Journal of the Egyptian Mathematical Society (2016) **24**, 348–354

Egyptian Mathematical Society
**Journal of the Egyptian Mathematical Society**
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

# A convexity of functions on convex metric spaces of Takahashi and applications

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Received 29 August 2015; accepted 4 October 2015
Available online 28 November 2015

**Keywords**
Convex metric space;
*W*-convex functions;
Metric projection

**Abstract** We show that Takahashi's idea of convex structures on metric spaces is a natural generalization of convexity in normed linear spaces and Euclidean spaces in particular. Then we introduce a concept of convex structure based convexity to functions on these spaces and refer to it as *W*-convexity. *W*-convex functions generalize convex functions on linear spaces. We provide illustrative examples of (strict) *W*-convex functions and dedicate the major part of this paper to proving a variety of properties that make them fit in very well with the classical theory of convex analysis. As expected, the lack of linearity forced us to make some compromises in terms of conditions on either the metric or the convex structure. Finally, we apply some of our results to the metric projection problem and fixed point theory.

**2010 Mathematics Subject Classification:** 26A51; 52A01; 46N10; 47N10



## 1. Introduction and preliminaries

There have been a few attempts to introduce the structure of convexity outside linear spaces. Kirk [1,2], Penot [3] and Takahashi [4], for example, presented notions of convexity for sets in metric spaces. Even in the more general setting of topological spaces there is the work of Liepiņš [5] and Taskovič [6]. Takahashi [4] introduced the following general concept of convexity in metric spaces:

**Definition 1** [4]**.** Let $(X, d)$ be a metric space and $I = [0, 1]$. A continuous function $W : X \times X \times I \to X$ is said to be a *convex structure* on $X$ if for each $x, y \in X$ and all $t \in I$,

$$d(u, W(x, y; t)) \leq (1 - t)\, d(u, x) + t\, d(u, y) \qquad (1)$$

for all $u \in X$. A metric space $(X, d)$ with a convex structure $W$ is called a *convex metric space* and is denoted by $(X, W, d)$. A subset $C$ of $X$ is called *convex* if $W(x, y; t) \in C$ whenever $x, y \in C$ and $t \in I$.

What makes Takahashi's notion of convexity solid is the invariance under taking intersections and convexity of closed

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

S1110-256X(15)00080-2 
http://dx.doi.org/10.1016/j.joems.2015.10.003

balls ([4], Propositions 1 and 2). The convex structure $W$ in Definition 1 has the following property which is stated in [4] without proof. For the sake of completeness, we give a proof of it here.

**Lemma 1.** *For any* $x$, $y$ *in a convex metric space* $(X, W, d)$ *and any* $t \in I$ *we have*

$$d(x, W(x, y; t)) = t\, d(x, y),$$
$$d(y, W(x, y; t)) = (1-t)\, d(x, y).$$

**Proof.** For simplicity, let $a$, $b$ and $c$ stand for $d(x, W(x, y; t))$, $d(y, W(x, y; t))$ and $d(x, y)$ respectively. Using (1) we get $a \le t\,c$ and $b \le (1-t)\,c$. But $c \le a + b$ by the triangle inequality. So $c \le a + b \le (1-t)\,c + t\,c = c$. This means $a + b = c$. If $a < t\,c$ then we would have $a + b < c$ which is a contradiction. Therefore, we must have $a = t\,c$ and consequently $b = (1-t)\,c$. □

The necessity for the condition (1) on $W$ to be a convex structure on a metric space $(X, d)$ is natural. To see this, assume that $(X, \|.\|_X)$ is a normed linear space. Then the mapping $W\colon X \times X \times I \to X$ given by

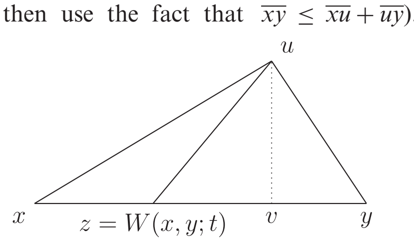
$$W(x, y; t) = (1-t)\,x + t\,y, \quad x, y \in X, \quad t \in I, \tag{2}$$

defines a convex structure on $X$. Indeed, if $\rho$ is the metric induced by the norm $\|.\|_X$ then

$$\begin{aligned}\rho(u, W(x, y; t)) &= \| u - ((1-t)x + ty) \|_X \\ &\le (1-t)\,\| u - x \|_X + t\,\| u - y \|_X \\ &= (1-t)\,\rho(u, x) + t\,\rho(u, y), \quad \forall u \in X,\ t \in I.\end{aligned}$$

The picture gets clearer in the linear space $\mathbb{R}^2$ with the Euclidean metric and the convex structure given by (2). In this case, given two points $x, y \in \mathbb{R}^2$ and a $t \in I$, $z = W(x, y; t)$ is a point that lies on the line segment joining $x$ and $y$. Moreover, Lemma 1 implies that if $\overline{xy} = L$ then $\overline{xz} = t\,L$ and $\overline{zy} = (1-t)\,L$ and we arrive at an interesting exercise of elementary trigonometry to show that $\overline{uz} \le (1-t)\,\overline{ux} + t\,\overline{uy}$ for any point $u$ in the plane. (Hint: apply the Pythagorean theorem to the triangles $\triangle uyv$, $\triangle uvz$ and $\triangle uvx$ in the figure below then use the fact that $\overline{xy} \le \overline{xu} + \overline{uy}$).

$$\frac{\overline{xz}}{\overline{zy}} = \frac{t}{1-t}$$




Takahashi's concept of convexity was used extensively in fixed point theory in metric spaces (cf. [7] and the references therein). One of its most important applications is probably iterative approximation of fixed points in metric spaces. There is quite huge literature on fixed point iterations (cf. [8,9]). Roughly speaking, the formation of most, if not all, known fixed point iterative procedures is based on that of the Mann iteration [10] and the Ishikawa iteration [11] as its very first generalization. All of these sequences require linearity and convexity of the ambient topological space. Although Takahashi's notion of convex metric spaces appeared in 1970, it was not until 1988 that Ding [12] exploited it to construct a fixed point iterative sequence and proved a convergence theorem in a convex metric space. To our best knowledge, this is the first time a fixed point iteration, other than the well-known Picard iteration, was introduced to metric spaces. Later, a lot of strong convergence results in convex metric spaces followed (see [8]).

