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Original Article

On statistical approximation properties of q-Baskakov–Szász–Stancu operators

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Abstract In the present paper, we consider Stancu type generalization of Baskakov–Szász operators based on the q-integers and obtain statistical and weighted statistical approximation properties of these operators. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function are also established for operators.

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1. Introduction

In the recent years several operators of summation-integral type have been proposed and their approximation properties have been discussed. In the present paper our aim is to investigate statistical approximation properties of a Stancu type q-Baskakov–Szász operators. Firstly, Baskakov–Szász operators based on q-integers was introduced by Gupta [1] and some approximation results were established. The q-Baskakov–Szász operators are defined as follows:

$$\mathcal{D}_n^q(f,x) = [n]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{q/1-q^n} q^{-k-1} s_{n,k}^q(t) f(t\,q^{-k}) \mathrm{d}_q t, \quad (1.1)$$

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where $x \in [0, \infty)$ and

$$p_{n,k}^q(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}}, \quad (1.2)$$

and

$$s_{n,k}^q(t) = E(-[n]_q t) \frac{([n]_q t)^k}{[k]_q!}. \quad (1.3)$$

In case $q = 1$, the above operators reduce to the Baskakov–Szász operators [2].

Later, Mishra and Sharma [3] introduced a new Stancu type generalization of q-Baskakov–Szász operators, which is defined as

$$\mathfrak{D}_n^{(\alpha,\beta)}(f; q; x) = [n]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{q/1-q^n} q^{-k-1} s_{n,k}^q(t) f \times \left(\frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta}\right) \mathrm{d}_q t, \quad (1.4)$$

where $p_{n,k}^q(x)$ and $s_{n,k}^q(t)$ are Baskakov and Szász basis function respectively, defined as above. The operators $\mathcal{D}_n^{(\alpha,\beta)}(f; q; x)$ in (1.4) are called q-Baskakov–Szász-Stancu operators. For $\alpha = 0$, $\beta = 0$ the operators (1.4) reduce to the operators (1.1).

In the recent years several researchers have worked on Stancu type generalization of different operators and they have obtained various approximation properties. We mention some of important papers as [4–8].

Before proceeding further, we recall certain notations of q-calculus as follows. Such notations can be found in [9,10]. We consider q as a real number satisfying $0 < q < 1$.

For

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q ... [1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

Then for $q > 0$ and integers $n, k, k \geq n \geq 0$, we have

$$[n+1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n [k-n]_q = [k]_q.$$

We observe that

$$(1+x)_q^n = (-x; q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x), & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases}$$

Also, for any real number α, we have

$$(1+x)_q^\alpha = \frac{(1+x)_q^\infty}{(1+q^\alpha x)_q^\infty}.$$

In special case, when α is a whole number, this definition coincides with the above definition.

The q-Jackson integral and q-improper integral defined as

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A},$$

provided sum converges absolutely.

The q-analogues of the exponential function e^x (see [10]), used here is defined as

$$E_q(z) = \prod_{j=0}^{\infty} (1 + (1-q) q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1 + (1-q)z)_q^\infty, \ |q| < 1,$$

where $(1-x)_q^\infty = \prod_{j=0}^{\infty} (1 - q^j x)$.

2. Moment estimates

Lemma 1. *[1] The following hold:*

1. $\mathcal{D}_n(1, q; x) = 1$,
2. $\mathcal{D}_n(t, q; x) = x + \frac{q}{[n]_q}$,
3. $\mathcal{D}_n(t^2, q; x) = \left(1 + \frac{1}{q[n]_q}\right) x^2 + \frac{x}{[n]_q}(1 + q(q+2)) + \frac{q^2(1+q)}{[n]_q^2}$.

Lemma 2 ([3]). *The following hold:*

1. $\mathfrak{D}_n^{(\alpha,\beta)}(1; q; x) = 1$,
2. $\mathfrak{D}_n^{(\alpha,\beta)}(t; q; x) = \frac{[n]_q x + q + \alpha}{[n]_q + \beta}$,
3. $\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q; x) = \left(\frac{[n]_q (q[n]_q + 1)}{q([n]_q + \beta)^2}\right) x^2 + \left(\frac{(1 + q(q+2))[n]_q + 2\alpha [n]_q}{([n]_q + \beta)^2}\right) x + \frac{q^2(1+q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2}$.

