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Original Article

Some fixed point theorems for *G*-isotone mappings in partially ordered metric spaces

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Abstract Fixed point theorems for *G*-isotone mappings, which extend some recent results for mixed monotone and isotone mappings in partially ordered metric spaces are proved. Moreover, the equivalence between unidimensional and multidimensional fixed point theorems is investigated.

MSC: Primary 47H10; 54H25

1. Introduction

Following paper [1], the problem of existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered a lot (see, e.g., [2–22] and the related references therein). Some fixed point theorems were proved in these papers and they are usually applied in discussing the existence and uniqueness of solution to matrix equations, periodic boundary value problems and nonlinear integral equations.

Recently, Roldán et al. [17] introduced the notion of coincidence point between mappings in any number of variables and extended several special notions of, so called, coupled, tripled, quadrupled and multidimensional fixed/coincidence points appeared in the literature see, for example, [3], [8], [14], [15], respectively. Results in [17] also extend some fixed points ones in the framework of partially ordered complete metric spaces. In order to guarantee the existence of coincidence point the authors of [17] constructed some Cauchy sequences using the properties of mixed monotone mappings and contractive conditions. The idea was used in a lot of paper (see, e.g., [16], [18], [19]). To prove that more than one sequences are simultaneously Cauchy's, seems not so easy. It is also known that the fixed point problems for isotone mappings are easier than that of mixed monotone mappings. Wang [21] obtained some multidimensional fixed point theorems for isotone mappings and extended some of the results in coupled, tripled, quadrupled and multidimensional fixed/coincidence points for mixed monotone and non-decreasing mappings in

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partially ordered complete metric spaces. She also gave a simple and unified approach to coupled, tripled, quadrupled and multidimensional fixed point theorems for mixed monotone mappings.

Motivated and inspired by the above results, we obtain some new fixed point theorems for G-isotone mappings and investigate the equivalence between unidimensional and multidimensional fixed point theorems.

2. Preliminaries

Let $n \in \mathbb{N}$, X be a non-empty set and X^n be the Cartesian product of n copies of X. For brevity, $g(x)$, $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, $(v_1, v_2, \ldots, v_n)$ and $(x_0^1, x_0^2, \ldots, x_0^n)$ will be denoted by gx, X, Y, Z, V and X_0, respectively.

Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$, $\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A$ and $\sigma(B) \subseteq B\}$ and $\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B$ and $\sigma(B) \subseteq A\}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be n mappings from Λ_n into itself. If $(X, \preceq)$ is a partially ordered space, $y, v \in X$ and $i \in \Lambda_n$, we use the next notation from [17]:

$$y \preceq_i v \Leftrightarrow \begin{cases} y \preceq v, & \text{if } i \in A, \\ y \succeq v, & \text{if } i \in B. \end{cases} \tag{1}$$

If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e. $x \preceq y$ or $y \preceq x$ holds) we will write $x \approx y$. The product space X^n is endowed with the following natural partial order: for $Y, V \in X^n$

$$Y \preceq_n V \iff y_i \preceq_i v_i, i \in \Lambda_n. \tag{2}$$

The mapping $\rho_n : X^n \times X^n \to [0, +\infty)$, given by:

$$\rho_n(X, Y) = \max_{1 \le i \le n} d(x_i, y_i), \tag{3}$$

defines a metric on X^n. We denote Γ the set of all continuous and strictly increasing functions φ: $[0, \infty) \to [0, \infty)$, and Ψ the set of all functions ψ: $[0, \infty) \to [0, \infty)$, such that $\lim_{t \to r} \psi(t) > 0$ for every $r > 0$ and $\psi(t) = 0 \iff t = 0$.

Definition 2.1 ([11]). A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.

Definition 2.2 ([17]). Let $g : X \to X$ be a mapping. If $(X, d, \preceq)$ is an ordered metric space, then X is said to have the sequential g-monotone property if it satisfies the following properties:

(i) If $(x_m)_{m \in \mathbb{N}}$ is a non-decreasing sequence and $\lim_{m \to \infty} x_m = x$, then $gx_m \preceq gx$ for all $m \in \mathbb{N}$.
(ii) If $(y_m)_{m \in \mathbb{N}}$ is a non-increasing sequence and $\lim_{m \to \infty} y_m = y$, then $gy_m \succeq gy$ for all $m \in \mathbb{N}$.

