Journal of the Egyptian Mathematical Society (2016) **24**, 373–380

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

A spectral Rayleigh–Ritz scheme for nonlinear partial differential systems of first order

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Received 11 October 2015; revised 17 November 2015; accepted 29 November 2015
Available online 12 January 2016

Keywords

Ultraspherical integral zeros method;
Rayleigh–Ritz method;
Boundary value problems;
Nonlinear first-order partial differential systems;
Multi-objective constrained optimization

Abstract Nonlinear systems of partial differential problems of first order with Dirichlet boundary conditions is considered. ultraspherical integral zero- boundary (UIZB) method is combined with Rayleigh–Ritz method to approximate the unknowns. The approach converts the problem to be a multi-objective constrained optimization problem which is easier to solve. Accurate results can be obtained by selecting a limited number of collocation points. Numerical examples are included to demonstrate the accuracy and efficiency of the propped method.

2010 Mathematics Subject Classification: 33C45; 65M60; 35F30; 90C29; 65K10

1. Introduction

For last few decades, spectral methods using expansion in orthogonal polynomials such as Chebyshev or ultraspherical polynomials (see for instance [1,2]) is well-known for its high accuracy. The pseudospectral method has been developed to obtain more accurate solutions in scientific computation. Doha et al. [3] constructed the Jacobi–Gauss–Lobatto pseudospectral schemes for numerically solving a certain nonlinear Schrodinger equations. Doha et al. [4] investigated a Chebyshev–Gauss–Radau collocation method in combination with the implicit Runge–Kutta scheme to obtain more accurate numerical solutions for hyperbolic systems of first order. Naher et al. [5] proposed extension of the generalized and improved (G'/G)-expansion method for constructing class of exact traveling wave solutions of nonlinear evolution equations. Demiray et al. [6] combine the (G0/G; 1/G)-expansion method with Maple to obtain exact travelling wave solutions of the nonlinear wave equations.

Rayleigh–Ritz method is used to convert differential equations to a minimization problem for certain criteria. Many papers discussed the use of this method to solve several problems, such as modeling the expansion of an elastic body [7], approximating part of the spectrum of an elliptic operator [8] and

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Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.11.001

obtaining results for the time period and deflection of certain modes of vibration of rectangular plates [9].

According to the close relation of boundary value problems of partial differential equations to physical applications, the theory of boundary value problems is very rich. Several technological processes and scientific applications yield boundary value problems for PDE's. Howison and Oliver [10] analyzed a free boundary problem arising in a model for inviscid, incompressible shallow water entry at small deadrise angles. El Dhaba et al. [11] used a boundary integral method to solve a problem of uncoupled magnetothermoelasticity for an infinite, elliptical cylindrical conductor carrying a steady axial and uniform electric current. Khanday [12] treated the temperature distribution in multi-layered human skin and subcutaneous tissues and suggested a model of the solution of parabolic heat equation.

Chen [13] studied a free boundary value problem of the Euler system arising in the inviscid steady supersonic flow past a symmetric curved cone. Tsai [14] combined the homotopy analysis method with the method of fundamental solutions and the augmented polyharmonic spline to solve certain nonlinear partial differential equations. Feng et al. [15] developed a new framework for designing and analyzing convergent finite difference methods for approximating both classical and viscosity solutions of second order fully nonlinear partial differential equations (PDEs) in 1-D. Hosseini et al. [16] applied the operational Tau method with arbitrary polynomial bases to approximate the solution of a class of nonlinear transient heat conduction equations with some supplementary conditions.

Khalil et al. [17] developed an operational matrix with shifted Legendre polynomials to approximate solution of fractional differential equations(FDEs) and coupled system of FDEs with variable coefficients. The proposed method converts the problem into a system of easily solvable algebraic equations. The authors discussed also the convergence of the scheme and solved some test problems to show the efficiency and applicability of the method. Khalil et al. [18] extended the idea of pseudo spectral method to approximate solution of time fractional order three-dimensional heat conduction equations on a cubic domain. They studied shifted Jacobi polynomials and provide a simple scheme to approximate function of multi variables in terms of these polynomials. They developed operational matrices for arbitrary order integrations as well as for arbitrary order derivatives.

In the present paper, we numerically solve partial differential systems of first order. In fact, we treat with this problem as follows: We use ultraspherical integral method zero- boundary (UIZB) method to approximate the unknowns. We apply Rayleigh–Ritz method to reformulate the problem to be multi-objective constrained optimization problem. The resulting constrained optimization problem is then solved by sequential minimization processes of the Penalty leap frog method.