3. Korovkin type statistical approximation properties

The idea of statistical convergence goes back to the first edition (published in Warsaw in 1935) of the monograph of Zygmund [11]. Formerly the concept of statistical convergence was introduced by Steinhaus [12] and Fast [13] and later reintroduced by Schoenberg [14]. Statistical convergence, while introduced over nearly 50 years ago, has only recently become an area of active research. Different mathematicians studied properties of statistical convergence and applied this concept in various areas.

In approximation theory, the concept of statistical convergence was used in the year 2002 by Gadjiev and Orhan [15]. They proved the Bohman–Korovkin type approximation theorem for statistical convergence. It was shown that the statistical versions are stronger than the classical ones.

Korovkin type approximation theory also has many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [16]).

Let us recall the concept of a limit of a sequence extended to a statistical limit by using the natural density δ of a set K of positive integers:

$$\delta(K) = \lim_n n^{-1} \{\text{the number } k \leq n \text{ such that } k \in K\}$$

whenever the limit exists (see [17], p. 407). So, the sequence $x = (x_k)$ is said to be statistically convergent to a number L, meaning that for every $\epsilon > 0$,

$$\delta\{k : |x_k - L| \geq \epsilon\} = 0$$

It is denoted by $st-\lim_n x_n = L$.

In [18] Doğru and Kanat defined the Kantorovich-type modification of Lupaş operators as follows:

$$\tilde{R}_n(f;q;x) = [n+1]\sum_{k=0}^{n}\left(\int_{\frac{[k]}{[n+1]}}^{\frac{[k+1]}{[n+1]}} f(t)\,d_q t\right)\binom{n}{k} \times \frac{q^{-k}q^{k(k-1)/2}x^k(1-x)^{(n-k)}}{(1-x+qx)\cdots(1-x+q^{n-1}x)}. \tag{3.1}$$

Doğru and Kanat [18] proved the following statistical Korovkin-type approximation theorem for operators (3.1).

Theorem 1. *Let $q=(q_n)$, $0<q<1$, be a sequence satisfying the following conditions:*

$$st-\lim_n q_n = 1,\ st-\lim_n q_n^n = a(a<1) and st-\lim_n \frac{1}{[n]_q} = 0, \tag{3.2}$$

then if f is any monotone increasing function defined on $[0,1]$, for the positive linear operators $\tilde{R}_n(f;q;x)$, then

$$st-\lim_n \|\tilde{R}_n(f;q;\cdot)-f\|_{C[0,1]} = 0$$

holds.

In [19] Doğru gave some examples so that (q_n) is statistically convergent to 1 but it may not be convergent to 1 in the ordinary case.

Now, we consider a sequence $q=(q_n)$, $q_n\in(0,1)$, such that

$$\lim_{n\to\infty} q_n = 1. \tag{3.3}$$

The condition (3.3) guarantees that $[n]_{q_n}\to\infty$ as $n\to\infty$.

Theorem 2. *Let $\mathfrak{D}_n^{(\alpha,\beta)}$ be the sequence of the operators (1.4) and the sequence $q=(q_n)$ satisfies (3.2). Then for any function $f\in C[0,\nu]\subset C[0,\infty)$, $\nu>0$, we have*

$$st-\lim_n \|\mathfrak{D}_n^{(\alpha,\beta)}(f;q;\cdot)-f\| = 0, \tag{3.4}$$

where $C[0,\nu]$ denotes the space of all real bounded functions f which are continuous in $[0,\nu]$.