In the light of Definition 1, it is tempting to identify convex functions on convex metric spaces. Based on the idea of convex structures on metric spaces, we define and illustrate by examples what we call $W$-convex functions. In linear metric spaces with $W$ defined by (2), $W$-convex functions coincide with traditional convex functions. We show throughout the paper that many of the main properties of convex functions on linear spaces are satisfied by $W$-convex functions. As expected some of these properties do not carry over automatically from linear spaces to convex metric spaces. In order to achieve such properties we had to require additional assumptions on the convex structure $W$. For instance, while midpoint convex continuous functions on normed linear spaces are convex, midpoint $W$-convexity on its own seems insufficient to obtain an analogous result in convex metric spaces. Another example appears when we study the equivalence between local boundedness from above and local Lipschitz continuity of $W$-convex functions. To achieve this equivalence we required the convex metric space to satisfy a certain property that is naturally satisfied in any linear space. Other properties necessitated providing a suitable framework to prove. For example, to investigate the relation between $W$-convexity of functions and the convexity of their epigraphs, we had to design a convex structure on product metric spaces to be able to define convex product metric spaces and characterize their convex subsets. Finally, we apply some of our results on $W$-convexity to the metric projection problem and fixed point theory. For this purpose, we give a definition for strictly convex metric spaces that generalizes strict convexity in Banach spaces and relate it to a certain class of strictly $W$-convex functions.

## 2. $W$-convex functions on convex metric spaces and their main properties

**Definition 2.** A realvalued function $f$ on a convex metric space $(X, W, d)$ is $W$-convex if for all $x$, $y \in X$ and $t \in I$, $f(W(x, y; t)) \le (1-t)\,f(x) + t\,f(y)$. We call $f$ strictly $W$-convex if $f(W(x, y; t)) < (1-t)\,f(x) + t\,f(y)$ for all distinct points $x, y \in X$ and every $t \in I^o = ]0, 1[$.

**Example 1.** Consider the Euclidean space $\mathbb{R}^3$ with the Euclidean norm $\|.\|$. Let $\mathcal{B}$ be the subset of $\mathbb{R}^3$ that consists of all closed balls $B(\xi, r)$ with center $\xi \in \mathbb{R}^3$ and radius $r > 0$. For any two balls $B(\xi_1, r_1)$, $B(\xi_2, r_2) \in \mathcal{B}$, define the distance function $d_{\mathcal{B}}(B(\xi_1, r_1), B(\xi_2, r_2)) = \| \xi_1 - \xi_2 \| + |r_1 - r_2|$. It is easy to check that $(\mathcal{B}, d_{\mathcal{B}})$ is a metric space. Let $W_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \times I \to \mathcal{B}$ be the continuous mapping given by

$$\begin{aligned}&W_{\mathcal{B}}(B(\xi_1, r_1),\ B(\xi_2, r_2); \theta) = B((1-\theta)\xi_1 + \theta\xi_2,\\ &\qquad (1-\theta)\,r_1 + \theta\,r_2), \quad \xi_i \in \mathbb{R}^3,\ r_i > 0,\ \theta \in I.\end{aligned}$$

Since for all $\theta \in I$ and any three balls $B(\xi_i, r_i) \in \mathcal{B}$, $i = 1, 2, 3$,

$$\begin{aligned}&d_{\mathcal{B}}\,(W_{\mathcal{B}}(B(\xi_1, r_1), B(\xi_2, r_2); \theta), B(\xi_3, r_3))\\ &= d_{\mathcal{B}}\,(B((1-\theta)\,\xi_1 + \theta\,\xi_2, (1-\theta)\,r_1 + \theta\,r_2), B(\xi_3, r_3))\\ &= \| (1-\theta)\,\xi_1 + \theta\,\xi_2 - \xi_3 \| + |(1-\theta)\,r_1 + \theta\,r_2 - r_3|\\ &\le (1-\theta)\,(\| \xi_1 - \xi_3 \| + |r_1 - r_3|) + \theta\,(\| \xi_2 - \xi_3 \| + |r_2 - r_3|)\\ &= (1-\theta)\,d_{\mathcal{B}}(B(\xi_1, r_1), B(\xi_3, r_3)) + \theta\,d_{\mathcal{B}}(B(\xi_2, r_2), B(\xi_3, r_3)).\end{aligned}$$

Then $(\mathcal{B}, W_{\mathcal{B}}, d_{\mathcal{B}})$ is a convex metric space. The function $f : \mathcal{B} \to \mathbb{R}$ defined by $f(B(\xi, r)) := \| \xi \| + |r|$ is $W_{\mathcal{B}}$-convex.

**Example 2.** Let $\mathcal{I}$ be the family of closed intervals $[a, b]$ such that $0 \le a \le b \le 1$ and define the mapping $W_{\mathcal{I}} : \mathcal{I} \times \mathcal{I} \times I$ by $W_{\mathcal{I}}(I_i, I_j; t) := \left[(1-t)a_i + ta_j, (1-t)b_i + tb_j\right]$ for $I_i = [a_i, b_i]$, $I_j = [a_j, b_j] \in \mathcal{I}$, $t \in I$. If $d_{\mathcal{I}}$ is the Hausdorff distance then $(\mathcal{I}, W_{\mathcal{I}}, d_{\mathcal{I}})$ is a convex metric space. This example of a convex metric space is given by Takahashi [4].

It is easy to verify that the Lebesgue measure defines a $W_{\mathcal{I}}$-convex function on $(\mathcal{I}, W_{\mathcal{I}}, d_{\mathcal{I}})$.

**Proposition 2** (*Composition with increasing convex functions*)**.** Assume that $f$ is a $W_X-$convex function on the convex metric space $(X, W_X, d_X)$. Let $g : f(X) \to \mathbb{R}$ be increasing and convex in the usual sense. Then $g \circ f$ is $W_X-$convex on $X$. The composition $g \circ f$ is strictly $W_X-$convex if $g$ is strictly convex or if $f$ is strictly $W_X$-convex and $g$ is strictly increasing.