If g is the identity mapping, then X is said to have the sequential monotone property (see [17]) and $(X, d, \preceq)$ is said to be regular (see [22]).

Definition 2.3 ([16]). Let $F : X^n \to X$ and $g : X \to X$ be two mappings. A point $(x_1, x_2, \ldots, x_n) \in X^n$ is a Y-coincidence point of F and g if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i$$

for $i \in \Lambda_n$. If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Y-fixed point of the mapping F.

Definition 2.4 ([19]). Let $(X, d, \preceq)$ be an ordered metric space. The mappings $F : X^n \to X$ and $g : X \to X$ are said to be O-compatible if, for all sequences $\{x_m^1\}_{m \ge 0}, \{x_m^2\}_{m \ge 0}, \ldots, \{x_m^n\}_{m \ge 0} \subset X$ such that $\{gx_m^1\}_{m \ge 0}, \{gx_m^2\}_{m \ge 0}, \ldots, \{gx_m^n\}_{m \ge 0}$ are monotone and the following limit exists: for all i,

$$\lim_{m \to \infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{m \to \infty} gx_m^i \in X,$$

we have

$$\lim_{m \to \infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})) = 0$$

for all i.

Definition 2.5 ([17]). Let $(X, \preceq)$ be a partially ordered space, and $F : X^n \to X$ and $g : X \to X$ be two mappings. It is said that F has the mixed g-monotone property if F is g-monotone nondecreasing in arguments with indices in A and g-monotone nonincreasing in arguments with indices in B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and each $i \in \{1, \ldots, n\}$,

$$gy \preceq gz \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Definition 2.6 ([20]). Let $(X^n, \preceq)$ be a partially ordered set, and T and G self-mappings of X^n. It is said that T is a G-isotone mapping if, for any $Y_1, Y_2 \in X^n$

$$G(Y_1) \preceq_n G(Y_2) \Rightarrow T(Y_1) \preceq_n T(Y_2).$$

Definition 2.7 ([20]). An element $Y \in X^n$ is called a coincidence point of the mappings $T : X^n \to X^n$ and $G : X^n \to X^n$ if $T(Y) = G(Y)$. Furthermore, if $T(Y) = G(Y) = Y$, then is said that Y is a common fixed point of T and G.

Remark 2.8. Note that if $G = I_{X^n}$ in Definitions 2.6 and 2.7, then T is an isotone mapping and Y is a fixed point of T (see [21]).

Definition 2.9. $\mathcal{C}$ a family functions $f : [0, \infty)^2 \to R$ is called C-class if it is continuous and satisfies following axioms:

(1) $f(s, t) \le s$;
(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$;

for all $s, t \in [0, \infty)$.

Example 2.10. The following functions $f : [0, \infty)^2 \to R$ are elements of $\mathcal{C}$. For each $s, t \in [0, \infty)$,

(1) $f(s, t) = ks$, $0 < k < 1$, $f(s, t) = s \Rightarrow s = 0$;
(2) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;
(3) $f(s, t) = \frac{s-t}{1+t}$, $f(s, t) = s \Rightarrow t = 0$;
(4) $f(s, t) = \frac{s}{1+t}$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$;
(5) $f(s, t) = \log \frac{t+a^s}{1+t}$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$;
(6) $f(s, t) = (s + l)^{(\frac{1}{1+t})} - l$, $l > 1$, $f(s, t) = s \Rightarrow t = 0$;
(7) $f(s, t) = s \log_{a+t} a$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ *or* $t = 0$.

Remark 2.11. Functions of C-class is a natural generalization for Banach contraction, as that can see in above example number (1).

3. Main results

Now, we state and prove our main results.