The outline of this paper is arranged as follows. In the next section, some properties of ultraspherical polynomials and ultraspherical integral matrix is investigated. In Section 3 Model of the problem is introduced. In Section 4, the proposed method, namely, the ultraspherical integral zero- boundary-Rayleigh–Ritz (UIZB-RR) method is constructed for solving the proposed problem. Error estimates and convergence index is investigated in Section 5. Some numerical examples are proposed in Section 6 to show the accuracy of our method. Finally, in Section 7, some observations and conclusions are presented.

2. Ultraspherical integral method

The ultraspherical polynomials $\{G_k(\lambda,\ x)\}_{k=0}^{\infty}$, where $\lambda > -0.5$ is a parameter, are defined by:

$$G_{k+1}(\lambda, x) = \frac{2(k+\lambda)}{k+2\lambda} x G_k(\lambda, x) - \frac{k}{k+2\lambda} G_{k-1}(\lambda, x), \quad k = 1, 2, \ldots, \tag{2.1}$$

$$G_k(\lambda, x) = \frac{d}{dx}\left[\frac{1}{2(k+1)} G_{k+1}(\lambda, x) - \frac{k}{2(k+2\lambda)(k+2\lambda-1)} G_{k-1}(\lambda, x)\right]. \tag{2.2}$$

Eq. (2.1) defines the ultraspherical polynomials starting with $G_1(\lambda,\ x) = x,\ G_0(\lambda,\ x) = 1,$ whereas Eq. (2.2) can be used to define the integration of the ultraspherical polynomials(by simple integration) see El-Hawary et al. [19].

We define the collocation points to be the ultraspherical zeros points combined with the two boundary points of the interval, that is:

$$\Lambda = \{x_j |\ G_N(\lambda, x_j) = 0, k_j = 1, 2, \ldots, N-1, x_0 = -1, x_N = 1\}. \tag{2.3}$$

With this definition, we have

$$\int_{-1}^{x_i} f(x)dx = \sum_{k_j=0}^{N} S_{ij}^{[\lambda]} f(x_j), \tag{2.4}$$

where the element of the ultraspherical integral matrix of first degree S, are defined by [17]:

$$S_{ij}^{[\lambda]} = \sum_{k=0}^{N} \frac{\varpi_j G_k(\lambda, x_j)}{\alpha_k} \int_{-1}^{x_i} G_k(x)dx, \quad i,\ j, = 0, \quad 1, \ldots N, \tag{2.5}$$

and $G_k(x)$ is the ultraspherical polynomial of degree k, where ϖ_j and α_k obtained by

$$\varpi_j = \frac{1}{\sum_{k=0}^{N} \frac{\varpi_j (G_k(x_j))^2}{\lambda_k}},$$
$$\alpha_k = \frac{\mathrm{j}!\Gamma(\lambda+.5)\Gamma(k+\lambda+.5)\Gamma(K+\lambda)\Gamma(2\lambda)}{2^{1-2k-2\lambda-\tau}\Gamma(2k+2\lambda+1)\Gamma(k+2\lambda)\Gamma(\lambda)}, \tag{2.6}$$

with

$$\tau = \begin{cases} 1, & if \quad \lambda = k = 0 \\ 0, & otherwise. \end{cases} \tag{2.7}$$

3. Model of the problem

We consider the general form of system of nonlinear boundary value problem of first order PDE. It can be defined by the following equations:

$$L_k\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \chi_k(x, y),$$
$$k = 1, 2 \quad and \quad (x, y) \in \varpi = [-1,\ 1] \times [-1,\ 1], \tag{3.1}$$

with Dirichlet boundary condition

$$u(-1, y) = f_1(y), \quad u(x, -1) = f_2(x), \tag{3.2a}$$

$$v(-1, y) = f_3(y), \quad v(x, -1) = f_4(x), \tag{3.2b}$$

$$u(1, y) = \overline{f_1}(y), \quad u(x, 1) = \overline{f_2}(x), \tag{3.3a}$$

$$v(1, y) = \overline{f_3}(y), \quad v(x, 1) = \overline{f_4}(x). \tag{3.3b}$$

Note that any problem defined on any interval $[a, b]$ can be transformed to the above problem making use of the change of variable:

$$\bar{x} = a + \frac{(b-a)(x+1)}{2}, \quad \bar{y} = a + \frac{(b-a)(y+1)}{2}. \tag{3.4}$$

4. The proposed method

The ultraspherical integral zero- boundary (UIZB-RR) method is constructed as follows:

4.1. Ultraspherical integral zero- boundary (UIZB) method

Each of the first partial derivative of the variables in the problem, $\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\}$, must be approximated by some variable, then the unknown variables in the problem $\{u, v\}$, can be approximated by simple integration and making use of (2.4)–(2.7).