**Proof.** Given $x, y \in X$ and $t \in I$, in the light of Definition 2, it follows from the monotonicity of $g$ that

$$\begin{aligned} g(f(W_X(x, y; t))) &\le g((1-t)f(x) + tf(y)) \\ &\le (1-t)\, g(f(x)) + t\, g(f(y)). \end{aligned}$$

□

**Example 3.** Let $(X, W_X, d_X)$ be a convex metric space and let $g : \mathbb{R} \to \mathbb{R}$ be increasing and (strictly) convex. Then the function $f : X \to \mathbb{R}$ defined by $f(x) := g(d_X(x, x_0))$ for some fixed $x_0 \in X$ is (strictly) $W_X$-convex. Examples of the function $g$ include $g(x) = x$,$g(x) = \chi_{[0,\infty[}(x)\, x^2$,$g(x) = \chi_{[0,\infty[}(x)\, |x|$ in the case of convexity and $g(x) = e^x$,$g(x) = \chi_{[0,\infty[}(x)\, |x|^\alpha$ with $\alpha > 1$ in the case of strict convexity.

**Proposition 3.** *Let $(X, W, d)$ be a convex metric space. Then*

1. *The restriction $g$ of a $W$-convex function $f$ on $X$ to a convex subset $C$ of $X$ is also $W$-convex.*
2. *If $f$ is a $W$-convex function on $X$ and $\alpha \ge 0$ then $\alpha f$ is also a $W$-convex function on $X$.*
3. *The finite sum of $W$-convex functions on $X$ is $W$-convex.*
4. *Conical combinations of $W$-convex functions is again $W$-convex.*
5. *The maximum of a finite number of $W$-convex functions is $W$-convex.*
6. *The pointwise limit of a sequence of $W$-convex functions is $W$-convex.*
7. *Suppose that $(Y_n)$ is a sequence of convex subsets of $X$ and that $f_n$ is a $W$-convex function on $Y_n$, $n \ge 1$. Let $S = \cap_n Y_n$ and $M = \{x \in X : \sup_n f_n(x) < \infty\}$. Then $M \cap S$ is convex and the upper limit of the family $(f_n)_{n \ge 1}$, the function $f = \sup_n f_n$, is $W$-convex on it.*
8. *If $f : X \to \mathbb{R}$ is a nontrivial strictly $W$-convex function then $f$ has at most one global minimizer on $X$.*

**Proof.**

1. By the convexity of $C$, the restriction of $f$ to $C$ makes sense and the $W$-convexity of $g$ on $C$ follows from the $W$-convexity of $f$ on $X$.
2. True since $\alpha f(W(x, y; t)) \le \alpha((1-t)f(x) + tf(y)) = (1-t)\alpha f(x) + t\alpha f(y)$.
3. Obvious from Definition 2 and the linearity of the summation operator.
4. Follows from 2 together with 3.
5. It suffices to show that $f = \max\{f_1, f_2\}$ is $W$-convex on $X$ given the $W$-convexity of both $f_1$ and $f_2$. For all $x, y \in X$ and $t \in I$ we have
$$\begin{aligned} f_i(W(x, y; t)) &\le (1-t)\, f_i(x) + t\, f_i(y) \\ &\le (1-t)\, f(x) + t\, f(y) \end{aligned}$$
which yields $f(W(x, y; t)) \le (1-t)\, f(x) + t\, f(y)$.
6. A consequence of the monotonicity of the limit.
7. Let $x, y \in M \cap S$. Then $x, y \in Y_n$ for all $n \ge 1$, $\sup_n f_n(x) < \infty$ and $\sup_n f_n(y) < \infty$. Fix $t \in I$ and $n \ge 1$. By the convexity of $Y_n$ we know that it contains $W(x, y; t)$. Hence $W(x, y; t) \in S$. To prove the convexity of $M \cap S$ it remains to show that $W(x, y; t) \in M$. This follows from the $W$-convexity of $f_n$ as $f_n(W(x, y; t)) \le (1-t)\, \sup_n f_n(x) + t\, \sup_n f_n(y) < \infty$. Finally, invoking the completeness axiom for the reals, the latter inequality implies $\sup_n f_n(W(x, y; t)) \le (1-t)\, \sup_n f_n(x) + t\, \sup_n f_n(y) < \infty$, which shows that $\sup_n f_n$ is $W$-convex on $M \cap S$.
8. Assume there are two distinct points $x, y \in X$ such that $f(x) = f(y) = \inf_{x \in X} f(x)$. By convexity of $X$ we have $W(x, y; \frac{1}{2}) \in X$. Since $f$ is strictly $W$-convex then $f\big(W(x, y; \frac{1}{2})\big) < \frac{1}{2} f(x) + \frac{1}{2} f(y) = \inf_{x \in X} f(x)$ which is a contradiction. □

## 3. *W*-convexity and continuity

We begin with proving Lipschitz continuity of $W$-convex functions on generalized segments in convex metric spaces.

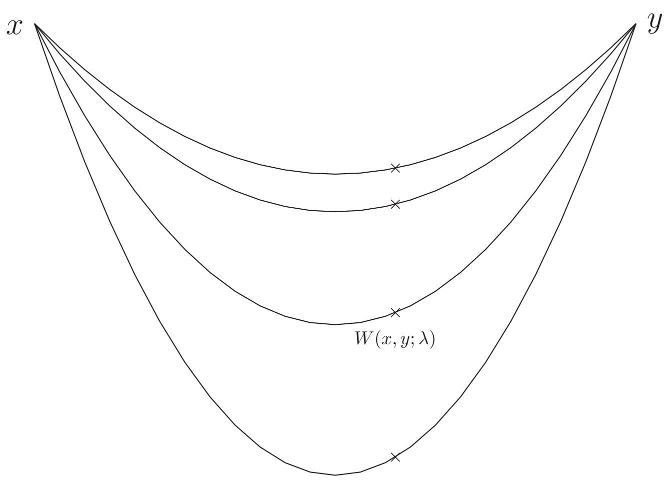
**Proposition 4.** *Let $(X, W, d)$ be a convex metric space. Let $x$ and $y$ be two distinct points in $X$. Then a $W$-convex function $f$ on the set $\mathcal{L}(x, y) = \{W(x, y; \lambda) : 0 \le \lambda \le 1\}$ is Lipschitz continuous on it with a Lipschitz constant that depends only on $x$ and $y$. Moreover, if $|f(x) - f(y)| \le \alpha\, d(x, y)$ for some $\alpha > 0$ then $|f(z) - f(w)| \le \alpha\, d(z, w)$ for all $z, w \in \mathcal{L}(x, y)$.*

Before proceeding with the proof of Proposition 4, we would like to make some remarks on the set $\mathcal{L}(x, y)$.