Theorem 3.1. *Let $(X, d, \preceq)$ be a complete ordered metric space, and let $T : X \to X$ and $G : X \to X$ be two mappings such that T is a G-isotone mapping, $T(X) \subseteq G(X)$ and G is continuous and O-compatible with T. Assume that there exist $h \in \mathcal{C}$, $\varphi \in \Gamma$ and $\psi \in \Psi$ such that, for all $y, v \in X$ with $G(y) \preceq G(v)$,*

$$\varphi(d(T(y), T(v))) \le h(\varphi(d(G(y), G(v))), \psi(d(G(y), G(v)))). \tag{4}$$

Suppose that either

(a) *T is continuous or;*

(b) *$G(x_m) \preceq G(x)$ for all $m \in \mathbb{N}$ when $(x_m)_{m\in\mathbb{N}}$ is a non-decreasing sequence in X such that $x_m \to x$; $G(x_m) \succeq G(x)$ for all $m \in \mathbb{N}$ when $(x_m)_{m\in\mathbb{N}}$ is a non-increasing sequence in X such that $x_m \to x$.*

If there exists $y_0 \in X$, such that $G(y_0) \approx T(y_0)$, then T and G have a coincidence point.

Proof. Since $T(X) \subseteq G(X)$, it follows that there is a $y_1 \in X$ such that $G(y_1) = T(y_0)$. Recursively, we obtain that for every $m \in \mathbb{N}_0$, there is a $y_{m+1} \in X$ such that $G(y_{m+1}) = T(y_m)$. Set $z_0 = G(y_0)$ and $z_{m+1} = G(y_{m+1}) = T(y_m)$ for every $m \in \mathbb{N}_0$.

Since $G(y_0) \approx T(y_0)$, we assume that $G(y_0) \preceq T(y_0)$, that is, $z_0 \preceq z_1$ (the case $G(y_0) \succeq T(y_0)$ is treated similarly). Assume that $z_{m-1} \preceq z_m$ for some $m \in \mathbb{N}_0$, that is, $G(y_{m-1}) \preceq G(y_m)$. Since T is a G-isotone mapping, we get

$$z_m = T(y_{m-1}) \preceq T(y_m) = z_{m+1}.$$

This actually means that the sequence $(z_m)_{m\in\mathbb{N}_0}$ is non-decreasing. If $z_{m_0+1} = z_{m_0}$ for some $m_0 \in \mathbb{N}_0$, then y_{m_0} is a coincidence point of T and G. Thus, we may assume that $z_{m+1} \ne z_m$ for every $m \in \mathbb{N}_0$.

By $G(y_{m-1}) \preceq G(y_m)$ and (4), we have that

$$\begin{aligned}\varphi(d(z_{m+1}, z_m)) &= \varphi(d(T(y_m), T(y_{m-1}))) \\ &\le h(\varphi(d(G(y_m), G(y_{m-1}))), \psi(d(G(y_m), G(y_{m-1})))) \\ &\le \varphi(d(z_m, z_{m-1})), \quad m \in \mathbb{N}.\end{aligned} \tag{5}$$

From (5), since φ is strictly increasing, we obtain

$$d(z_{m+1}, z_m) \le d(z_m, z_{m-1}), \quad m \in \mathbb{N}.$$

Hence, the sequence $(\delta_m)_{m\in\mathbb{N}_0}$ given by $\delta_m = d(z_{m+1}, z_m)$ is non-increasing and bounded below. Therefore, there exists some $\delta \ge 0$ such that $\lim_{m\to\infty} \delta_m = \delta$. We shall prove that $\delta = 0$. Assume that $\delta > 0$. Using the properties of φ and ψ, we have $\varphi(\delta) > \varphi(0) \ge 0$ and $\lim_{m\to\infty} \psi(\delta_{m-1}) > 0$. Using Definition 2.9, we know that when $h(s, t) = s$, then $s = 0$ or $t = 0$ and $h(s, t) < s$ when $s > 0$ and $t > 0$. Then, by letting $m \to \infty$ in (5) and using the properties of h, we have

$$\varphi(\delta) \le h(\varphi(\delta), \lim_{m\to\infty} \psi(\delta_{m-1})) = h(\varphi(\delta), \lim_{r\to\delta} \psi(r)) < \varphi(\delta),$$

which is a contradiction. Thus, $\lim_{m\to\infty} \delta_m = 0$.

