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Original Article

Attractivity of the recursive sequence $x_{n+1} = (A - Bx_{n-2})/(C + Dx_{n-1})$

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Abstract In this paper, we investigate the global attractivity of the difference equation

$$x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}, n = 0, 1, \ldots,$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $C + Dx_{n-1} \neq 0$ for all $n \geq 0$.

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1. Introduction

Difference equations appear naturally as discrete analogues and numerical solutions of differential equations and delay differential equations having applications in biology, ecology, physics, etc. The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

R. Abo-Zeid [1] investigated the attractivity of two nonlinear third order difference equations

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \qquad n = 0, 1, \ldots,$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $C + Dx_{n-2} \neq 0$ for all $n \geq 0$.

El-Owaidy et al. [2] investigated the global attractivity of the difference equation

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + x_n}, \qquad n = 0, 1, \ldots,$$

where α, β, γ are non-negative real numbers and $\gamma + x_n \neq 0$ for all $n \geq 0$.

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M. A. El-Moneam [3] studied the global behavior of the higher order nonlinear rational difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{|dx_{n-k} - ex_{n-l}|},\quad n = 0, 1, \ldots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \ldots, x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$.

A. E. Hamza et al. [4] investigated the global asymptotic stability of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \qquad n = 0, 1, \ldots,$$

where $\alpha, \beta, \gamma \geq 0$. For other related results, see [5,6]. In this paper we study the global attractivity of the difference equations

$$x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where A, B are non negative real numbers, C, D are positive real numbers and $C + Dx_{n-1} \neq 0$ for all $n \geq 0$.

Theorem 1.1 ([6]). *Consider the third-degree polynomial equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{1.2}$$

where a_1, a_0 and a_2 are real numbers. Then a necessary and sufficient condition for all roots of Eq. (1.2) to lie inside the open disk $|\lambda| < 1$ is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1 \text{ and } a_0^2 + a_1 - a_0a_2 < 1.$$

The change of variables $x_n = \frac{C}{D}y_n$ reduces Eq. (1.1) to the difference equation

$$y_{n+1} = \frac{p - qy_{n-2}}{1 + y_{n-1}}, \qquad n = 0, 1, \ldots. \tag{1.3}$$

where $p = \frac{AD}{C^2}$ and $q = \frac{B}{C}$.

2. The recursive sequence $y_{n+1} = (p - qy_{n-2})/(1 + y_{n-1})$

In this section we study the global attractivity of the difference equation

$$y_{n+1} = \frac{p - qy_{n-2}}{1 + y_{n-1}}, \qquad n = 0, 1, \ldots, \tag{2.1}$$

where p and q are positive real numbers.

The equilibrium points of Eq. (2.1) are the zeros of the function

$$f(\overline{y}) = \overline{y}^2 + (1 + q)\overline{y} - p.$$

That is

$$\overline{y}_1 = \frac{1}{2}(-(1 + q) + \sqrt{(1 + q)^2 + 4p})$$

and

$$\overline{y}_2 = \frac{1}{2}(-(1 + q) - \sqrt{(1 + q)^2 + 4p}).$$

The linearized equation associated with Eq. (2.1) about $\overline{y}_i$ is

$$z_{n+1} + \frac{\overline{y}_i}{1 + \overline{y}_i}z_{n-1} + \frac{q}{1 + \overline{y}_i}z_{n-2} = 0, \quad n = 0, 1, 2, \ldots.$$

Its associated characteristic equation is

$$\lambda^3 + \frac{\overline{y}_i}{1 + \overline{y}_i}\lambda + \frac{q}{1 + \overline{y}_i} = 0.$$

Suppose that

$$g_i(\lambda) = \lambda^3 + \frac{\overline{y}_i}{1 + \overline{y}_i}\lambda + \frac{q}{1 + \overline{y}_i}, \quad i = 1, 2. \tag{2.2}$$

Theorem 2.1.