**Remark 1.** If $X$ is a linear space and $W$ is defined by (2) then $W(x, y; \lambda)$ is a unique vector in $X$ for each $\lambda \in I$ and the set $\mathcal{L}(x, y)$ is known ([13]) as the line segment joining the two vectors $x$ and $y$. Clearly, the Euclidean geometry justifies this notion. In metric spaces the situation is different as, for $\lambda \in I^o$, $W(x, y; \lambda)$ is not necessarily a unique point. In fact the continuity of $W$ in $\lambda$ required by Definition 1 is to be understood in the sense of continuity of multivalued functions. And if $\xi \in X$, the distance $d(\xi, W(x, y; \lambda))$ should be thought of as a point-set distance, but this is just a technicality. Nevertheless Lemma 1 assures that every point in the set $W(x, y; \lambda)$ belongs to $S(x, (1-\lambda)\, d(x, y)) \cap S(y, \lambda\, d(x, y))$ where $S(x_0, r)$ is the usual sphere with center $x_0$ and radius $r > 0$. Moreover, in the linear setting we have the symmetry $W(x, y; \lambda) = W(y, x; 1-\lambda)$ which leads to the symmetry $\mathcal{L}(x, y) = \mathcal{L}(y, x)$. While, from Definition 1 and Lemma 1 deduced from it, we have

$$\begin{aligned} d\big(W(x, y; \lambda), W(y, x; 1-\lambda)\big) &\le (1-\lambda)\, d\big(x, W(y, x; \lambda)\big) \\ &\quad + \lambda\, d\big(y, W(y, x; \lambda)\big) \\ &= \big((1-\lambda)^2 + \lambda^2\big)\, d(x, y). \end{aligned}$$

So, all that can be inferred in the convex metric space $(X, W, d)$ is $d\big(W(x, y; \lambda), W(y, x; 1-\lambda)\big) < 2d(x, y)$, $\lambda \in I^o$. Consequently $\mathcal{L}(x, y)$ is not to be assumed symmetric in general. Finally, observe that $\mathcal{L}(x, y)$ is closed. Indeed, it follows from Lemma 1 that any $u \in \mathcal{L}(y, x)$ can be written as $u = W(x, y, d(x, u)/d(x, y))$. So if $(z_n)$ is a sequence of elements of $\mathcal{L}(x, y)$ then $z_n = W(x, y, d(x, z_n)/d(x, y)), n \geq 1$. If in addition $z_n \to z$ as $n \to \infty$ then, by the continuity of $d$ and $W$, we formally get $z = \lim_{n\to\infty} W(x, y, d(x, z_n)/d(x, y)) = W(x, y, d(x, z)/d(x, y))$. Since $d(x, z_n) \leq d(x, y)$ then, passing to the limit, we also have $d(x, z) \leq d(x, y)$. This shows that $z \in \mathcal{L}(x, y)$.




$L(x, y) = \bigcup_{\lambda\in I} W(x, y; \lambda)$.

Now we prove Proposition 4.

**Proof.** Fix $x, y \in X$ so that $d(x, y) > 0$. Let $z, w \in \mathcal{L}(y, x)$ be such that $z \neq w$. Then, by $W$-convexity of $f$, we have

$$f(z) = f\left(W\left(x, y, \frac{d(x, z)}{d(x, y)}\right)\right) \leq \left(1 - \frac{d(x, z)}{d(x, y)}\right) f(x) + \frac{d(x, z)}{d(x, y)} f(y). \tag{3}$$

Similarly

$$f(w) = f\left(W\left(x, y, \frac{d(x, w)}{d(x, y)}\right)\right) \leq \left(1 - \frac{d(x, w)}{d(x, y)}\right) f(x) + \frac{d(x, w)}{d(x, y)} f(y). \tag{4}$$

Considering (3) and (4), we have only two possibilities. Either

$$f(z) - f(w) \leq (d(x, y))^{-1} (d(x, w) - d(x, z)) (f(x) - f(y)) \leq (d(x, y))^{-1} |f(x) - f(y)| d(z, w). \tag{5}$$

Or

$$f(z) - f(w) \leq (d(x, y))^{-1} (d(x, z) - d(x, w)) (f(x) - f(y)) \leq (d(x, y))^{-1} |f(x) - f(y)| d(z, w). \tag{6}$$

Interchanging $z$ and $w$ in both sides of (5) or (6) we immediately get

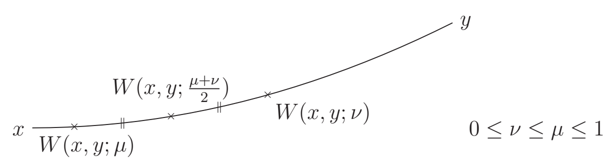
$$|f(z) - f(w)| \leq (d(x, y))^{-1} |f(x) - f(y)| d(z, w) \tag{7}$$

which proves that $f$ is Lipschitz continuous on $\mathcal{L}(y, x)$ with the Lipschitz constant $(d(x, y))^{-1} |f(x) - f(y)|$. The inequality (7) demonstrates the second assertion of the proposition as well. □

**Corollary 5.** *Let $(X, W, d)$ be a convex metric space. If a $W$-convex function $f$ on the set $\mathcal{L}(x, y) = \{W(x, y; \lambda) : 0 \leq \lambda \leq 1\}$ is such that $f(x) = f(y)$ then $f$ is constant on $\mathcal{L}(x, y)$.*

Continuous functions on convex metric spaces are $W$-convex provided that they are midpoint $W$-convex in a certain sense. We prove this in the following proposition.