To apply UIZB method for the problem (3.1)–(3.3), We put

$$\frac{\partial u}{\partial x} = \zeta \quad \text{and} \quad \frac{\partial u}{\partial y} = \eta. \tag{4.1}$$

Integrating (4.1) and using conditions (3.2a), we get respectively:

$$u(x, y) = \int_{-1}^{x} \zeta(\check{x}, y)d\check{x} + c_1, \quad u(x, y) = \int_{-1}^{y} \eta(x, \check{y})d\check{y} + c_2, \tag{4.2}$$

thus, $c_1 = u(-1, y) = f_1(y)$, $c_2 = u(x, -1) = f_2(x)$.

Making use of ultraspherical integral approximations (2.4)–(2.7), we obtain

$$u(x_i, y_j) = \sum_{k=0}^{N} S_{ik}\, \zeta(x_k, y_j) + f_1(y_j), \quad \tilde{u}(x_i, y_j) = \sum_{(k=0)}^{N} S_{ik}\eta(x_i, y_k) + f_2(x_i). \tag{4.3}$$

The resulting two approximate solution obtained in (4.3) must be equal. So, we take as a condition to be satisfied that:

$$u(x_i, y_j) = \tilde{u}(x_i, y_j). \tag{4.4}$$

For the unknown variable v, we have similar approximations, that is:

$$\frac{\partial v}{\partial x} = \bar{\zeta} \quad \text{and} \quad \frac{\partial v}{\partial y} = \bar{\eta}. \tag{4.5}$$

Then with similar calculations, we have

$$v(x_i, y_j) = \sum_{k=0}^{N} S_{ik}\bar{\zeta}(x_k, y_j) + f_3(y_j),$$

$$\tilde{v}(x_i, y_j) = \sum_{(k=0)}^{N} S_{ik}\bar{\eta}(x_i, y_k) + f_4(x_i). \tag{4.6}$$

$$v(x_i, y_j) = \tilde{v}(x_i, y_j). \tag{4.7}$$

4.2. Applying Rayleigh–Ritz method

The Rayleigh–Ritz Method for problem (3.1)–(3.3) is to find the minimum of [20]

$$F_1 = \int_{-1}^{1}\int_{-1}^{1}\left(L_1\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) - 2\chi_1\right)u\,dxdy, \tag{4.8}$$

$$F_2 = \int_{-1}^{1}\int_{-1}^{1}\left(L_2\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) - 2\chi_2\right)v\,dxdy. \tag{4.9}$$

Making use of (2.4)-(2.7), we have:

$$F_1 = \sum_{i=0}^{N}\sum_{j=0}^{N} S_{Ni}^{[\lambda]}S_{Nj}^{[\lambda]}[L_1(u(x_i, y_j), \zeta(x_i, y_j), \eta(x_i, y_j), v(x_i, y_j), \bar{\zeta}(x_i, y_j), \bar{\eta}(x_i, y_j)) - 2\chi_1(x_i, y_j)]u(x_i, y_j), \tag{4.10}$$

$$F_2 = \sum_{i=0}^{N}\sum_{j=0}^{N} S_{Ni}^{[\lambda]}S_{Nj}^{[\lambda]}[L_2(u(x_i, y_j), \zeta(x_i, y_j), \eta(x_i, y_j), v(x_i, y_j), \bar{\zeta}(x_i, y_j), \bar{\eta}(x_i, y_j)) - 2\chi_2(x_i, y_j)]v(x_i, y_j) \tag{4.11}$$

This is a multi-objective function of the optimization problem. We can reformulate it as one objective function as follows [21]:

$$R = F_1 + F_2. \tag{4.12}$$

So, to obtain the unknown values ζ, η, $\bar{\zeta}$ and $\bar{\eta}$, we construct the following constrained optimization problem making use of (4.12), (4.4), (4.7), (3.3a), (4.3), (3.3b) and (4.6):