Proof. Let $f_i = t^i$, where $i=0,1,2$. Using $\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)=1$, it is clear that

$$st-\lim_n \|\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)-1\| = 0.$$

Now by Lemma (2) (ii), we have

$$\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_n;x)-x\| = \left\|\frac{[n]_q x+q+\alpha}{[n]_q+\beta}-x\right\| \le \frac{(q+\alpha)}{[n]_q+\beta}+\frac{\beta}{([n]_q+\beta)}x.$$

For given $\epsilon>0$, we define the following sets:

$$L=\{k:\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_k;x)-x\|\ge\epsilon\},$$

and

$$L'=\left\{k:\frac{(q+\alpha)}{[k]_q+\beta}+\frac{\beta}{[k]_q+\beta}x\right\}. \tag{3.5}$$

It is obvious that $L\subset L'$, it can be written as

$$\delta\big(\{k\le n:\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_k;x)-x\|\ge\epsilon\}\big) \le \delta\left(\left\{k\le n:\frac{(q+\alpha)}{[k]_q+\beta}+\frac{\beta}{[k]_q+\beta}x\right\}\right).$$

By using (3.2), we get

$$st-\lim_n\left(\frac{(q+\alpha)}{[n]_q+\beta}+\frac{\beta}{[n]_q+\beta}x\right)=0.$$

So, we have

$$\delta\left(\left\{k\le n:\frac{(q+\alpha)}{[n]_q+\beta}+\frac{\beta}{[n]_q+\beta}x\right\}\right)=0,$$

then

$$st-\lim_n\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_n;x)-x\|=0.$$

Similarly, by Lemma (2) (iii), we have

$$\begin{aligned}
\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2;q_n;x)-x^2\| &= \Bigg\|\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}\right)x^2 + \left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right)x + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}-x^2\Bigg\| \\
&\le \left|\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}\right)-1\right|\nu^2 + \left|\left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right)\right|\nu + \left|\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\right| \\
&\le \mu^2\Bigg[\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}-1\right) + \left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right) + \left(\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\right)\Bigg]
\end{aligned}$$

where $\mu^2=\max\{\nu^2,\nu,1\}=\nu^2$.

Now, if we choose

$$\alpha_n=\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}-1\right),$$

$$\beta_n=\left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right),$$

$$\gamma_n=\left(\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\right),$$

now using (3.2), we can write

$$st-\lim_{n\to\infty}\alpha_n = 0 = st-\lim_{n\to\infty}\beta_n = st-\lim_{n\to\infty}\gamma_n. \tag{3.6}$$

Now for given $\epsilon>0$, we define the following four sets

$$\mathcal{U}=\{k:\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2;q_k;x)-x^2\|\ge\epsilon\},$$

$$\mathcal{U}_1=\left\{k:\alpha_k\ge\frac{\epsilon}{\mu^2}\right\},$$

$$\mathcal{U}_2=\left\{k:\beta_k\ge\frac{\epsilon}{\mu^2}\right\},$$

$$\mathcal{U}_3=\left\{k:\gamma_k\ge\frac{\epsilon}{\mu^2}\right\}.$$

It is obvious that $\mathcal{U} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$. Then, we obtain

$$\delta\left(\left\{k \leq n : \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\| \geq \epsilon\right\}\right)$$
$$\leq \delta\left(\left\{k \leq n : \alpha_k \geq \frac{\epsilon}{\mu^2}\right\}\right) + \delta\left(\left\{k \leq n : \beta_k \geq \frac{\epsilon}{\mu^2}\right\}\right) + \delta\left(\left\{k \leq n : \gamma_k \geq \frac{\epsilon}{\mu^2}\right\}\right).$$

Using (3.6), we get

$$st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\| = 0.$$

Since,

$$\|\mathfrak{D}_n^{(\alpha,\beta)}(f; q_n; x) - f\| \leq \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\| + \|\mathfrak{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\| + \|\mathfrak{D}_n^{(\alpha,\beta)}(1; q_n; x) - 1\|,$$

we get

$$st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(f; q_n; x) - f\| \leq st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\| + st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\| + st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(1; q_n; x) - 1\|,$$

which implies that

$$st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(f; q_n; x) - f\| = 0.$$

This completes the proof of theorem. □

4. Weighted statistical approximation

In this section, we obtain the Korovkin type weighted statistical approximation by the operators defined in (1.4). A real function ρ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x|\to\infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$. Let $B_\rho(\mathbb{R})$ denote the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f \rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by $C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}): f \text{ continuous on } \mathbb{R}\}$. Note that $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with $\|f\|_\rho = \sup_{x\in R} \frac{|f(x)|}{\rho(x)}$. In case of weight function $\rho_0 = 1 + x^2$, we have $\|f\|_{\rho_0} = \sup_{x\in R} \frac{|f(x)|}{1+x^2}$. Now we are ready to prove our main result as follows:

Theorem 3. *Let $\mathfrak{D}_n^{(\alpha,\beta)}$ be the sequence of the operators (1.4) and the sequence $q = (q_n)$ satisfies (3.2). Then for all nondecreasing function $f \in C_\rho$*, we have

$$st - \lim_{n\to\infty} \|\mathfrak{D}_n^{(\alpha,\beta)}(f; q_n; .) - f\|_{\rho_0} = 0.$$

Proof. By Lemma (2) (iii), we have $\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) \leq Cx^2$, where C is a positive constant, $\mathfrak{D}_n^{(\alpha,\beta)}(f; q_n; x)$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

Using $\mathfrak{D}_n^{(\alpha,\beta)}(1; q_n; x) = 1$, it is clear that

$$st - \lim_{n} \|\mathfrak{D}_n^{(\alpha,\beta)}(1; q_n; x) - 1\|_{\rho_0} = 0.$$

Now, by Lemma (2) (ii), we have

$$\|\mathfrak{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\|_{\rho_0} = \sup_{x\in[0,\infty)} \frac{|\mathfrak{D}_n^{(\alpha,\beta)}(t; q_n; x) - x|}{1+x^2} \leq \frac{(q+\alpha)}{[n]_q+\beta} + \frac{\beta}{([n]_q+\beta)}.$$

Using (3.2), we get

$$st - \lim_{n} \left(\frac{(q+\alpha)}{[n]_q+\beta} + \frac{\beta}{([n]_q+\beta)}\right) = 0,$$

then

$$st - \lim_{n} \|\mathfrak{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\|_{\rho_0} = 0.$$

Finally, by Lemma (2) (iii), we have

$$\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\|_{\rho_0} \leq \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2} - 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} + \left(\frac{(1+q(q+2))[n]_q + 2\alpha[n]_q}{([n]_q+\beta)^2}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}$$
$$\leq \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2} - 1\right) + \left(\frac{(1+q(q+2))[n]_q + 2\alpha[n]_q}{([n]_q+\beta)^2}\right) + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.$$

Now, If we choose

$$\alpha_n = \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2} - 1\right),$$
$$\beta_n = \left(\frac{(1+q(q+2))[n]_q + 2\alpha[n]_q}{([n]_q+\beta)^2}\right),$$
$$\gamma_n = \left(\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\right),$$

then by (3.2), we can write

$$st - \lim_{n\to\infty} \alpha_n = 0 = st - \lim_{n\to\infty} \beta_n = st - \lim_{n\to\infty} \gamma_n. \tag{4.1}$$

Now for given $\epsilon > 0$, we define the following four sets:

$$S = \{k : \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_k; x) - x^2\|_{\rho_0} \geq \epsilon\},$$
$$S_1 = \left\{k : \alpha_k \geq \frac{\epsilon}{3}\right\},$$
$$S_2 = \left\{k : \beta_k \geq \frac{\epsilon}{3}\right\},$$
$$S_3 = \left\{k : \gamma_k \geq \frac{\epsilon}{3}\right\}.$$

It is obvious that $S \subseteq S_1 \cup S_2 \cup S_3$. Then, we obtain

$$\delta\left(\left\{k \leq n : \|\mathfrak{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\|_{\rho_0} \geq \epsilon\right\}\right)$$
$$\leq \delta\left(\left\{k \leq n : \alpha_k \geq \frac{\epsilon}{3}\right\}\right) + \delta\left(\left\{k \leq n : \beta_k \geq \frac{\epsilon}{3}\right\}\right) + \delta\left(\left\{k \leq n : \gamma_k \geq \frac{\epsilon}{3}\right\}\right).$$

Using (4.1), we get

$$st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2;q_n;x)-x^2\|_{\rho_0}=0.$$

