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Original Article

# A new class defined by subordination for $\gamma$-spirallike functions

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**Abstract** In this paper we shall introduce and study some subordination results for the class of $\gamma$-spirallike univalent functions defined by convolution.

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## 1. Introduction

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in \mathcal{A}$ be given by (1.1) and $g$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

**Definition 1.** Let a function $f$ defined by (1.1) and $g$ defined by (1.2), the Hadamard product (or convolution) $(f{*}g)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

**Definition 2 ([1]).** A function $f(z) \in \mathcal{A}$ is in $S^{\gamma}(\alpha)$, the class of $\gamma$-spirallike functions of order $\alpha$ $(0 \le \alpha < 1, |\gamma| < \frac{\pi}{2})$, if and only if

$$\text{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\gamma \quad (z \in U). \tag{1.4}$$

We note that $S^{\gamma}(0) = S^{\gamma}$ (the class of $\gamma$ -spirallike functions) was introduced by Spacek [2] (also see [3]) and $S^{0}(0) = S^{*}$ (see

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Silverman [4]). Further, a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $C^{\gamma}(\alpha)$ if and only if

$$\mathrm{Re}\left\{e^{i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha\cos\gamma \quad (z \in U). \tag{1.5}$$

We note that $C^{\gamma}(0) = C^{\gamma}$, the class of functions $f(z)$, for which $zf'(z)$ is $\gamma$ -spirallike in $U$ introduced by Robertson [5] and the class $C^{\gamma}(\alpha)$ was introduced and studied by Chichra [6] and Sizuk [7]. From (1.4) and (1.5) it follows that:

$$f(z) \in C^{\gamma}(\alpha) \iff zf'(z) \in \mathcal{S}^{\gamma}(\alpha).$$

**Definition 3** [8]. (Subordination Principle). For two functions $f(z)$ and $F(z)$, analytic in $U$, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec F(z) (z \in U),$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $U$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 (z \in U)$$

such that

$$f(z) = F(\omega(z)) (z \in U).$$

Indeed it is known that

$$f(z) \prec F(z) (z \in U) \Rightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

In particular, if the function $F(z)$ is univalent in $U$, we have the following equivalence

$$f(z) \prec F(z) (z \in U) \iff f(0) = F(0) \text{ and } f(U) \subset F(U).$$

Dziok and Srivastava [9] defined a linear operator $H_{q,s}(\alpha_1) : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$H_{q,s}(\alpha_1)f(z) = z + \sum_{k=2}^{\infty}\Gamma_k(\alpha_1)a_k z^k, \tag{1.6}$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}....(\beta_s)_{k-1}} \cdot \frac{1}{(1)_{k-1}} \; (k \geq 2).$$

The linear operator $H_{q,s}(\alpha_1)$ includes (as its special cases) various other linear operators for example Carlson and Shaffer [10], Ruscheweyh [11] and others.

For fixed $A, B(-1 \leq B < A \leq 1)$, $0 \leq \lambda < 1$ and $|\gamma| < \frac{\pi}{2}$, we define the subclass $S_{\lambda}^{\gamma}(f, g; A, B)$ of $\mathcal{A}$ consisting of functions $f$ of the form (1.1) and functions $g$ is given by (1.2), with $b_k \geq 0$ as follows:

$$e^{i\gamma}\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)} \prec \cos\gamma\frac{1+Az}{1+Bz} + i\sin\gamma, \tag{1.7}$$

where

$$zF_{\lambda}'(f,g)(z) = z(f*g)'(z) + \lambda z^2(f*g)''(z),$$

and

$$F_{\lambda}(f,g)(z) = (1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)$$

From (1.7) and the definition of subordination, we obtain

$$e^{i\gamma}\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)} = \cos\gamma\frac{1+A\omega(z)}{1+B\omega(z)} + i\sin\gamma, \omega(z) \in \Omega$$

and hence

$$\left|\frac{\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)} - 1}{B\frac{zF_{\lambda}'(f,g)(z)}{F_{\lambda}(f,g)(z)} - [B + (A-B)\cos\gamma e^{-i\gamma}]}\right| < 1. \tag{1.8}$$