We claim that $(z_m)_{m\in\mathbb{N}_0}$ is a Cauchy sequence. Indeed, if it was false, then there would exist an $\epsilon > 0$ and the subsequences $(z_{m(l)})_{l\in\mathbb{N}}$ and $(z_{n(l)})_{l\in\mathbb{N}}$ of $(z_m)_{m\in\mathbb{N}_0}$ such that $n(l)$ is the minimal in the sense that $n(l) > m(l) \ge l$, $d(z_{m(l)}, z_{n(l)}) > \epsilon$, and $d(z_{m(l)}, z_{n(l)-1}) \le \epsilon$.

Using the triangle inequality, we obtain

$$\begin{aligned}\epsilon < d(z_{m(l)}, z_{n(l)}) &\le d(z_{m(l)}, z_{m(l)-1}) + d(z_{m(l)-1}, z_{n(l)-1}) \\ &\quad + d(z_{n(l)-1}, z_{n(l)}) \\ &\le d(z_{m(l)}, z_{m(l)-1}) + d(z_{m(l)-1}, z_{m(l)}) + d(z_{m(l)}, z_{n(l)-1}) \\ &\quad + d(z_{n(l)-1}, z_{n(l)}) \\ &\le 2d(z_{m(l)}, z_{m(l)-1}) + \epsilon + d(z_{n(l)-1}, z_{n(l)}).\end{aligned}$$

Letting $l \to \infty$ in the above inequality, we get

$$\lim_{l\to\infty} d(z_{m(l)}, z_{n(l)}) = \lim_{l\to\infty} d(z_{m(l)-1}, z_{n(l)-1}) = \epsilon. \tag{6}$$

Since $n(l) > m(l)$, we have $z_{m(l)-1} \preceq z_{n(l)-1}$, i.e., $G(y_{m(l)-1}) \preceq G(y_{n(l)-1})$. From (4), it follows that

$$\begin{aligned}\varphi(d(z_{n(l)}, z_{m(l)})) &= \varphi(d(T(y_{n(l)-1}), T(y_{m(l)-1}))) \\ &\le h(\varphi(d(G(y_{n(l)-1}), G(y_{m(l)-1}))), \psi(d(G(y_{n(l)-1}), \\ &\quad G(y_{m(l)-1})))) \\ &= h(\varphi(d(z_{n(l)-1}, z_{m(l)-1})), \psi(d(z_{n(l)-1}, z_{m(l)-1}))).\end{aligned}$$

Using the properties of φ and ψ, we have $\varphi(\epsilon) > 0$ and $\lim_{l\to\infty}\psi(r_l) > 0$, where $r_l = d(z_{n(l)-1}, z_{m(l)-1})$. Letting $l \to \infty$ in the above inequality and using (6), it follows that

$$\varphi(\epsilon) \le h(\varphi(\epsilon), \lim_{l\to\infty} \psi(r_l)) < \varphi(\epsilon),$$

which is a contradiction. Hence, the sequence $(z_m)_{m\in\mathbb{N}_0}$ is Cauchy's in the metric space (X, d). Since (X, d) is a complete metric space, then there exists $z \in X$ such that $\lim_{m\to\infty} z_m = z$, that is,

$$\lim_{m\to\infty} T(y_m) = \lim_{m\to\infty} G(y_m) = z. \tag{7}$$

As G is continuous, we have

$$\lim_{m\to\infty} G(G(y_m)) = G(z). \tag{8}$$

By the O-compatibility of T and G, we have

$$\begin{aligned}&\lim_{m\to\infty} d(G(G(y_{m+1})), T(G(y_m))) \\ &\quad = \lim_{m\to\infty} d(G(T(y_m)), T(G(y_m))) = 0.\end{aligned} \tag{9}$$

Now suppose that T is continuous. It follows from (7)-(9) that z is a coincidence point of T and G.