(1) *The sufficient condition for the equilibrium point $\overline{y}_1$ to be locally asymptotically stable is $q \leq 1$.*
(2) *If $q > \frac{1}{3} + \sqrt{\frac{4}{3}p + \frac{4}{9}}$, then $\overline{y}_1$ is unstable.*
(3) *$\overline{y}_2$ is saddle equilibrium point.*

Proof.

(1) If $q \leq 1$, then by using Theorem 1.1 with $a_0 = \frac{q}{1+\overline{y}_i}$, $a_1 = \frac{\overline{y}_i}{1+\overline{y}_i}$, $a_2 = 0$. We can easily show that $\overline{y}_1$ is locally asymptotically stable.
(2) If $q > \frac{1}{3} + \sqrt{\frac{4}{3}p + \frac{4}{9}}$, then $g_1(\lambda)$ has a zero λ_1 in $(-\infty, -1)$, which implies that the equilibrium point $\overline{y}_1$ is unstable.
(3) It is clear that $g_2(\lambda)$ has a zero $\lambda_1 \in (0, 1)$, and $g_2(-\frac{q}{1+\overline{y}_2}) = \frac{q}{(1+\overline{y}_2)^3}[\overline{y}_2 + 1 - q^2]$.

It is clear that $g_2(\lambda)$ is an increasing function. Since $g_2(-\frac{q}{1+\overline{y}_2}) > 0$, then $\lambda_1 < -\frac{q}{1+\overline{y}_2} \Longrightarrow |\lambda_2\lambda_3| > 1 \Longrightarrow |\lambda_2| = |\lambda_3| > 1$, which implies that $\overline{y}_2$ is unstable equilibrium point (saddle). □

Lemma 1. *Assume that $q \leq 1$. Then the interval $[0, \frac{p}{q}]$ is an invariant interval for Eq. (2.1).*

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2.1) with $y_{-2}, y_{-1}, y_0 \in [0, \frac{p}{q}]$.

Consider the function $U_1(x, y) = \frac{p-qy}{1+x}$, U_1 is decreasing in x and y on $(-1, \infty) \times (-\infty, \frac{p}{q})$.

Hence

$0 = U_1(\frac{p}{q}, \frac{p}{q}) \leq y_1 = U_1(y_{-1}, y_{-2}) < U_1(0, 0) = p < \frac{p}{q}$,

by induction we obtain $0 \leq y_n \leq \frac{p}{q}\ \forall n \geq 1$.

Assume that there exists $k \geq 2$ such that the following conditions hold

$p \geq kq^2$ and $1 \geq \frac{kp}{q}$. □

Lemma 2. *Assume that condition (2.3) hold for some $k \geq 2$. Let $\{y_n\}$ be a solution of Eq. (2.1)*

If $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{p}{q}]$ for some $n \geq -2$, then $y_{n+i} \in [0, \frac{p}{q}]\ \forall i \geq 3$.

Proof.

(i) If $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{p}{q}]$ for some $n \geq -2$, then

$$0 = U_1\left(\frac{p}{q}, \frac{p}{q}\right) \leq y_{n+3} = U_1(y_{n+1}, y_n) \leq U_1\left(-(k-1)\frac{p}{q}, -(k-1)\frac{p}{q}\right) \leq \frac{p}{q}.$$

$$0 = U_1\left(\frac{p}{q}, \frac{p}{q}\right) \leq y_{n+4} = U_1(y_{n+2}, y_{n+1}) \leq U_1\left(-(k-1)\frac{p}{q}, -(k-1)\frac{p}{q}\right) \leq \frac{p}{q}.$$

As we know $-(k-1)\frac{p}{q} \leq 0 \leq y_{n+3} \leq \frac{p}{q}$, then we can write

$$0 = U_1\left(\frac{p}{q}, \frac{p}{q}\right) \leq y_{n+5} = U_1(y_{n+3}, y_{n+2}) \leq U_1\left(-(k-1)\frac{p}{q}, -(k-1)\frac{p}{q}\right) \leq \frac{p}{q}.$$