We note that:

(i) Putting $g(z) = \frac{z}{1-z}$ and $\lambda = 0$, we have $S_0^{\gamma}(f, \frac{z}{1-z}; A, B) = S^{\gamma}(A, B)$ (see Aouf [12], with $\alpha = 0$);
(ii) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 0$, $A = 1$ and $B = -1$, we have $S_0^{\gamma}(f, \frac{z}{1-z}; 1, -1) = S^{\gamma}$ (see Spacek [2]);
(iii) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 0$, $A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$, we have $S_0^{\gamma}(f, \frac{z}{1-z}; 1-2\alpha, -1) = S^{\gamma}(\alpha)$ (see Libera [1] and Kwon and Owa [13]);
(iv) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 1$, $A = 1$ and $B = -1$, we have $S_1^{\gamma}(f, \frac{z}{1-z}; 1, -1) = C^{\gamma}$ (see Robertson [5]);
(v) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 1$, $A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$, we have $S_1^{\gamma}(f, \frac{z}{1-z}; 1-2\alpha, -1) = C^{\gamma}(\alpha)$ (see Chichra [6] and Sizuk [7]);
(vi) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 0$, $A = (1-2\alpha)\beta$ and $B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$, we have $S^{\gamma}(f, \frac{z}{1-z}; (1-2\alpha)\beta, -\beta) = S^{\gamma}(\alpha, \beta)$;
(vii) Putting $g(z) = \frac{z}{1-z}$, $\lambda = 1$, $A = (1-2\alpha)\beta$ and $B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$, we have $S^{\gamma}(f, \frac{z}{1-z}; (1-2\alpha)\beta, -\beta) = C^{\gamma}(\alpha, \beta)$.

Also we note that:

(i) Putting $g(z) = z + \sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, we have $S_{\lambda}^{\gamma}(f, z + \sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k; A, B) = S_{\lambda}^{\gamma}(f, H_{q,s}(\alpha_1); A, B)$

$$= \left\{f \in \mathcal{A} : e^{i\gamma}\frac{z(H_{q,s}(\alpha_1)f(z))' + \lambda z^2(H_{q,s}(\alpha_1)f(z))''}{(1-\lambda)(H_{q,s}(\alpha_1)f(z)) + \lambda z(H_{q,s}(\alpha_1)f(z))'} \prec \cos\gamma\frac{1+Az}{1+Bz} + i\sin\gamma, z \in U\right\},$$

where $H_{q,s}(\alpha_1)$ is given by (1.6);

(ii) Putting $g(z) = z + \sum_{k=2}^{\infty}(\frac{1+\ell+\delta(k-1)}{1+\ell})^m z^k$, where $\delta \geq 0$; $\ell \geq 0$ and $m \in \mathbb{N}_0$, we have $S_{\lambda}^{\gamma}(f, z + \sum_{k=2}^{\infty}(\frac{1+\ell+\delta(k-1)}{1+\ell})^m z^k; A, B) = S_{\lambda}^{\gamma}(f, I_{\delta,\ell}^m; A, B)$

$$= \left\{f \in \mathcal{A} : e^{i\gamma}\frac{z(I_{\delta,\ell}^m f(z))' + \lambda z^2(I_{\delta,\ell}^m f(z))''}{(1-\lambda)(I_{\delta,\ell}^m f(z)) + \lambda z(I_{\delta,\ell}^m(\alpha_1)f(z))'} \prec \cos\gamma\frac{1+Az}{1+Bz} + i\sin\gamma, z \in U\right\},$$

where $I_{\delta,\ell}^m$ is Catas operator (see [14]);