**Proposition 6.** *Let $(X, W, d)$ be a convex metric space. Every continuous function $f : X \to \mathbb{R}$ such that $f(W(x, y; \frac{\mu+\nu}{2})) \leq \frac{1}{2} f(W(x, y; \mu)) + \frac{1}{2} f(W(x, y; \nu))$, $x, y \in X$, $\mu, \nu \in I$, is $W$-convex.*




**Proof.** Let $n$ be a nonnegative integer and let $\Lambda_n = \{m/2^n, m = 0, 1, \ldots, 2^n\}$. By induction on $n$, we show that

$$f(W(x, y; \lambda)) \leq (1 - \lambda) f(x) + \lambda f(y), \quad \text{for every } x, y \in X, \ \lambda \in \Lambda_n. \tag{8}$$

Since, by Lemma 1, $x = W(x, y; 0)$ and $y = W(x, y; 1)$ then (8) is valid when $n = 0$ as $\Lambda_0 = \{0, 1\}$. Assume that (8) is satisfied for any $\lambda \in \Lambda_k$ for some natural number $k$. Now let $x, y \in X$ and suppose that $\lambda \in \Lambda_{k+1}$. Obviously, there exist $s, t \in \Lambda_k$ such that $\lambda = (s + t)/2$. The induction hypothesis implies that

$$f(W(x, y; u)) \leq (1 - u) f(x) + u f(y), \quad u \in \{s, t\}. \tag{9}$$

By our assumption on $f$ we have

$$f(W(x, y; \lambda)) \leq \frac{1}{2} f(W(x, y; s)) + \frac{1}{2} f(W(x, y; t)). \tag{10}$$

Using (9) in (10) we obtain

$$\begin{aligned} f(W(x, y; \lambda)) &\leq \frac{1}{2} \sum_{u\in\{s,t\}} \big((1 - u) f(x) + u f(y)\big) \\ &= \left(1 - \frac{s + t}{2}\right) f(x) + \frac{s + t}{2} f(y) \\ &= (1 - \lambda) f(x) + \lambda f(y). \end{aligned}$$

This proves (8). Let $r \in I$ be arbitrary. Since the set $\Lambda = \cup_{n\geq 0}\Lambda_n$ is dense in $I$ then there exists a sequence $(r_n) \subset \Lambda$ that converges to $r$. Thus

$$f(W(x, y; r)) = f\Big(W(x, y; \lim_{n\to\infty} r_n)\Big) = \lim_{n\to\infty} f(W(x, y; r_n)) \tag{11}$$

by the continuity of both the convex structure $W$ and the function $f$. Since $r_n \in \Lambda$ then there exists an integer $m \geq 0$ such that $r_n \in \Lambda_m$. By (8), $f(W(x, y; r_n)) \leq (1 - r_n) f(x) + r_n f(y)$. From the latter inequality, the monotonicity of the limit and (11) we obtain

$$f(W(x,y;r)) \le (1-\lim_{n\to\infty} r_n)\, f(x) + \lim_{n\to\infty} r_n\, f(y) = (1-r)\, f(x) + r\, f(y).$$

□

The next lemma paves the way to Proposition 8 where we show that boundedness of $W$-convex functions on certain convex metric spaces is a necessary and sufficient condition for their continuity. In fact, our discussion in the rest of this section is confined to convex metric spaces $(X, W, d)$ that enjoy the property that for every two distinct points $x, y \in X$ and every $\lambda \in ]0, 1[$ there exists $\xi \in X$ such that $x = W(y, \xi; \lambda)$ or there exists $\eta \in X$ such that $y = W(x, \eta; \lambda)$. This property is naturally satisfied if $X$ is a linear space with $W$ defined as in (2). In that case, $\xi = \lambda^{-1}(x-y)+y$ and $\eta = \lambda^{-1}(y-x)+x$.

**Lemma 7.** Let $B(x_0, r)$ be an open ball centered at $x_0$ with radius $r > 0$ that is contained in $X$. If $f : X \to \mathbb{R}$ is $W$-convex such that $|f(x)| \le M$ on $B(x_0, r)$ then $f$ is $\frac{2M}{\rho}$–Lipschitz on $B(x_0, r-\rho)$, $0 < \rho < r$.

**Proof.** Let $x$ and $y$ be two distinct points in $B(x_0, r)$. Then, by our assumption on $(X, W, d)$, there exists $\xi \in X$ such that $x = W(y, \xi; \frac{d(x,y)}{\rho+d(x,y)})$ or there exists $\eta \in X$ such that $y = W(x, \eta; \frac{\rho}{\rho+d(x,y)})$. We shall deal with the first case and the second one can be treated analogously. First, since $f$ is $W$-convex then

$$f(x) \le \frac{\rho}{\rho+d(x,y)} f(y) + \frac{d(x,y)}{\rho+d(x,y)} f(\xi).$$

This implies

$$f(x) - f(y) \le \frac{f(\xi)-f(x)}{\rho} d(x,y). \tag{12}$$

Assume for the moment that $\xi \in B(x_0, r)$. Using the boundedness of $f$ on $B(x_0, r)$ the inequality (12) takes the form

$$f(x) - f(y) \le \frac{2M}{\rho} d(x,y). \tag{13}$$

Interchanging the roles of $x$ and $y$ then exploiting the symmetry of the metric, we deduce from (13) that

$$|f(x) - f(y)| \le \frac{2M}{\rho} d(x,y).$$

To complete the proof, it remains to show that $\xi \in B(x_0, r)$. From Lemma 1, we have

$$d(\xi, x) = d\left(\xi, W\left(y, \xi; \frac{d(x,y)}{\rho+d(x,y)}\right)\right) = \frac{\rho d(\xi,y)}{\rho+d(x,y)} \le \frac{\rho d(\xi,x)}{\rho+d(x,y)} + \frac{\rho d(x,y)}{\rho+d(x,y)}.$$