Minimize R, subject to

$$I_{ij}^{[1]} = \left[\sum_{k=0}^{N} S_{ik}\zeta(x_k, y_j) + f_1(y_j)\right] - \left[\sum_{k=0}^{N} S_{ik}\eta(x_i, y_k) + f_2(x_i)\right] = 0, \quad i, j = 0, 1, \ldots, N, \tag{4.13}$$

$$I_{ij}^{[2]} = \left[\sum_{k=0}^{N} S_{ik}\bar{\zeta}(x_k, y_j) + f_3(y_j)\right] - \left[\sum_{k=0}^{N} S_{ik}\,\bar{\eta}(x_i, y_k) + f_4(x_i)\right] = 0, \quad i, j = 0, 1, \ldots, N, \tag{4.14}$$

$$I_j^{[3]} = \left[\sum_{k=0}^{N} S_{ik}\zeta(1, y_k) + f_1(y_j)\right] - \overline{f_1}(y_j) = 0, \; j = 0, 1, \ldots, N, \tag{4.15a}$$

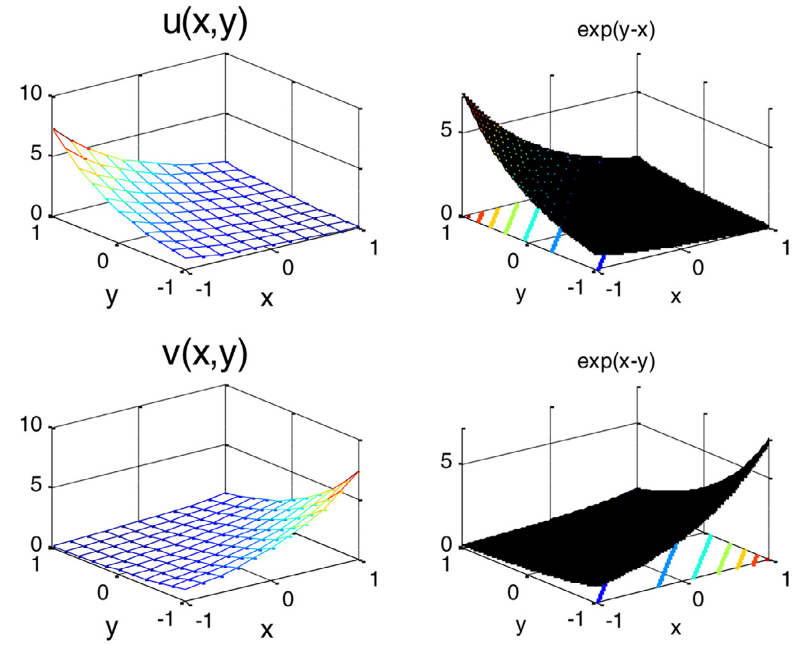

Fig. P1.1 Approximation solution with $N = 10$ (Left) and analytic solution (Right) for problem 1.

$$I_i^{[4]} = \left[\sum_{k=0}^{N} S_{ik}\eta(x_k, 1) + f_2(x_i)\right] - \overline{f_2}(x_i) = 0, \quad i = 0, 1, \ldots, N, \tag{4.15b}$$

$$I_j^{([5])} = \left[\sum_{k=0}^{N} S_{ik}\bar{\zeta}(1, y_k) + f_3(y_j)\right] - \bar{f}_3(y_j) = 0, \quad j = 0, 1, \ldots, N, \tag{4.16a}$$

$$I_i^{([6])} = \left[\sum_{k=0}^{N} S_{ik}\bar{\eta}(x_k, 1) + f_4(x_i)\right] - \bar{f}_4(x_i) = 0, \quad i = 0, 1, \ldots, N. \tag{4.16b}$$

This problem can be solved by Penalty leap frog method [22].

5. Error estimates

Not all of the nonlinear partial differential equations have available analytic solutions. If the analytic solution is found for the problem, we use the following error definitions to measure the difference between the numerical and analytic solutions:

$$E_u = E(u_{ij}) = \frac{1}{N^2}\left[\sum_{i=0}^{N}\sum_{j=0}^{N}\left(u_{ij}^e - u_{ij}^a\right)^2\right]^{0.5}, \quad i = 0, 1, \ldots, N, \tag{5.1}$$

$$E_v = E(v_{ij}) = \frac{1}{N^2}\left[\sum_{i=0}^{N}\sum_{j=0}^{N}\left(v_{ij}^e - v_{ij}^a\right)^2\right]^{0.5}, \quad i = 0, 1, \ldots, N. \tag{5.2}$$

where u_{ij}^e and u_{ij}^a the exact and approximate solutions, respectively. In the other hand, If the analytic solution is not found for the problem, we estimate the error in the optimization process by two indices, the first is the value of the minimized cost function R of Eq. (4.12). The second is to ensure satisfying the conditions (4.13)–(4.16), that is, the value of

$$J = \sum_{i=0}^{N}\sum_{j=0}^{N}\left[I_{ij}^{[1]} + I_{ij}^{[2]}\right] + \sum_{i=0}^{N}\left[I_i^{[4]} + I_i^{[6]}\right] + \sum_{j=0}^{N}\left[I_j^{[3]} + I_j^{[5]}\right]. \tag{5.3}$$

Must be small enough.