(iii) Putting $g(z) = z + \sum_{k=2}^{\infty}\binom{k+\eta-1}{\eta}z^k$, where $\eta > -1$, we have $S_{\lambda}^{\gamma}(f, z + \sum_{k=2}^{\infty}\binom{k+\eta-1}{\eta}z^k; A, B) = S_{\lambda}^{\gamma}(f, D^{\eta}; A, B)$

$$= \left\{f \in \mathcal{A} : e^{i\gamma}\frac{z(D^{\eta}f(z))' + \lambda z^2(D^{\eta}f(z))''}{(1-\lambda)(D^{\eta}f(z)) + \lambda z(D^{\eta}f(z))'} \prec \cos\gamma\frac{1+Az}{1+Bz} + i\sin\gamma, z \in U\right\},$$

where $D^{\eta}$ is Ruscheweyh derivative [11] defined by

$$D^{\eta}f(z) = \frac{z(z^{\eta-1}f(z))^{\eta}}{\eta!} = \frac{z}{(1-z)^{\eta+1}} * f(z);$$

(iv) Putting $g(z) = z + \sum_{k=2}^{\infty}k^n z^k$, where $n \in \mathbb{N}_0$, we have $S_{\lambda}^{\gamma}(f, z + \sum_{k=2}^{\infty}k^n z^k; A, B) = S_{\lambda}^{\gamma}(f, D^n; A, B)$

$$= \left\{f \in \mathcal{A} : e^{i\gamma}\frac{z(D^n f(z))' + \lambda z^2(D^n f(z))''}{(1-\lambda)(D^n f(z)) + \lambda z(D^n f(z))'}\right.$$

$$\prec \cos\gamma \frac{1+Az}{1+Bz} + i\sin\gamma, z \in U\Bigg\},$$

where $D^n$ is Salagean operator [15] defined by

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k;$$

(v) Putting $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k$, where $m \in \mathbb{N}_0$ and $\ell \geq 0$ we have $S_\lambda^\gamma(f, z + \sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k; A, B) = S_\lambda^\gamma(f, I_\ell^m; A, B)$

$$= \Bigg\{f \in \mathcal{A} : e^{i\gamma} \frac{z\left(I_\ell^m f(z)\right)' + \lambda z^2\left(I_\ell^m f(z)\right)''}{(1-\lambda)\left(I_\ell^m f(z)\right) + \lambda z\left(I_\ell^m f(z)\right)'}$$
$$\prec \cos\gamma \frac{1+Az}{1+Bz} + i\sin\gamma, z \in U\Bigg\},$$

where $I_\ell^m$ is Cho and Kim operator [16], defined by

$$I_\ell^m f(z) = z + \sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m a_k z^k.$$

**Definition 4** [17]**.** (Subordination Factor Sequence). A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $U$, we have the subordination is given by

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) (z \in U; a_1 = 1). \tag{1.9}$$

## 2. Main results

To prove our main results we need the following lemmas.

**Lemma 1** [17]**.** *The sequence $\{c_k\}_{k=1}^{\infty}$ is subordinating factor sequence if and only if*

$$\mathbf{Re}\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \qquad (z \in U).$$

*Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $S_\lambda^\gamma(f, g; A, B)$.*

**Lemma 2.** *A function $f(z)$ of the form (1.1) is in the class $S_\lambda^\gamma(f, g; A, B)$ if*

$$\sum_{k=2}^{\infty}(1-\lambda+\lambda k)[(k-1)(1-B) + (A-B)\cos\gamma]|a_k||b_k|$$
$$\leq (A-B)\cos\gamma, \tag{2.1}$$

*where $-1 \leq B < A \leq 1, 0 \leq \lambda \leq 1, |\gamma| < \frac{\pi}{2}$ and $b_k \geq b_2 (k \geq 2)$.*