Solving this inequality for $d(\xi, x)$ we find $d(\xi, x) \le \rho$. Finally

$$d(\xi, x_0) \le d(\xi, x) + d(x, x_0) < \rho + r - \rho = r. \qquad □$$

**Proposition 8.** *A $W$-convex function $f$ on $X$ is locally bounded if and only if it is locally Lipschitz.*

**Proof.** Of course a locally Lipschitz function is continuous and therefore locally bounded. Let $f$ be locally bounded and let $x_0 \in X$. Then there exists $r > 0$ such that $f$ is bounded on $B(x_0, r)$ and, by Lemma 7, $f$ is Lipschitz on $B(x_0, r/2)$. Since $x_0$ was arbitrary then $f$ is locally Lipschitz. □

**Remark 2.** The local boundedness assumption on the $W$-convex function $f$ in Lemma 7, and consequently in Proposition 8, can be weakened to local boundedness from above. To prove this, assume that there exists $c > 0$ such that $f(\xi) \le c$ for every $\xi \in B(x_0, r)$ and let $x \in B(x_0, r)$. Then there exists $y \in X$ such that $x_0 = W(x, y, \frac{1}{2})$. Since, by Lemma 1, $d(y, x_0) = d(x, x_0) < r$ then $y \in B(x_0, r)$. Furthermore, by $W$-convexity of $f$, $2f(x_0) \le f(x) + f(y)$. So

$$2f(x_0) - c \le 2f(x_0) - f(y) \le f(x) \le c \implies |f(x)| \le c + 2|f(x_0)|.$$

**Remark 3.** Recall that a function $f : X \to \mathbb{R}$ is lower semicontinuous at $x_0$ if for every $t < f(x_0)$ there is an open neighborhood $\mathcal{N}_{x_0}$ of $x_0$ such that $f(x) > t$ for every $x \in \mathcal{N}_{x_0}$, and if $\forall\, t > f(x_0)\ \exists\, \mathcal{N}_{x_0} :\ f(x) < t,\ \forall\, x \in \mathcal{N}_{x_0}$ then $f$ is upper semicontinuous at $x_0$. It follows from Proposition 8 is that upper semicontinuous $W$-convex functions on open sets are continuous. The same applies to lower semicontinuous functions if and only if $X$ is complete. Furthermore, a family of continuous $W$-convex pointwise bounded functions on an open convex subset of a complete metric space is locally equi-bounded and locally equi-Lipschitz. The most important consequence of these facts is that pointwise convergence of sequences of continuous $W$-convex functions on open convex subsets of complete metric spaces is uniform on compact sets and preserves continuity. Since the proofs of these results (cf. [13–15] ) is indifferent to the topology of the space and does not depend on linearity, we find it redundant to give them here.

## 4. Epigraphs and sublevel sets of $W$-convex functions

The epigraph of a realvalued function $f$ on a set $C$ is the set $Epi(f) = \{(x,s) \in C \times \mathbb{R} : f(x) \le s\}$ and the sublevel set of $f$ of height $h$ is is the set $S_h(f) = \{x \in C : f(x) \le h\}$. In Proposition 11 below we show how $W$-convexity of functions is related to the convexity of their epigraphs and sublevel sets. First, let $(X, W_X, d_X)$ and $(Y, W_Y, d_Y)$ be two convex metric spaces. The mapping $d_p$: $(X \times Y)^2 \to [0, \infty[$,

$$d_p((x_1,y_1),(x_2,y_2)) = \begin{cases} \left((d_X(x_1,x_2))^p + (d_Y(y_1,y_2))^p\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}, & p = \infty, \end{cases}$$

is a metric on the Cartesian product $X \times Y$ and $(X \times Y, d_p)$ is called a product metric space. Now, let $(X, W_X, d_X)$ and $(Y, W_Y, d_Y)$ be two convex metric spaces. We note the following:

**Lemma 9.** *The mapping $W_{X\times Y}$: $(X \times Y)^2 \times I \to X \times Y$ given by $W_{X\times Y}((x_1,y_1),(x_2,y_2);t) = (W_X(x_1,x_2;t), W_Y(y_1,y_2;t))$ is continuous and defines a convex structure on the product metric space $(X \times Y, d_1)$.*

**Proof.** The continuity of $W_{X\times Y}$ follows from the continuity of the convex structures $W_X$ and $W_Y$. Let $t \in I$ and $(x_i, y_i) \in X \times Y$, $i = 1, 2, 3$. By the definition of $W_{X\times Y}$, it remains to prove that

$$d_1((x_3,y_3),(W_X(x_1,x_2;t), W_Y(y_1,y_2;t))) \le (1-t)\, d_1((x_1,y_1),(x_3,y_3)) + t\, d_1((x_2,y_2),(x_3,y_3)). \tag{14}$$

However, we shall pretend that we need to prove the inequality (14) for the metric $d_p$ with $1 \le p < \infty$. This enables us to demonstrate the difficulty in the proof for the case $p > 1$ and explain why the assertion of Lemma 9 is limited to the case $p = 1$. The metric $d_\infty$ is excluded for the same reason. Of course we could simply construct counterexamples for those cases but that would take us outside the scope of this paper. Now, for $1 \le p < \infty$, we exploit the following facts:

(i) $W_X$ and $W_Y$ are convex structures on $X$ and $Y$ respectively.
(ii) The map $x \mapsto x^p$ is monotonically increasing on $[0, \infty[$.
(iii) $(\mu + \nu)^p \le 2^{p-1}(\mu^p + \nu^p)$, for all $\mu, \nu \ge 0$.