Now suppose that condition (b) holds. Since $(z_m)_{m\in\mathbb{N}_0}$ is a non-decreasing sequence and $z_m \to z$ $(m \to \infty)$, then $G(G(y_m)) \preceq G(z)$ for every $m \in \mathbb{N}_0$. From (4), we obtain

$$\begin{aligned}&\varphi(d(T(G(y_m)), T(z))) \\ &\quad \le h(\varphi(d(G(G(y_m)), G(z))), \psi(d(G(G(y_m)), G(z)))) \\ &\quad \le \varphi(d(G(G(y_m)), G(z)))\end{aligned} \tag{10}$$

for $m \in \mathbb{N}_0$. From (10), since φ is strictly increasing, we have

$$d(T(G(y_m)), T(z)) \le d(G(G(y_m)), G(z)), \quad m \in \mathbb{N}_0. \tag{11}$$

Letting $m \to \infty$ in (11), we obtain

$$\lim_{m\to\infty} T(G(y_m)) = T(z). \tag{12}$$

From (8), (9) and (12), we have that z is also a coincidence point of T and G in this case. □

Corollary 3.2. *Under hypothesis of Theorem 3.1, if $y \in X$ is a coincidence point of T and G, then $G(y)$ also is a coincidence point of T and G.*

Proof. Suppose that $G(y) = T(y)$. Then we can choose $y_m = y$ for all $m \geq 0$ as in the previous proof. We have just prove that $G(y_m) \to z$ as $m \to \infty$ and z is a coincidence point of T and G. In this case, $G(y_m) = G(y) \to G(y)$ as $m \to \infty$ and so $G(y)$ also is a coincidence point of T and G. This completes the proof. □

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1, suppose that for all coincidence points $y, v \in X$ of mappings T and G, there exists $u \in X$ such that $G(u)$ is comparable to $G(y)$ and $G(v)$. Then T and G have a unique coincidence point z such that $G(z) = z$.*

Proof. Put $u_1 = u$ and define the sequence $(G(u_m))_{m\in\mathbb{N}}$ by: $G(u_{m+1}) = T(u_m)$ for $m \in \mathbb{N}$. We may assume that $G(y) \preceq G(u_1)$ (the case $G(y) \succeq G(u_1)$ is treated similarly). Since T is a G-isotone mapping, we have $G(y) = T(y) \preceq T(u_1) = G(u_2)$. By induction we obtain $G(y) \preceq G(u_m)$, for every $m \in \mathbb{N}$. From (4), we have that

$$\begin{aligned}\varphi(d(G(u_{m+1}), G(y))) &= \varphi(d(T(u_m), T(y)))\\ &\leq h(\varphi(d(G(u_m), G(y))), \psi(d(G(u_m), G(y))))\\ &\leq \varphi(d(G(u_m), G(y)))\end{aligned} \tag{13}$$

for $m \in \mathbb{N}$. Then, since φ is strictly increasing, we have

$$d(G(u_{m+1}), G(y)) \leq d(G(u_m), G(y)), \quad m \in \mathbb{N},$$

that is, the sequence $(\beta_m)_{m\in\mathbb{N}}$ defined by $\beta_m = d(G(u_m), G(y))$ is non-increasing. Hence, there exists $\beta \geq 0$ such that $\lim_{m\to\infty} \beta_m = \beta$. We prove that $\beta = 0$. Suppose, conversely, that $\beta > 0$. Using the properties of φ and ψ, we have $\varphi(\beta) > 0$ and $\lim_{\beta_m\to\beta} \psi(\beta_m) > 0$. Letting $m \to \infty$ in (13), we get

$$\varphi(\beta) \leq h(\varphi(\beta), \lim_{m\to\infty} \psi(\beta_m)) \leq h(\varphi(\beta), \lim_{\beta_m\to\beta} \psi(\beta_m)) < \varphi(\beta),$$

which is a contradiction. Thus $\beta = 0$, that is,

$$\lim_{m\to\infty} d(G(u_m), G(y)) = 0. \tag{14}$$

Similarly, we find that

$$\lim_{m\to\infty} d(G(u_m), G(v)) = 0. \tag{15}$$

From (14) and (15), we obtain

$$G(y) = G(v). \tag{16}$$

By Corollary 3.2, we find that $z := G(y)$ is a coincidence point of the mappings T and G. Using (16) with $v = z$, we obtain

$$z = G(y) = G(z). \tag{17}$$

To prove the uniqueness, assume that z' is another coincidence point of mappings T and G. Then by (17) we get $z' = G(z') = G(z) = z$, as claimed. □