**Proof.** From (1.7), we obtain

$$e^{i\gamma}\frac{zF_\lambda'(f,g)(z)}{F_\lambda(f,g)(z)} = \cos\gamma\frac{1+A\omega(z)}{1+B\omega(z)} + i\sin\gamma, \omega \in \Omega$$

which implies that

$$\left|\frac{zF_\lambda'(f,g)(z) - F_\lambda(f,g)(z)}{BF_\lambda'(f,g)(z) - [B + (A-B)\cos\gamma e^{-i\gamma}]F_\lambda(f,g)(z)}\right| < 1,$$

we have

$$|zF_\lambda'(f,g)(z) - F_\lambda(f,g)(z)|$$
$$< |BF_\lambda'(f,g)(z) - [(A-B)\cos\gamma e^{-i\gamma} + B]F_\lambda(f,g)(z)|$$
$$= \left|\sum_{k=2}^{\infty}(1-\lambda+\lambda k)(k-1)a_k b_k z^k\right|$$
$$- | -[(A-B)\cos\gamma e^{-i\gamma}]z + \sum_{k=2}^{\infty}[Bk(1-\lambda+\lambda k)]a_k b_k z^k$$
$$- \sum_{k=2}^{\infty}(1-\lambda+\lambda k)[(A-B)\cos\gamma e^{-i\gamma} + B]a_k b_k z^k|$$
$$\leq \sum_{k=2}^{\infty}(1-\lambda+\lambda k)[(k-1) - Bk$$
$$+ (A-B)\cos\gamma + B]|a_k|b_k - (A-B)\cos\gamma < 0$$
$$\sum_{k=2}^{\infty}(1-\lambda+\lambda k)[(k-1)(1-B) + (A-B)\cos\gamma]|a_k|b_k$$
$$\leq (A-B)\cos\gamma,$$

and hence the proof of Lemma 2 is completed. □

**Remark 1.**

(i) Putting $g(z) = \frac{z}{1-z}, \lambda = 0, A = 1$ and $B = -1$ in Lemma 2, we obtain the result obtained by Silverman [18] (see also Singh [3]);
(ii) Putting $g(z) = \frac{z}{1-z}, \lambda = 0, A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$ in Lemma 2, we obtain the result obtained by Kwon and Owa [13], Theorem 2.3].

Let us denote by $S_\lambda^{*\gamma}(f, g; A, B)$, the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the inequality (2.1). We note that $S_\lambda^{*\gamma}(f, g; A, B) \subset S_\lambda^\gamma(f, g; A, B)$.

Employing the technique used earlier by Attiya [19] and Srivastava and Attiya [20], we prove:

**Theorem 1.** *Let $f(z) \in \mathcal{A}$ satisfies the inequality (2.1), and $K$ denote the class of the convex univalent functions in $U$. Then for every $h \in K$, we have*

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2 + (A-B)\cos\gamma\}}$$
$$\times (f * h)(z) \prec h(z) (z \in U), \tag{2.2}$$

*and*

$$\mathbf{Re}(f(z)) > -\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2 + (A-B)\cos\gamma}{[1-B+(A-B)\cos\gamma](1+\lambda)b_2},$$
$$(z \in U). \tag{2.3}$$

*The constant factor $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}$ in the subordination result (2.2) can not be replaced by any larger one.*

**Proof.** Let $f(z) \in S_\lambda^{*\gamma}(f, g; A, B)$ and let $h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in K$. Then we have

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2 + (A-B)\cos\gamma\}}(f * h)(z)$$
$$= \frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2 + (A-B)\cos\gamma\}}$$
$$\times \left(z + \sum_{k=2}^{\infty} a_k d_k z^k\right). \tag{2.4}$$

Thus, by Definition 3, the subordination result (2.2) will hold true if the sequence

$$\left\{\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2 + (A-B)\cos\gamma\}} a_k\right\}_{k=1}^{\infty} \tag{2.5}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalence to the following inequality:

$$\mathbf{Re}\left\{1+2\sum_{k=1}^{\infty}\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}a_k z^k\right\}$$
$$>0 (z\in U). \tag{2.6}$$