We then see that

$$\begin{aligned} d_p^p&((x_3, y_3), (W_X(x_1, x_2; t), W_Y(y_1, y_2; t))) \\ &= \left(d_X(x_3, W_X(x_1, x_2; t))\right)^p + \left(d_Y(y_3, W_Y(y_1, y_2; t))\right)^p \\ &\le \left((1-t)d_X(x_1, x_3) + t d_X(x_2, x_3)\right)^p \\ &\quad + \left((1-t)d_Y(y_1, y_3) + t d_Y(y_2, y_3)\right)^p \\ &\le 2^{p-1}(1-t)^p\left[\left(d_X(x_1, x_3)\right)^p + \left(d_Y(y_1, y_3)\right)^p\right] \\ &\quad + 2^{p-1}t^p\left[\left(d_X(x_2, x_3)\right)^p + \left(d_Y(y_2, y_3)\right)^p\right] \\ &= 2^{p-1}\left[(1-t)^p d_p^p\left((x_1, y_1), (x_3, y_3)\right)\right. \\ &\quad \left. + t^p d_p^p\left((x_2, y_2), (x_3, y_3)\right)\right] \end{aligned}$$

which gives the desired inequality (14) when $p = 1$. □

Using Lemma 9, we can describe convex subsets of convex product metric spaces.

**Definition 3.** A subset $Z$ of the convex product metric space $(X \times Y, W_{X \times Y}, d_1)$ is convex if $W_{X \times Y}((x_1, y_1), (x_2, y_2); t) \in Z$ for all points $(x_1, y_1), (x_2, y_2) \in Z$ and all $t \in I$.

In the light of Definition 3 one can easily verify Lemma 10 below.

**Lemma 10.** *The intersection of any collection of convex subsets of the convex product metric space $(X \times Y, W_{X \times Y}, d_1)$ is convex.*

**Proposition 11.** *Let $f$ be a realvalued function on a convex metric space $(X, W_X, d_X)$. Then*

1. *The function $f$ is $W_X$-convex if and only if $Epi(f)$ is a convex subset of the convex product metric space $(X \times R, W_{X\times\mathbb{R}}, d_X + d_{\mathbb{R}})$, where $W_{\mathbb{R}}$ and $d_{\mathbb{R}}$ are the usual convex structure and metric on $\mathbb{R}$ respectively.*
2. *If $f$ is $W_X-$convex then the sublevel set $S_h(f)$ is a convex subset of $X$ for every $h \in \mathbb{R}$.*

**Proof.**

1. Suppose that $f$ is $W_X-$convex on $X$ and let $(x, s), (y, t) \in Epi(f)$. Then

$$f(W_X(x, y; \lambda)) \le (1-\lambda)f(x) + \lambda f(y) \le (1-\lambda)s + \lambda t$$

for all $\lambda \in I$. Therefore $(W_X(x, y; \lambda), (1-\lambda)s + \lambda t) \in Epi(f)$. That is

$$\begin{aligned} &W_{X\times\mathbb{R}}((x, s), (y, t); \lambda) \\ &\quad = (W_X(x, y; \lambda), W_{\mathbb{R}}(s, t; \lambda)) \in Epi(f), \quad \lambda \in I. \end{aligned}$$

Hence $Epi(f)$ is a convex subset of $X \times R$. Conversely, suppose $Epi(f)$ is convex. Fix $x, y \in X$ and $t \in I$. Since $(x, f(x))$, $(y, f(y)) \in Epi(f)$, then

$$\begin{aligned} &(W_X(x, y; t), W_{\mathbb{R}}(f(x), f(y); t)) \\ &\quad = W_{X\times\mathbb{R}}((x, f(x)), (y, f(y)); t) \in Epi(f). \end{aligned}$$

Thus $f(W_X(x, y; t)) \le W_{\mathbb{R}}(f(x), f(y); t) = (1-t)f(x) + t f(y)$, which is to say that $f$ is $W_X-$convex.
2. Let $t \in I$ and let $x, y \in S_h(f)$ so that $f(x) \le h$ and $f(y) \le h$. Since $f$ is $W_X-$convex then $f(W_X(x, y; t)) \le (1-t)f(x) + t f(y) \le h$. Therefore $W_X(x, y; t) \in S_h(f)$ and $S_h(f)$ is convex. □

The following theorem is an application of Lemma 10 and Proposition 11.

**Theorem 12.** *The pointwise supremum of an arbitrary collection of W-convex functions is W-convex.*

**Proof.** Let $(X, W, d)$ be a convex metric space. Let $J$ be some index set and assume that $\{f_i\}_{i\in J}$ is a collection of $W$-convex functions on $X$. Then, by Proposition 11, $Epi(f_i)$ is a convex subset of the convex product metric space $(X \times R, W_{X\times\mathbb{R}}, d_X + d_{\mathbb{R}})$ for every $i \in J$. If $f : X \to \mathbb{R}$ is such that $f(x) = \sup_{i\in J} f_i(x)$, $x \in X$, then $Epi(f) = \cap_{i\in J} Epi(f_i)$. By Lemma 10, $Epi(f)$ is a convex subset of $(X \times R, W_{X\times\mathbb{R}}, d_X + d_{\mathbb{R}})$ and, using Proposition 11, it follows that $f$ is $W$-convex on $X$. □

## 5. Applications to the projection problem and fixed point theory

Let $Y$ be a nonempty subset of a convex metric space $(X, W, d)$. The distance map (cf. [16]) $d_Y: X \to [0, \infty[$ is defined by $d_Y(x) = \inf_{y\in Y} d(x, y)$. The distance map $d_Y$ is $W$-convex. Indeed, if $x_1, x_2 \in X$, $y \in Y$ and $t \in I$ then, by the definition of $d_Y$, we have

$$\begin{aligned} d_Y(W(x_1, x_2; t)) &\le d(W(x_1, x_2; t), y) \\ &\le (1-t)\, d(x_1, y) + t\, d(x_2, y) \end{aligned}$$

for every $y \in Y$. Hence, by positive homogeneity and subadditivity of the infimum,

$$\begin{aligned} d_Y(W(x_1, x_2; t)) &\le \inf_{y\in Y}\left((1-t)\, d(x_1, y) + t\, d(x_2, y)\right) \\ &\le (1-t) \inf_{y\in Y} d(x_1, y) + t \inf_{y\in Y} d(x_2, y) \\ &= (1-t)\, d_Y(x_1) + t\, d_Y(x_2). \end{aligned}$$

If $Y$ is convex, then the metric projection operator (also called the nearest point mapping) (cf. [17]) $P_Y: X \to 2^Y$ is given by $P_Y(x) := \{y \in Y : d(x, y) = d_Y(x)\}$. If $P_Y(x) \ne \emptyset$ for every $x \in X$ then $Y$ is called proximal. $P_Y(x)$ is convex ([18], Lemma 3.2) and if $Y$ is closed then it is proximal. The proof of the proximality of $Y$ in this case is standard and given, in the setting of normed spaces, in many books (cf. [13,16]). We briefly sketch it here. There exists a minimizing sequence $(y_n) \subset Y$ such that $d(x, y_n) \to d_Y(x), x \in X$, as $n \to \infty$. So the sequence $(y_n)$ is bounded and, up to replacing it by a subsequence, it converges to $y$, say. Consequently, $d(x, y_n) \to d(x, y)$ as $n \to \infty$. Hence $d(x, y) = d_Y(x)$. Since $Y$ is closed then $y \in P_Y(x)$.