Remark 3.4. Note that if there exists $u \in X$ such that $T(u)$ is comparable to $T(y)$ and $T(v)$, then Theorem 3.3 still holds. Indeed, using a similar argument to the proof Theorem 3.3, we only have to check that $G(y) \preceq G(u_m)$ for $m \geq 2$. We assume that $T(y) \preceq T(u_1)$ (the case $T(y) \succeq T(u_1)$ is treated similarly). Since y is a coincidence point of T and G, we have $G(y) = T(y) \preceq T(u_1) = G(u_2)$. By the G-isotone property of T, we have $G(y) = T(y) \preceq T(u_2) = G(u_3)$. So, $G(y) \preceq G(u_m)$ for $m \geq 2$ by induction, as claimed.

Theorem 3.5. *Let $(X, d, \preceq)$ be a complete ordered metric space, and let $T : X^n \to X^n$ and $G = (g, g, \ldots, g) : X^n \to X^n$ be two mappings such that T is a G-isotone mapping, $T(X^n) \subseteq G(X^n)$ and G is continuous and O-compatible with T. Assume that there exist $h \in \mathcal{C}$, $\varphi \in \Gamma$ and $\psi \in \Psi$ such that, for all $Y, V \in X^n$ with $G(Y) \preceq_n G(V)$,*

$$\varphi(\rho_n(T(Y), T(V))) \leq h(\varphi(\rho_n(G(Y), G(V))),\ \psi(\rho_n(G(Y), G(V))))$$

where ρ_n is defined by (3). Suppose that either

(a) *T is continuous or;*
(b) *X has the sequential g-monotone property.*

If there exists $Y_0 \in X^n$, such that $G(Y_0) \approx T(Y_0)$, then T and G have a coincidence point. Furthermore, suppose that for all coincidence points $Y, V \in X^n$ of mappings T and G, there exists $U \in X^n$ such that $G(U)$ is comparable to $G(Y)$ and $G(V)$. Then T and G have a unique coincidence point Z such that $G(Z) = Z$.

Proof. Since $(X, d, \preceq)$ is a complete ordered metric space, so is $(X^n, \rho_n, \preceq_n)$. Now we shall prove that condition (b) of Theorem 3.1 holds with respect to $(X^n, \rho_n, \preceq_n)$. Suppose that $(Z_m)_{m\in\mathbb{N}_0}$ is a non-decreasing sequence in X^n such that $Z_m \to Z$ $(m \to \infty)$. That is, $Z_m \preceq_n Z_{m+1}$ for all $m \in \mathbb{N}_0$ and $z^i_m \to z^i (m \to \infty)$ for all $i \in \Lambda_n$. Thus, $(z^i_m)_{m\in\mathbb{N}_0}$ is a non-decreasing sequence when $i \in A$ and $(z^i_m)_{m\in\mathbb{N}_0}$ is a non-increasing sequence when $i \in B$. If $i \in A$, as X has the sequential g-monotone property, then we have $gz^i_m \preceq gz^i$ for all $m \in \mathbb{N}_0$. Similarly, if $i \in B$, then we deduce that $gz^i_m \succeq gz^i$ for all $m \in \mathbb{N}_0$. Since $G = (g, g, \ldots, g)$, then $G(Z_m) \preceq_n G(Z)$ for every $m \in \mathbb{N}_0$. The other case is treated similarly.