Now, since

$$\Psi(k) = (1-\lambda+\lambda k)[(k-1)(1-B)+(A-B)\cos\gamma]b_k$$

is an increasing function of $k (k \geq 2)$, we have

$$\mathbf{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}a_k z^k\right\}$$
$$= \mathbf{Re}\left\{1+\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}z\right.$$
$$+\frac{1}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}$$
$$\left.\times\sum_{k=2}^{\infty}[1-B+(A-B)\cos\gamma](1+\lambda)b_2 a_k z^k\right\}$$
$$\geq 1-\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}r$$
$$-\frac{1}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}$$
$$\times\sum_{k=2}^{\infty}[1-B+(A-B)\cos\gamma](1+\lambda)b_k|a_k|r^k$$
$$>1-\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}r$$
$$-\frac{(A-B)\cos\gamma}{\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}r$$
$$>0\left(|z|=r<1, |\gamma|<\frac{\pi}{2}\right),$$

where we have also made use of assertion (2.1) of Lemma 2. Thus (2.6) holds true in $U$. This proves the inequality (2.2). The inequality (2.3) follows from (2.2) by taking the convex function $h(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$. To prove the sharpness of the constant $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}$, we consider the function $f_0(z) \in S_{\lambda}^{*\gamma}(f, g; A, B)$ is given by

$$f_0(z) = z - \frac{(A-B)\cos\gamma}{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}z^2. \tag{2.7}$$

Thus from the relation (2.2) we obtain

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}(f_0)(z)$$
$$\prec \frac{z}{1-z}(z\in U). \tag{2.8}$$

It can easily be verified that

$$\min_{|z|\leq 1}\mathbf{Re}\left\{\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}(f_0)(z)\right\}$$
$$=-\frac{1}{2}. \tag{2.9}$$

This shows that the constant $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)b_2}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)b_2+(A-B)\cos\gamma\}}$ is the best possible. This completes the proof of Theorem 1. □

**Remark 2.**

(i) Putting $g(z) = \frac{z}{1-z}, \lambda = 0, A = 1$ and $B = -1$ in Theorem 1, we obtain the result obtained by Singh [[3], Theorem 2.1];
(ii) Putting $g(z) = \frac{z}{1-z}, \lambda = 0, A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$ in Theorem 1, we obtain the result obtained by Kwon and Owa [[13], Theorem 2.4].

Also, we establish subordination results for the associated subclasses, $C^{*\gamma}, C^{*\gamma}(\alpha), S^{*\gamma}(\alpha,\beta), C^{*\gamma}(\alpha,\beta)$, $S_{\lambda}^{*\gamma}(f, H_{q,s}(\alpha_1); A, B)$, $S_{\lambda}^{*\gamma}(f, I_{\delta,\ell}^{m}; A, B)$, $S_{\lambda}^{*\gamma}(f, D^{\eta}; A, B)$, $S_{\lambda}^{*\gamma}(f, D^{n}; A, B), S_{\lambda}^{*\gamma}(f, I_{\ell}^{m}; A, B)$.

Putting $g(z) = \frac{z}{1-z}, \lambda = A = 1$ and $B = -1$ in Theorem 1, we obtain the following corollary

**Corollary 1.** *Let the function f(z) defined by (1.1) be in the class* $C^{*\gamma}$ *and suppose that* $h(z) \in K$. *Then*

$$\frac{1+\sec\gamma}{3+2\sec\gamma}(f*h)(z) \prec h(z)(z\in U), \tag{2.10}$$

*and*

$$\mathbf{Re}(f(z)) > -\frac{3+2\sec\gamma}{2(1+\sec\gamma)}, \qquad (z\in U).$$

*The constant factor* $\frac{1+\sec\gamma}{3+2\sec\gamma}$ *in the subordination result (2.10) is the best possible.*