6. Numerical experiments

In this section some numerical examples are presented.

Problem 1

We now consider the inhomogeneous nonlinear system

$$\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u = 1, \quad \frac{\partial v}{\partial x} - u\frac{\partial v}{\partial y} - v = 1, \quad (x, y) \in \varpi = [-1, 1] \times [-1, 1], \tag{P1.1}$$

with the boundary conditions

$$u(-1, y) = e^{y+1}, \quad u(1, y) = e^{y-1}, \quad u(x, -1) = e^{-(x+1)}, \quad u(x, 1) = e^{-(x-1)}, \tag{P1.2}$$

$$v(-1, y) = e^{-(1+y)}, \quad v(1, y) = e^{1-y}, \quad v(x, -1) = e^{x+1}, \; v(x, 1) = e^{x-1}. \tag{P1.3}$$

The exact solutions of the problem (3.1)–(3.3) are

$$(x, y) = e^{y-x}, \quad v(x, y) = e^{x-y}. \tag{P1.4}$$

The approximate solution of $u(x, y)$ of (4.3), $v(x, y)$ of (4.6) and the exact solution related to (P1.4) are introduced in

Table P1.1 Optimization indices and error estimates of problem 1 with $N = 10$.

λ	R	J	E_u	E_v
0.0	6.76e−14	4.81e−12	6.18e−09	7.81e−09
0.5	3.86e−14	6.78e−12	4.28e−09	5.21e−09
1.0	6.34e−14	8.65e−12	7.68e−09	8.84e−09
$\lambda^* = 0.523$	1.45e−14	9.85e−13	1.99e−09	2.65e−09

Fig. P1.1. Moreover, optimization indices R of (4.12), J of (5.3) and error estimates E_u, E_v of (5.1) and (5.2) are introduced in Table P1.1. These amounts are evaluated at $\lambda = 0,\ 1.0,\ 0.5$,which corresponds to Chebyshev approximation of first, second kind, and Legendre approximation. The evaluation at λ^*, the best experimental evaluation is included. The convergence of the proposed method is displayed in Fig. P1.2, while in Fig. P1.3, the approximate solution for $(-1, y)$, $(1, y)$, $(x, -1)$, $(x, 1)$ are we presented. These values

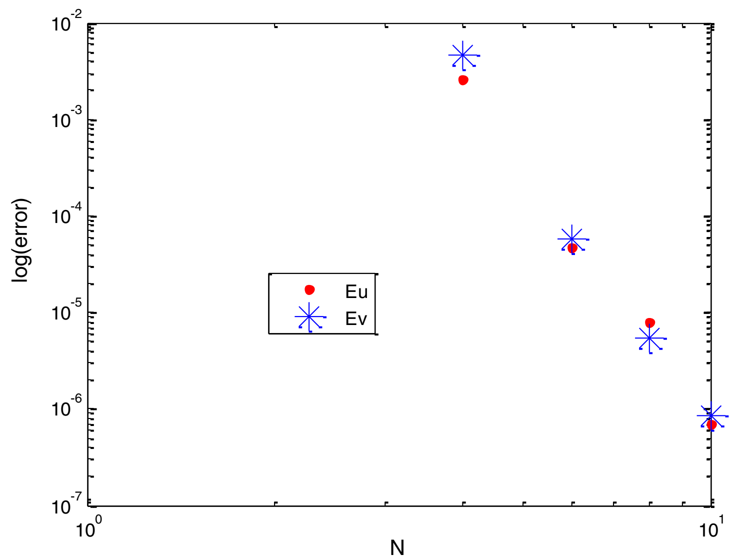

Fig. P1.2 Convergence of the proposed method for problem 1.