Therefore by Lemma 1, we have $0 \leq y_{n+i} \leq \frac{p}{q}, \ \forall\, i \geq 3.$ □

Lemma 3. *Assume that condition (2.3) hold for some $k \geq 2$ and $q \leq 1$. Let $\{y_n\}$ be a solution of Eq (2.1). If*

$$y_n, y_{n+1}, y_{n+2} \in \left[-(k-1)\frac{p}{q}, \frac{mkp}{q}\right] \text{ for some } n \geq -2,$$

$$m \in Z^+ \text{ then } y_{n+i} \in \left[0, \frac{p}{q}\right] \forall i \geq 6.$$

Proof. We prove the theorem by induction. For $m = 1$, let $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{kp}{q}]$ for some $n \geq -2$, then

$$\frac{-(k-1)p}{q} \leq \frac{-(k-1)pq}{q+kp} \leq \frac{p-kp}{1+\frac{kp}{q}} \leq y_{n+3} = \frac{p-qy_n}{1+y_{n+1}} \leq \frac{p-kp+p}{\frac{p}{q}} \leq kq \leq \frac{p}{q}.$$

One may show that y_{n+4} and $y_{n+5} \in [-(k-1)\frac{p}{q}, \frac{p}{q}]$. Then by Lemma 2 we have $0 \leq y_{n+i} \leq \frac{p}{q}, \forall\, i \geq 6$.

Assume that if $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{Skp}{q}]$, for some $n \geq -2$, $S \in Z^+$, then $0 \leq y_{n+i} \leq \frac{p}{q}, \ \forall i \geq 6$.

Now, assume that $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{(S+1)kp}{q}]$, then

$$\frac{-(k-1)p}{q} \leq \frac{-((S+1)k-1)pq}{q+(S+1)kp} \leq \frac{p-(S+1)kp}{1+\frac{(S+1)kp}{q}} \leq y_{n+3} = \frac{p-qy_n}{1+y_{n+1}} \leq \frac{p-kp+p}{\frac{p}{q}} \leq \frac{p}{q}.$$

By using the same steps we obtain $y_{n+4}, y_{n+5} \in [-(k-1)\frac{p}{q}, \frac{p}{q}]$, then by Lemma 2 we have $0 \leq y_{n+i} \leq \frac{p}{q}, \ \forall\, i \geq 6$. This complete the proof. □

Theorem 2.2. *Suppose that $q \leq 1$ and there exists $k \geq 2$, $m \geq 1$ such that conditions (2.3) hold. Then the positive equilibrium point $\bar{y}_1$ of Eq (2.1) is a global attractor with a basin*

$$S = \left[-(k-1)\frac{p}{q}, \frac{mkp}{q}\right]^3$$

Proof. Suppose that $q < 1$ and let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2.1) with $y_{-2}, y_{-1}, y_0 \in S$. Then by Lemma 2 and Lemma 3 we have $y_n \in [0, \frac{p}{q}]$, $n \geq 6$.

Set $\lambda = \liminf y_n$ and $\Lambda = \limsup y_n$.

Let $\epsilon > 0$ such that $\epsilon < \min\{(p/q) - \Lambda, \lambda\}$. There exists $n_0 \in \mathbb{N}$ such that $\lambda - \epsilon < y_n < \Lambda + \epsilon \ \ \forall\, n \geq n_0$.