*Putting* $g(z) = \frac{z}{1-z}, \lambda = 1, A = 1-2\alpha (0 \leq \alpha < 1)$ *and* $B = -1$ *in Theorem 1, we obtain the following corollary*

**Corollary 2.** *Let the function f(z) defined by (1.1) be in the class* $C^{*\gamma}(\alpha)$ *and suppose that* $h(z) \in K$. *Then*

$$\frac{(1-\alpha)+\sec\gamma}{3(1-\alpha)+2\sec\gamma}(f*h)(z) \prec h(z)(z\in U), \tag{2.11}$$

*and*

$$\mathbf{Re}(f(z)) > -\frac{3(1-\alpha)+2\sec\gamma}{2[(1-\alpha)+\sec\gamma]}, \qquad (z\in U).$$

*The constant factor* $\frac{(1-\alpha)+\sec\gamma}{3(1-\alpha)+2\sec\gamma}$ *in the subordination result (2.11) is the best possible.*

*Putting* $g(z) = \frac{z}{1-z}, \lambda = 0, A = (1-2\alpha)\beta$ *and* $B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$, *in Theorem 1, we obtain the following corollary.*

**Corollary 3.** *Let the function f(z) defined by (1.1) be in the class* $S^{*\gamma}(\alpha,\beta)$ *and suppose that* $h(z) \in K$. *Then*

$$\frac{1+\beta+2\beta(1-\alpha)\cos\gamma}{2[1+\beta+4\beta(1-\alpha)\cos\gamma]}(f*h)(z) \prec h(z)(z\in U), \tag{2.12}$$

*and*

$$\mathbf{Re}(f(z)) > -\frac{1+\beta+4\beta(1-\alpha)\cos\gamma}{1+\beta+2\beta(1-\alpha)\cos\gamma}, \qquad (z\in U).$$

*The constant factor* $\frac{1+\beta+2\beta(1-\alpha)\cos\gamma}{2[1+\beta+4\beta(1-\alpha)\cos\gamma]}$ *in the subordination result (2.12) is the best possible.*

*Putting* $g(z) = \frac{z}{1-z}, \lambda = 1, A = (1-2\alpha)\beta$ *and* $B = -\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$, *in Theorem 1, we obtain the following corollary.*

**Corollary 4.** *Let the function f(z) defined by (1.1) be in the class* $C^{*\gamma}(\alpha,\beta)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{1+\beta+2\beta(1-\alpha)\cos\gamma}{2[1+\beta+3\beta(1-\alpha)\cos\gamma]}(f*h)(z)\prec h(z)(z\in U), \quad (2.13)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{1+\beta+3\beta(1-\alpha)\cos\gamma}{1+\beta+2\beta(1-\alpha)\cos\gamma}, \quad (z\in U).$$

*The constant factor* $\frac{1+\beta+2\beta(1-\alpha)\cos\gamma}{2[1+\beta+3\beta(1-\alpha)\cos\gamma]}$ *in the subordination result (2.13) is the best possible.*

*Putting* $g(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$, *where* $\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_s)_{k-1}}\cdot\frac{1}{(1)_{k-1}}(k\geq 2)$, *in Theorem 1, we obtain the following corollary.*

**Corollary 5.** *Let the function f(z) defined by (1.1) be in the class* $S_\lambda^{*\gamma}(f,H_{q,s}(\alpha_1);A,B)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)+(A-B)\cos\gamma\}}\times(f*h)(z)\prec h(z), \quad (z\in U), \quad (2.14)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)+(A-B)\cos\gamma}{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)}, \quad (z\in U).$$

*The constant factor* $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)\Gamma_2(\alpha_1)+(A-B)\cos\gamma\}}$ *in the subordination result (2.14) is the best possible.*

*Putting* $g(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+\ell+\delta(k-1)}{1+\ell}\right)^m z^k$, *where* $\delta\geq 0;\ \ell\geq 0$ *and* $m\in\mathbb{N}_0$, *in Theorem 1, we obtain the following corollary.*