Since

$$\begin{aligned}&\|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f\|_{\rho_0}\\&\quad\le\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2;q_n;x)-x^2\|_{\rho_0}+\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_n;x)-x\|_{\rho_0}\\&\qquad+\|\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)-1\|_{\rho_0},\end{aligned}$$

we get

$$\begin{aligned}&st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f\|_{\rho_0}\\&\quad\le st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(t^2;q_n;x)-x^2\|_{\rho_0}\\&\qquad+st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(t;q_n;x)-x\|_{\rho_0}\\&\qquad+st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)-1\|_{\rho_0},\end{aligned}$$

which implies that

$$st-\lim_{n\to\infty}\|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f\|_{\rho_0}=0.$$

This completes the proof of the theorem. □

5. Rates of statistical convergence

In this section, by using the modulus of continuity, we will study rates of statistical convergence of operators (1.4) and Lipschitz type maximal functions are introduced.

Lemma 3. *Let $0<q<1$ and $a\in[0,bq]$, $b>0$. The inequality*

$$\int_a^b|t-x|\mathrm{d}_qt\le\left(\int_a^b|t-x|^2\mathrm{d}_qt\right)^{1/2}\left(\int_a^b\mathrm{d}_qt\right)^{1/2}\tag{5.1}$$

is satisfied.

Let $C_B[0,\infty)$, the space of all bounded and continuous functions on $[0,\infty)$ and $x\ge0$. Then, for $\delta>0$, the modulus of continuity of f denoted by $\omega(f;\delta)$ is defined to be

$$\omega(f;\delta)=\sup_{|t-x|\le\delta}|f(t)-f(x)|,\ t\in[0,\infty).$$

It is known that $\lim_{\delta\to0}\omega(f;\delta)=0$ for $f\in C_B[0,\infty)$ and also, for any $\delta>0$ and each $t,x\ge0$, we have

$$|f(t)-f(x)|\le\omega(f;\delta)\left(1+\frac{|t-x|}{\delta}\right).\tag{5.2}$$

Theorem 4. *Let (q_n) be a sequence satisfying (3.2). For every non-decreasing $f\in C_B[0,\infty)$, $x\ge0$ and $n\in\mathbb{N}$, we have*

$$|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f(x)|\le2\omega(f;\sqrt{\delta_n(x)}),$$

where

$$\begin{aligned}\delta_n(x)=&\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}+1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2\\&+\left(\frac{[n]_q+q^2[n]_q-2\alpha\beta-2q\beta}{([n]_q+\beta)^2}\right)x\\&+\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.\end{aligned}$$

Proof. Let $f\in C_B[0,\infty)$ be a non-decreasing function and $x\ge0$. Using linearity and positivity of the operators $\mathfrak{D}_n^{(\alpha,\beta)}$ and then applying (5.2), we get for $\delta>0$

$$\begin{aligned}|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f(x)|&\le\mathfrak{D}_n^{(\alpha,\beta)}(|f(t)-f(x)|;q_n;x)\\&\le\omega(f,\delta)\left\{\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)+\frac{1}{\delta}\mathfrak{D}_n^{(\alpha,\beta)}(|t-x|;q_n;x)\right\}.\end{aligned}$$

Taking $\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)=1$ and using Cauchy–Schwartz inequality, we have

$$\begin{aligned}&|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f(x)|\\&\le\omega(f;\delta)\left\{1+\frac{1}{\delta}\left(\mathfrak{D}_n^{(\alpha,\beta)}((t-x)^2;q_n;x)^{1/2}\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)^{1/2}\right)\right\}\\&\le\omega(f;\delta)\Bigg[1+\frac{1}{\delta}\Bigg\{\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}+1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2\\&\qquad+\left(\frac{[n]_q+q^2[n]_q-2\alpha\beta-2q\beta}{([n]_q+\beta)^2}\right)x\\&\qquad+\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\Bigg\}^{1/2}\Bigg].\end{aligned}$$

Taking $q=(q_n)$, a sequence satisfying (3.2) and choosing $\delta=\delta_n(x)$ as in Theorem 4, the theorem is proved. □

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions.