By our assumptions, all conditions of Theorem 3.1 and Theorem 3.3 hold with respect to $(X^n, \rho_n, \preceq_n)$. Using Theorem 3.1, T and G have a coincidence point. Moreover, it follows from Theorem 3.3 that T and G have a unique coincidence point $Z \in X^n$ such that $G(Z) = Z$. □

Remark 3.6. The metric ρ_n in Theorem 3.5 can be replaced by some other metrics on X^n, for example, by the next one:

$$\rho_n(Y, V) = \frac{1}{n}\big[d(y_1, v_1) + d(y_2, v_2) + \cdots + d(y_n, v_n)\big], \tag{18}$$

and the result will be also true. As the proof of Theorem 3.5, Theorem 3.5 is a consequence of Theorems 3.1 and 3.3. Note also that taking $n = 1$ in Theorem 3.5, we can obtain Theorems 3.1 and 3.3 immediately. So, Theorem 3.5 is equivalent to Theorems 3.1 and 3.3.

Taking $h(s,t) = s - t$ and $g = I_X$ in Theorem 3.5, we obtain the following result.

Corollary 3.7 ([21]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X^n \to X^n$ be an isotone mapping for which there exist $\varphi \in \Gamma$ and $\psi \in \Psi$ such that, for all $Y, V \in X^n$ with $Y \preceq_n V$,*

$$\varphi(\rho_n(T(Y), T(V))) \leq \varphi(\rho_n(Y, V)) - \psi(\rho_n(Y, V)),$$

where ρ_n is defined by (18). Suppose that either

(a) *T is continuous or;*
(b) *$(X, d, \preceq)$ is regular.*

If there exists $Y_0 \in X^n$, such that $Y_0 \approx T(Y_0)$, then T has a fixed point. Furthermore, suppose that for all fixed points $Y, V \in X^n$ of T, there exists $U \in X^n$ such that U is comparable to Y and V. Then T has a unique a fixed point.

Corollary 3.8. *Let $(X, d, \preceq)$ be a complete ordered metric space. Let $\Upsilon = (\sigma_1, \ldots, \sigma_n)$ be a n-tuple of self-mappings of Λ_n such that σ_i is a permutation for all $i \in \Lambda_n$, $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$, g is continuous and O-compatible with F. Assume that there exists $\varphi \in \Gamma$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \varphi(\max_{1 \leq i \leq n} d(gx_i, gy_i)) - \psi(\max_{1 \leq i \leq n} d(gx_i, gy_i)) \tag{19}$$

for which $gx_i \preceq_i gy_i$ for all $i \in \Lambda_n$. Suppose that either F is continuous or X has the sequential g-monotone property. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ such that:

$$gx_0^i \preceq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) \tag{20}$$

for all $i \in \Lambda_n$, then F and g have, at least, one Υ-coincidence point.

Furthermore, assume that for all pairs of Υ-coincidence points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$ of F and g there exists $(u_1, u_2, \ldots, u_n) \in X^n$ such that $(gu_1, gu_2, \ldots, gu_n)$ is comparable, at the same time, to $(gx_1, gx_2, \ldots, gx_n)$ and to $(gy_1, gy_2, \ldots, gy_n)$. Then F and g have a unique Υ-coincidence point $(z_1, z_2, \ldots, z_n) \in X^n$ such that $gz_i = z_i$ for $i \in \Lambda_n$.

Proof. Consider the mappings $T : X^n \to X^n$ and $G : X^n \to X^n$ defined by

$$\begin{aligned} T(Y) = &(F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, \ldots, y_{\sigma_1(n)}), \ldots, \\ &F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), \ldots, \\ &F(y_{\sigma_n(1)}, y_{\sigma_n(2)}, \ldots, y_{\sigma_n(n)})) \end{aligned} \tag{21}$$

and

$$G(Y) = (gy_1, gy_2, \ldots, gy_n) \tag{22}$$

for $Y \in X^n$. Note that T and G are O-compatible with respect to $(X^n, \rho_n, \preceq_n)$. Indeed, suppose that $\{Y_m\}_{m \geq 0} \subset X^n$ such that $\{G(Y_m)\}_{m \geq 0}$ is monotone and the following limit exists:

$$\lim_{m\to\infty} T(Y_m) = \lim_{m\to\infty} G(Y_m) \in X^n.$$

From (21) and (22), we see that, for sequences $\{y_m^1\}_{m \geq 0}$, $\{y_m^2\}_{m \geq 0}, \ldots, \{y_m^n\}_{m \geq 0} \subset X$ such that $\{gy_m^1\}_{m \geq 0}, \{gy_m^2\}_{m \geq 0}, \ldots, \{gy_m^n\}_{m \geq 0}$ are monotone and the following limit exists: for all i,

$$\lim_{m\to\infty} F(y_m^{\sigma_i(1)}, y_m^{\sigma_i(2)}, \ldots, y_m^{\sigma_i(n)}) = \lim_{m\to\infty} gy_m^i \in X.$$