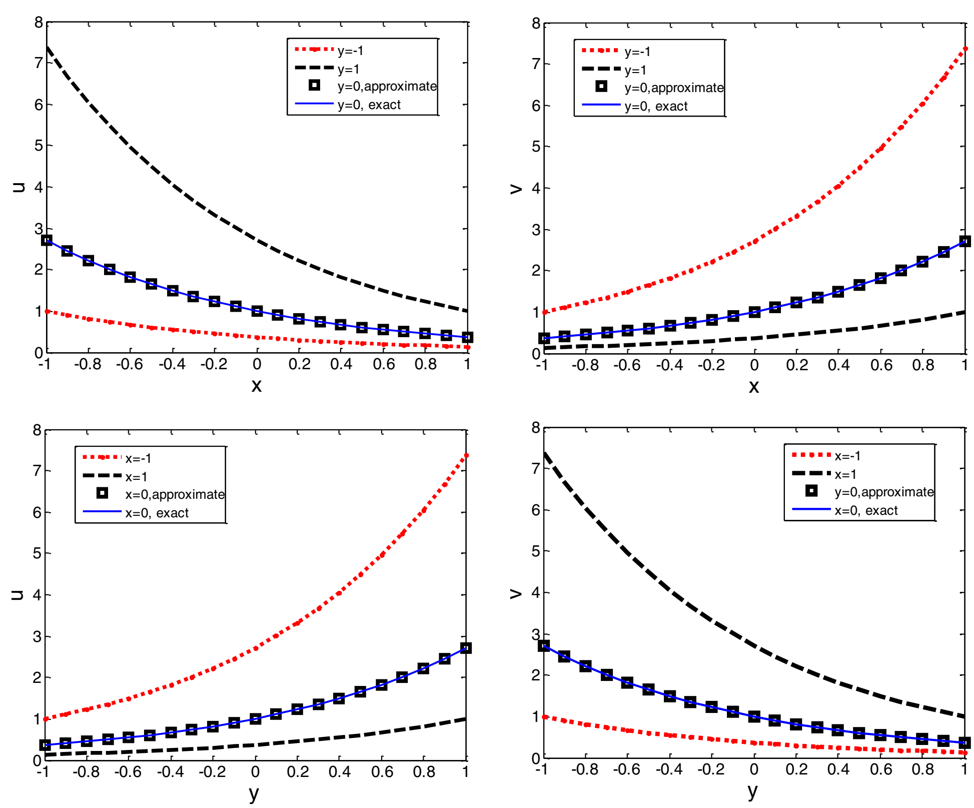

Fig. P1.3 The solution for problem 1.

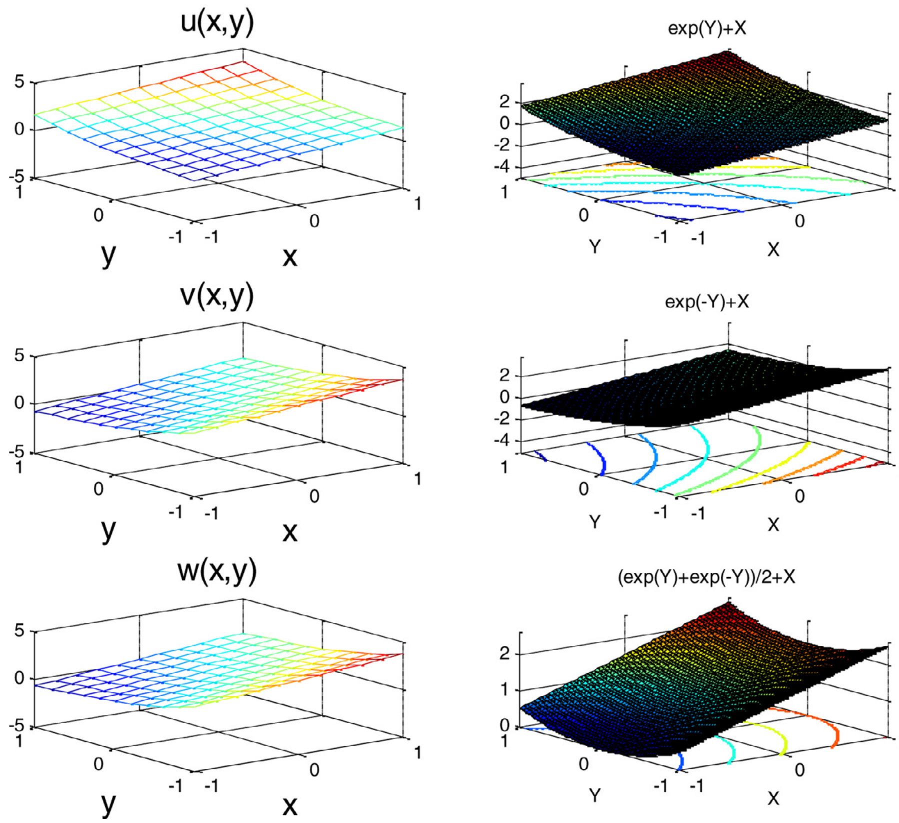

Fig. P2.1 Approximation solution with $N = 10$(Left) and analytic solution (Right) for problem 2.