In [20], Lenze introduced a Lipschitz type maximal function as

$$f_\alpha(x,y)=\sup_{t>0,t\ne x}\frac{|f(t)-f(x)|}{|t-x|^\alpha}.$$

In [21], the Lipschitz type maximal function space on $E\subset[0,\infty)$ is defined as follows

$$\begin{aligned}\tilde{V}_{\alpha,E}=&\Big\{f:\sup(1+x)^\alpha f_\alpha(x,y)\\&\le M\frac{1}{(1+y)^\alpha};\ x\ge0\text{ and }y\in E\Big\},\end{aligned}$$

where f is bounded and continuous function on $[0,\infty)$, M is a positive constant and $0<\alpha\le1$.

Also, let $d(x,E)$ be the distance between x and E, that is,

$$\mathrm{d}(x,E)=\inf\{|x-y|;y\in E\}.$$

Theorem 5. *If $\mathfrak{D}_n^{(\alpha,\beta)}$ be defined by (1.4), then for all $f\in\tilde{V}_{\alpha,E}$*

$$|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)-f(x)|\le M(\delta_n^{\frac{\alpha}{2}}+d^\alpha(x,E)),\tag{5.3}$$

where

$$\begin{aligned}\delta_n(x)=&\left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}+1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2\\&+\left(\frac{[n]_q+q^2[n]_q-2\alpha\beta-2q\beta}{([n]_q+\beta)^2}\right)x\\&+\frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.\end{aligned}\tag{5.4}$$

Proof. Let $x_0\in\bar{E}$, where $\bar{E}$ denote the closure of the set E such that $|x-x_0|=d(x,E)$, where $x\in[0,\infty)$. Then we have

$$|f(t)-f(x)|\le|f(t)-f(x_0)|+|f(x_0)-f(x)|.$$

Since $\mathfrak{D}_n^{(\alpha,\beta)}$ is a positive and linear operators, $f \in \tilde{V}_{\alpha,E}$ and using the above inequality

$$|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x) - f(x)| \le \mathfrak{D}_n^{(\alpha,\beta)}(|f(t)-f(x_0)|;q_n;x) + (|f(x_0)-f(x)|)\mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x) \le M\big(\mathfrak{D}_n^{(\alpha,\beta)}(|t-x_0|^\alpha;q_n;x) + |x-x_0|^\alpha \mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)\big). \tag{5.5}$$

Therefore, we have

$$\mathfrak{D}_n^{(\alpha,\beta)}(|t-x_0|^\alpha;q_n;x) \le \mathfrak{D}_n^{(\alpha,\beta)}(|t-x|^\alpha;q_n;x) + |x-x_0|^\alpha \mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x).$$

Now, we take $p = \frac{2}{\alpha}$ and $q = \frac{2}{(2-\alpha)}$ and by using the Hölder's inequality, one can write

$$\mathfrak{D}_n^{(\alpha,\beta)}((t-x)^\alpha;q_n;x) \le \mathfrak{D}_n^{(\alpha,\beta)}((t-x)^2;q_n;x)^{\alpha/2} \times \mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x)^{(2-\alpha)/2} + |x-x_0|^\alpha \mathfrak{D}_n^{(\alpha,\beta)}(1;q_n;x) = \delta_n^{\frac{\alpha}{2}} + |x-x_0|^\alpha.$$

Substituting this in (5.5), we get (5.3). This completes the proof of the theorem. □

Remark 1. Observe that by the conditions in (3.2),

$$st - \lim_n \delta_n = 0.$$

By (5.2), we have

$$st - \lim_n \omega(f;\delta_n) = 0.$$

This gives us the pointwise rate of statistical convergence of the operators $\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x)$ to $f(x)$.

Remark 2. If we take $E = [0,\infty)$ in Theorem 5, since $d(x,E) = 0$, then we get for every $f \in \tilde{V}_{\alpha,[0,\infty)}$

$$|\mathfrak{D}_n^{(\alpha,\beta)}(f;q_n;x) - f(x)| \le M\delta_n^{\frac{\alpha}{2}}$$

where δ_n is defined as in (5.4).

Remark 3. By using (4.1), It is easy to verify that

$$st - \lim_{n\to\infty} \delta_n = 0.$$

That is, the rate of statistical convergence of operators (1.4) to the function f are estimated by means of Lipschitz type maximal functions.

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