Hence

$$\frac{p-q(\Lambda+\epsilon)}{1+\Lambda+\epsilon} < y_{n+1} < \frac{p-q(\lambda-\epsilon)}{1+\lambda-\epsilon}, \ \forall\, n \geq n_0 + 2.$$

We get the inequality

$$\frac{p-q(\Lambda+\epsilon)}{1+\Lambda+\epsilon} \leq \lambda \leq \Lambda \leq \frac{p-q(\lambda-\epsilon)}{1+\lambda-\epsilon}.$$

Hence we have

$$\frac{p-q\Lambda}{1+\Lambda} \leq \lambda \leq \Lambda \leq \frac{p-q\lambda}{1+\lambda}.$$

This implies

$p - q\Lambda \leq \lambda + \lambda\Lambda$ and $\Lambda + \lambda\Lambda \leq p - q\lambda$, then $p - q\Lambda \leq \lambda + \lambda\Lambda \leq \lambda + p - q\lambda - \Lambda$.

That is $\lambda(q-1) \leq \Lambda(q-1)$. This is contradiction and therefore, $\Lambda = \lambda = \bar{y}_1$. □

Theorem 2.3. *Assume the initial conditions $y_{-2}, y_{-1}, y_0 \in [0, \frac{p}{q}]$. If they are not both equal to $\bar{y}_1$, then the following statements are true*

(1) *$\{y_n\}$ cannot have three consecutive terms equal to $\bar{y}_1$.*
(2) *Every negative semicycle of $\{y_n\}$ has at most three terms.*
(3) *Every positive semicycle of $\{y_n\}$ has at most four terms.*
(4) *$\{y_n\}$ is strictly oscillatory.*

Proof.

(1) If $y_{l-1} = y_l = y_{l+1} = \bar{y}_1$for some $l \in \mathbb{N}$, then $y_{l-2} = \bar{y}_1$, which implies that $y_{l-1} = y_{l-2} = y_{l-3} = \cdots = y_0 = y_{-1} = y_{-2} = \bar{y}_1$ which is impossible.
(2) Assume a negative semicycle starts with three terms y_{l-2}, y_{l-1}, y_l, then $y_{l-2}, y_{l-1}, y_l < \bar{y}_1$, and $y_{l+1} = f(y_{l-1}, y_{l-2}) > f(\bar{y}_1, \bar{y}_1) = \bar{y}_1$.
(3) Assume a positive semicycle starts with three terms y_{l-2}, y_{l-1}, y_l, then $y_{l-2}, y_{l-1}, y_l \geq \bar{y}_1$, provided that $y_{l-2}, y_{l-1}, y_l \neq \bar{y}_1$ at the same time. $y_{l+1} = f(y_{l-1}, y_{l-2}) \leq f(\bar{y}_1, \bar{y}_1) = \bar{y}_1$ and $y_{l+1} = \bar{y}_1$ iff $y_{l-1} = y_{l-2} = \bar{y}_1$ which implies that $y_l > \bar{y}_1$. In all cases $y_{l+2} = f(y_l, y_{l-1}) < f(\bar{y}_1, \bar{y}_1) = \bar{y}_1$.
(4) From (1)–(3), we obtain $\{y_n\}$ is strictly oscillatory.

□

3. The recursive sequence $y_{n+1} = (-qy_{n-2})/(1 + y_{n-1})$

In this section we study the global behavior of the difference equation

$$y_{n+1} = \frac{-qy_{n-2}}{1+y_{n-1}}, \qquad n = 0, 1, \ldots. \tag{3.1}$$

Eq. (3.1) has two equilibrium points $\bar{y}_1 = 0$ and $\bar{y}_2 = -1 - q$.

The linearized equation associated with Eq. (3.1) about $\bar{y}_i$, $i = 1, 2$ is

$$z_{n+1} - \frac{q\bar{y}_i}{(1+\bar{y}_i)^2} z_{n-1} + \frac{q}{1+\bar{y}_i} z_{n-2} = 0, \; i = 1, 2, \quad n = 0, 1, \ldots . \tag{3.2}$$

The characteristic equation associated with Eq. (3.2) is

$$\lambda^3 - \frac{q\bar{y}_i}{(1+\bar{y}_i)^2}\lambda + \frac{q}{1+\bar{y}_i} = 0, \quad i = 1, 2.$$

Let

$$g_i(\lambda) = \lambda^3 - \frac{q\bar{y}_i}{(1+\bar{y}_i)^2}\lambda + \frac{q}{1+\bar{y}_i}, \quad i = 1, 2.$$

Theorem 3.1.