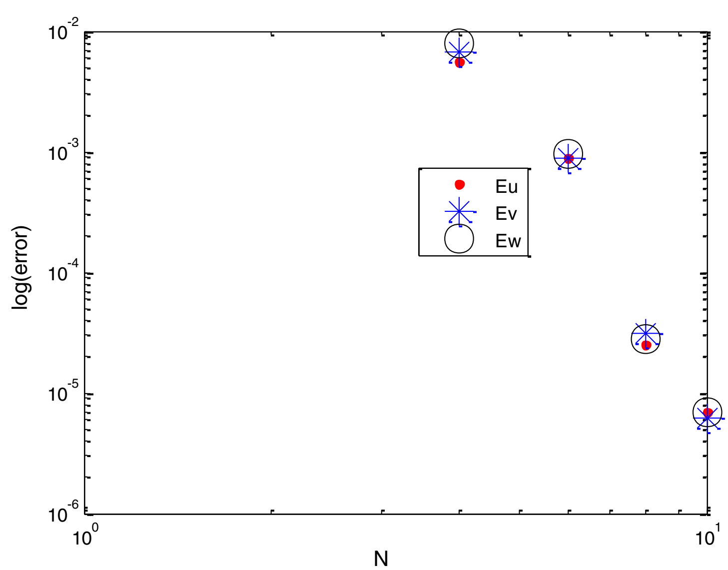

Fig. P2.2 Convergence of the proposed method for problem 2.

Table P2.1 Optimization indices and error estimates of problem 2 with $N = 10$.

λ	R	J	E_u	E_v	E_w
0.0	5.51e−13	9.85e−12	8.98e−06	8.71e−06	9.92e−06
0.5	3.79e−13	7.98e−12	7.68e−06	7.61e−06	7.16e−06
1.0	8.64e−13	9.75e−12	8.62e−06	7.94e−09	8.74e−06
$\lambda^* = 0.487$	2.45e−13	8.45e−12	6.96e−06	6.15e−06	6.85e−06

are clearly close to the exact solution by Table P1.1. The approximate solution at the middle of the interval, namely, $(0, y)$, $(x, 0)$ are plotted compared with the exact solution to ensure the accuracy of the approximate solution.

Problem 2

We now consider the nonlinear system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} = \frac{3}{2} - \frac{1}{2}e^{-2x},$$

