBN 3-89820-736-6. Available at: http://www.diss.fu-berlin.de/2004/168/index.html.

[15] W. Feller, An Introduction to Probability Theory and its Applications, vol. 2, Johon Wiley and Sons, New York, London, Sydney, Toronto, 1971.

[16] W. Feller, On a generalization of Mercel Riesz' potentials and the semi-groups generated by them, Medd. Lunds Univ. Mat. Sem. (Comm. Sém. Math. Univ. Lund), Tome suppl. dédié á M. Riesz, Lund, 1952, pp. 73–81.

[17] W.R. Schneider, Stable distributions: Fox function representation and generalization, in: S. Albeverio, G. Casati, D. Merlini (Eds.), Stochastic Processes in Classical and Quantum Systems, Lecture Notes in Physics, vol. 262, Springer-Verlag, Berlin, Heidelberg, 1986, pp. 497–511.

[18] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics, vol. 301, Longman, Harlow, 1994.

[19] R. Gorenflo, Y. Luchko, M. Mainard, Analytical properties and applications of the Wright function, Fract. Calc. Appl. Anal. 2 (1999) 383–414.

[20] R. Gorenflo, Y. Luchko, M. Mainard, Wright functions as scale-invariant solutions of the diffusion wave equation, J. Comput. Appl. Math. 118 (2000) 175–191.

[21] R. Gorenflo, M. Mainardi, On the distinguished role of the Mittag–Leffler and Wright functions in fractional calculus, in: Proceedings of International Conference on Special Functions in the 21st Century: Theory & Applications, Washington DC, USA, April 2011, pp. 6–8.

[22] Talat S. El Danaf, Numerical solution for the linear time and space fractional diffusion equation, J. Vib. Control (2013) 1–9.

[23] F. Liu, V. Anh, I. Turner, Numerical solution to the space fractional Fokker–Planck equation, J. Comput. Appl. Math. 166 (1) (2004) 209–219.