Since F and g are O-compatible, we have

$$\begin{aligned} &\lim_{m\to\infty} \rho_n(GT(Y_m), TG(Y_m)) \\ &= \lim_{m\to\infty} \max_{1 \leq i \leq n} d(gF(y_m^{\sigma_i(1)}, y_m^{\sigma_i(2)}, \ldots, y_m^{\sigma_i(n)}), \\ &F(gy_m^{\sigma_i(1)}, gy_m^{\sigma_i(2)}, \ldots, gy_m^{\sigma_i(n)})) = 0. \end{aligned}$$

By our assumptions, we deduce that $T(X^n) \subseteq G(X^n)$ and G is continuous.

Now, we shall deduce that T is a G-isotone mapping. Indeed, suppose that $G(X) \preceq_n G(Y)$, $\forall X, Y \in X^n$. By (2) and (22), we have $gx_t \preceq gy_t$ when $t \in A$ and $gx_t \succeq gy_t$ when $t \in B$. For each $i \in A$, we have $\sigma_i \in \Omega_{A,B}$. So, $gx_{\sigma_i(t)} \preceq gy_{\sigma_i(t)}$, $\forall t \in A$ and $gx_{\sigma_i(t)} \succeq gy_{\sigma_i(t)}$, $\forall t \in B$. Thus, by the mixed g-monotonicity of F, we have $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \preceq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})$, for all $i \in A$. Similarly, we have $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) \succeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})$, for all $i \in B$. Thus, by (2) and (21), we deduce that T is a G-isotone mapping.

Since σ_i is a permutation for all $i \in \Lambda_n$, we have

$$\max_{1 \leq t \leq n} d(gy_{\sigma_i(t)}, gv_{\sigma_i(t)}) = \max_{1 \leq t \leq n} d(gy_t, gv_t) = \rho_n(G(Y), G(V)) \tag{23}$$

for all $i \in \Lambda_n$. From (19) and (23), we have

$$\begin{aligned} &\varphi(\rho_n(T(Y), T(V))) \\ &= \varphi(\max_{1 \leq i \leq n} d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), \\ &\quad F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \ldots, v_{\sigma_i(n)}))) \\ &= \max_{1 \leq i \leq n} \varphi(d(F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)}), \\ &\quad F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, \ldots, v_{\sigma_i(n)}))) \\ &\leq \max_{1 \leq i \leq n} [(\varphi - \psi)(\max_{1 \leq j \leq n} d(gx_{\sigma_i(j)}, gy_{\sigma_i(j)}))] \\ &\leq \varphi(\rho_n(G(Y), G(V))) - \psi(\rho_n(G(Y), G(V))) \end{aligned} \tag{24}$$

for $i \in \Lambda_n$ and $G(Y) \preceq_n G(V)$. It follows from (24) that

$$\varphi(\rho_n(T(Y), T(V))) \leq h(\varphi(\rho_n(G(Y), G(V))), \psi(\rho_n(G(Y), G(V)))), \tag{25}$$

for $G(Y) \preceq_n G(V)$, where $h(s,t) = s - t$.

It follows from (20) that $G(X_0) \preceq_n T(X_0)$. If F is continuous, then T is continuous.

Using Theorem 3.5, we deduce that T and G have a coincidence point and $Z \in X^n$ is a unique coincidence point such that $G(Z) = Z$. That is, F and g have a Υ-coincidence point and $(z_1, z_2, \ldots, z_n) \in X^n$ is a unique Υ-coincidence point of F and g such that $gz_i = z_i$ for $i \in \Lambda_n$. □

Remark 3.9. As an application, we give a simple proof of Corollary 3.8, which is similar to Theorems 14 and 20 in [19]. The techniques are used in [23–25].

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