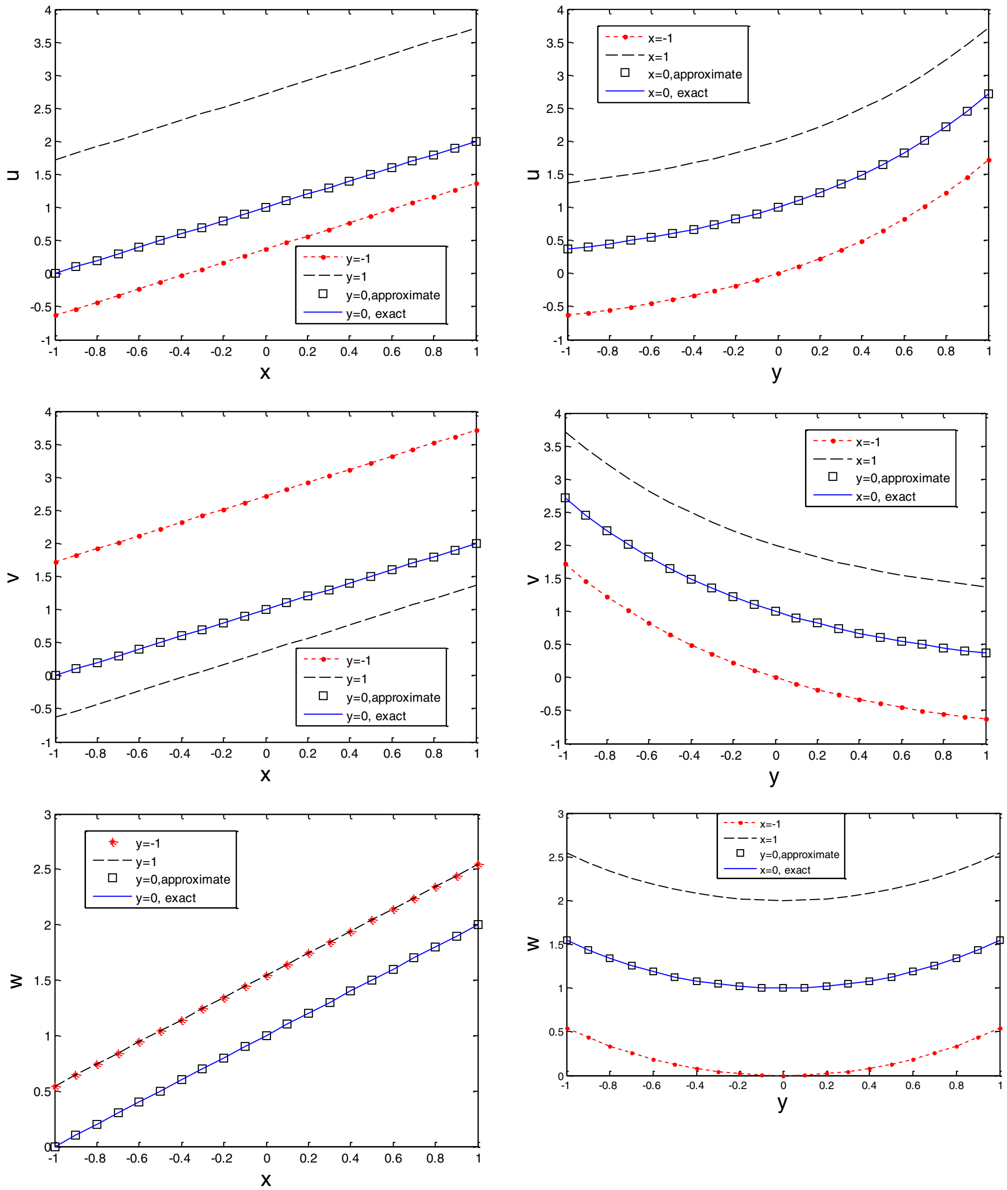

Fig. P2.3 The solution for problem 2.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} = \frac{3}{2} - \frac{1}{2}e^{2x},$$
$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 2,$$
$$\textit{and } x,\ y \in \varpi = [-1, 1] \times [-1, 1], \tag{P2.1}$$

with the boundary conditions

$$u(-1, y) = e^{y} - 1, \quad u(1, y) = e^{y} + 1,$$
$$u(x, -1) = e^{-1} + x, \quad u(x, 1) = e^{1} + x, \tag{P2.2}$$

$$v(-1, y) = e^{-y} - 1, \quad v(1, y) = e^{-y} + 1,$$
$$v(x, -1) = e^{1} + x, \quad v(x, 1) = e^{-1} + x, \tag{P2.3}$$

$$w(-1, y) = \frac{1}{2}(e^{y}+e^{(-y)}) - 1, \ \ w(1, y) = \frac{1}{2}(e^{y}+e^{(-y)}) + 1,$$
$$w(x, -1) = \frac{1}{2}(e^{-1} + e^{1}) + x, \ \ w(x, 1) = \frac{1}{2}(e^{1} + e^{-1}) + x. \tag{P2.4}$$

The exact solutions of the problem (P2.1)–(P2.4) are

$$u(x, y) = e^{y} + x,\ v(x, y) = e^{-y} + x,\ w(x, y) = \frac{1}{2}\left(e^{y} + e^{-y}\right) + x.$$

The results are introduced in Fig. P2.1, Table P2.1, Figs. P2.2 and P2.3 with the same introduction the previous example.

7. Conclusion

In this paper, we have proposed a numerical algorithm to solve the nonlinear systems of partial differential problems of first order using ultraspherical integral approximation with Rayleigh–Ritz(UIZB-RR) method and an optimization technique for obtaining the solution of the resulting nonlinear algebraic equation system.

The numerical results given above ensure the good accuracy of proposed method. Moreover, The procedure discussed here can be used for approximating solution of linear and nonlinear partial differential equations.

Acknowledgment

The author thanks the referees for the careful reviews and the valuable comments.

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