The set of metric projections $P_Y(x)$, if nonempty, is not necessarily a singleton. If $P_Y(x)$ is a singleton for each $x \in X$ then the convex set $Y$ is called a Chebyshev set. It is well-known (cf. [15]) that every closed convex subset of a strictly convex and reflexive Banach space is a Chebyshev set.

We would like to describe sufficient conditions for a point $x \in X$ to have a unique projection in $Y$. We begin with introducing a definition for strict convexity in convex metric spaces.

**Definition 4.** A convex metric space $(X, W, d)$ is strictly convex if for each $x_0 \in X$ and any two distinct points $x, y \in S(x_0, \rho)$ with $\rho > 0$, we have $W(x, y; t) \in B(x_0, \rho), \forall t \in I^o$.

**Remark 4.** If $X$ is a linear space endowed with a norm that induces the metric $d$ and $W$ is given by (2) then Definition 4, after normalizing and translating to the origin, coincides with the known definition of strictly convex normed spaces [15].

**Definition 5** ((Strict) $W$-convexity w.r.t spheres)**.** Let $(X, W, d)$ be a convex metric space. Fix $x_0 \in X$, $\rho > 0$ and $\sigma \in ]0, \rho[$. We call a realvalued function $f$ on $\overline{B(x_0, \rho)}$ $W$-convex w.r.t the sphere $S(x_0, \sigma)$ if

$$f(W(x, y; t)) \le (1-t)\, f(x) + t\, f(y),$$
$$\forall\, x, y \in S(x_0, \sigma),\ t \in I,$$

and we call it strictly $W$-convex w.r.t the sphere $S(x_0, \sigma)$ if

$$f(W(x, y; t)) < (1-t)\, f(x) + t\, f(y),$$
$$\forall\, x, y \in S(x_0, \sigma) \text{ with } x \ne y,\ \forall\, t \in I^o.$$

The following proposition is a direct consequence of Definitions 4 and 5.

**Proposition 13.** *Let $(X, W, d)$ be a convex metric space. If for each $x_0 \in X$ and $\rho > 0$, the function $f$: $X \to [0, \infty[$ defined by $f(x) := d(x, x_0)$ is strictly $W$-convex w.r.t the sphere $S(x_0, \rho)$ then the space $X$ is strictly convex.*

The following theorem asserts that closed convex subsets of strictly convex metric spaces are Chebyshev sets.

**Theorem 14.** *Assume that $Y$ is a closed convex subset of a strictly convex metric space $(X, W, d)$. Then every $x \in X$ has a unique projection on $Y$.*

**Proof.** Since $Y$ is closed then $P_Y(x) \ne \emptyset, \forall\, x \in X$ by the discussion above. If $x \in Y$ then $P_Y = \{x\}$. Let $x \in X - Y$ have two distinct projections $y_1, y_2 \in Y$. Then $d(x, y_1) = d(x, y_2) = d_Y(x)$. Let $t \in I^0$. Since $Y$ is convex then $W(y_1, y_2; t) \in Y$, and since $X$ is strictly convex then

$$d(W(y_1, y_2; t), x) < (1-t)\, d(y_1, x) + t\, d(y_2, x) = d_Y(x),$$

which is a contradiction. □

**Theorem 15.** *Let $Y$ be a compact convex subset of a strictly convex complete metric space. If $f$: $Y \to Y$ is continuous then it has a fixed point in $Y$.*

**Proof.** Since $Y$ is compact then it is closed and, by Theorem 14 above a Chebyshev set. The rest of the proof follows from Theorem 3.4 and Corollary 3.5 in [18]. □

**Theorem 16.** *Let $(X, W, d)$ be a convex metric space and let $T$: $X \to X$ is a nonexpansive mapping. Assume that the function $f$: $X \to [0, \infty[$ defined by $f(x) := d(x, Tx)$ is strictly $W$-convex with a local minimum at $\xi \in X$. Then $\xi$ is a fixed point of $T$.*

**Proof.** By Proposition 3, the point $\xi$ is the unique global minimizer of $f$. Suppose that $T\xi \ne \xi$. Since $X$ is convex then $W(\xi, T\xi; t) \in X \forall\, t \in I$, and since $f$ is strictly $W$-convex on $X$ then, for all $t \in I^o$, we have

$$\begin{aligned} f(W(\xi, T\xi; t)) &< (1-t)\, f(\xi) + t\, f(T\xi) \\ &= (1-t)\, d(\xi, T\xi) + t\, d(T\xi, T^2\xi) \\ &\le (1-t)\, d(\xi, T\xi) + t\, d(\xi, T\xi) \\ &= d(\xi, T\xi) = f(\xi), \end{aligned} \tag{15}$$

where we used nonexpansiveness of $f$ in estimating $d(T\xi, T^2\xi) \le d(\xi, T\xi)$. The strict inequality (15) contradicts the fact that $f(\xi) = \min_{x \in X} f(x)$. Therefore we must have $T\xi = \xi$. □

**Remark 5.** The function $f$ is continuous by the continuity of $T$. Hence, if $X$ is compact then there does exist a point $\xi \in X$ such that $f(\xi) = \min_{x \in X} f(x)$, and we do not need to make such an assumption on $f$.

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