**Corollary 6.** *Let the function f(z) defined by (1.1) be in the class* $S_\lambda^{*\gamma}(f,I_{\delta,\ell}^m;A,B)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{\lambda}{1+\ell})^m}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{\lambda}{1+\ell})^m+(A-B)\cos\gamma\}}\times(f*h)(z)\prec h(z), \quad (z\in U), \quad (2.15)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m+(A-B)}{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m}, \quad (z\in U).$$

*The constant factor* $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{\lambda}{1+\ell})^m}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{\lambda}{1+\ell})^m+(A-B)\cos\gamma\}}$ *in the subordination result (2.15) is the best possible.*

*Putting* $g(z)=z+\sum_{k=2}^{\infty}\binom{k+\eta-1}{\eta}z^k$, *where* $\eta>-1$ *in Theorem 1, we obtain the following corollary.*

**Corollary 7.** *Let the function f(z) defined by (1.1) be in the class* $S_\lambda^{*\gamma}(f,D^\eta;A,B)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)+(A-B)\cos\gamma\}}\times(f*h)(z)\prec h(z), \quad (z\in U), \quad (2.16)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)+(A-B)\cos\gamma}{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)}, \quad (z\in U).$$

*The constant factor* $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\eta)+(A-B)\cos\gamma\}}$ *in the subordination result (2.16) is the best possible.*

*Putting* $g(z)=z+\sum_{k=2}^{\infty}k^n z^k$, *where* $n\in\mathbb{N}_0$ *in Theorem 1, we obtain the following corollary.*

**Corollary 8.** *Let the function f(z) defined by (1.1) be in the class* $S_\lambda^{*\gamma}(f,D^n;A,B)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{2^n[1-B+(A-B)\cos\gamma](1+\lambda)}{2\{2^n[1-B+(A-B)\cos\gamma](1+\lambda)+(A-B)\cos\gamma\}}\times(f*h)(z)\prec h(z), \quad (z\in U), \quad (2.17)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{2^n[1-B+(A-B)\cos\gamma](1+\lambda)+(A-B)\cos\gamma}{2^n[1-B+(A-B)\cos\gamma](1+\lambda)}, \quad (z\in U).$$

*The constant factor* $\frac{2^n[1-B+(A-B)\cos\gamma](1+\lambda)}{2\{2^n[1-B+(A-B)\cos\gamma](1+\lambda)+(A-B)\cos\gamma\}}$ *in the subordination result (2.17) is the best possible.*

*Putting* $g(z)=z+\sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k$, *where* $m\in\mathbb{N}_0$ *and* $\ell\geq 0$ *in Theorem 1, we obtain the following corollary.*

**Corollary 9.** *Let the function f(z) defined by (1.1) be in the class* $S_\lambda^{*\gamma}(f,I_\ell^m;A,B)$ *and suppose that* $h(z)\in K$. *Then*

$$\frac{[1-B+(A-B)\cos\gamma](1+\lambda)\left(1+\frac{1}{1+\ell}\right)^m}{2\left\{[1-B+(A-B)\cos\gamma](1+\lambda)\left(1+\frac{1}{1+\ell}\right)^m+(A-B)\cos\gamma\right\}}\times(f*h)(z)\prec h(z) \quad (z\in U), \quad (2.18)$$

*and*

$$\mathrm{Re}(f(z))>-\frac{[1-B+(A-B)\cos\gamma](1+\lambda)\left(1+\frac{1}{1+\ell}\right)^m+(A-B)\cos\gamma}{[1-B+(A-B)\cos\gamma](1+\lambda)\left(1+\frac{1}{1+\ell}\right)^m} \quad (z\in U).$$

*The constant factor* $\frac{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{1}{1+\ell})^m}{2\{[1-B+(A-B)\cos\gamma](1+\lambda)(1+\frac{1}{1+\ell})^m+(A-B)\cos\gamma\}}$ *in the subordination result (2.18) is the best possible.*

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