(1) *The equilibrium point $\bar{y}_1 = 0$ is locally asymptotically stable iff $q < 1$, saddle point iff $q > 1$, and non hyperbolic if $q = 1$.*
(2) *The equilibrium point $\bar{y}_2$ is unstable.*

Proof.

(1) The linearized equation associated with Eq. (3.2) about $\bar{y}_1 = 0$ is

$$z_{n+1} + qz_{n-2} = 0, \quad n = 0, 1, \ldots .$$

Its associated characteristic equation is $\lambda^3 + q = 0$. Easily one can show that $\bar{y}_1 = 0$ is locally asymptotically stable iff $q < 1$, saddle point iff $q > 1$, and non hyperbolic if $q = 1$.
(2) The characteristic equation about $\bar{y}_2 = -1 - q$ is

$$\lambda^3 + \frac{1}{q}(1+q)\lambda - 1 = 0,$$

$g_2(\lambda)$ has a root $\lambda_1 \in (0, 1)$, then $|\lambda_2\lambda_3| > 1$, which implies $\bar{y}_2 = -1 - q$ is unstable (saddle point). □

Theorem 3.2. *Assume that $q < \frac{1}{2}$. Then the interval $[-q, q]$ is an invariant interval of Eq. (3.1)*

Proof. Suppose that $q < \frac{1}{2}$ and let $|y_{-i}| < q$, $i = 0, 1, 2$. we show that $y_n \in [-q, q]$, $n = 1, 2, \ldots$ for Eq. (3.1).

$0 < 1 - q < 1 + y_{-i} < q + 1.$

Hence

$$|y_1| = \left|\frac{-qy_{-2}}{1+y_{-1}}\right| = \frac{q|y_{-2}|}{|1+y_{-1}|} < \frac{q|y_{-2}|}{1-q} < |y_{-2}| < q,$$

by induction we obtain $|y_{n+1}| < \dfrac{q|y_{n-2}|}{1-q} < |y_{n-2}| < q$, $n \geq 0$. □

Theorem 3.3. *Assume that $q < \frac{1}{2}$ and let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (3.1) with $y_{-2}, y_{-1}, y_0 \in [-q, q]$. Then $\{y_n\}_{n=-2}^{\infty}$ oscillate with semicycles of length at most three.*

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (3.1) with $y_{-2}, y_{-1}, y_0 \in [-q, q]$. Then by Theorem 3.2., we get $1 + y_n > 0$, $n \geq -2$, that is $sgn(y_{n+1}) = -sgn(y_{n-2})$, $n \geq 0$.

This implies that the subsequences $\{y_{3n-2}\}_{n=0}^{\infty}$, $\{y_{3n-1}\}_{n=0}^{\infty}$ and $\{y_{3n}\}_{n=0}^{\infty}$ oscillate with semicycles of length one. Therefore, the solution $\{y_n\}_{n=-2}^{\infty}$ oscillates with semicycles of length at most three. □

Theorem 3.4. *If $q < \frac{1}{2}$, then the equilibrium point $\bar{y}_1 = 0$ is a global attractor with basin $[-q, q]^3$.*

Proof. According to Theorem 3.2, we have

$$|y_{3n+1}| < \left(\frac{q}{1-q}\right)^{n+1}|y_{-2}|, \; |y_{3n+2}| < \left(\frac{q}{1-q}\right)^{n+1}|y_{-1}|, \text{ and}$$

$$|y_{3n+3}| < \left(\frac{q}{1-q}\right)^{n+1}|y_0|,$$

since $q < \frac{1}{2}$, we have $\lim_{n\to\infty} y_n = 0$. □

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