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Original Article

Weakly semi-preopen "semi-preclosed" functions in *L*-double fuzzy topological spaces

A. Ghareeb*

Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

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Abstract In this paper, we introduce a new class of functions called *L*-double fuzzy weakly semi-preopen (semi-preclosed) functions in *L*-double fuzzy topological spaces. Some characterizations of this class and its properties and the relationship with other classes in *L*-double fuzzy topological spaces are also discussed.

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1. Introduction

The fuzzy concept has overrun almost all branches of mathematics since the definition of the concept by Zadeh [1]. Fuzzy sets have applications in many fields such as information [2] and control [3]. The theory of fuzzy topological spaces was defined and developed in the first time by Chang [4] and since then various notions in general topology have been generalized to Chang's fuzzy topological spaces. Šostak [5] and Kubiak [6] introduced the fuzzy topology as an extension of Chang's fuzzy topology. It has been developed in many directions. Šostak [7] also published a survey article of the developed areas of fuzzy topological spaces. The topologistes used to call Chang's fuzzy topology by "*L*-topology" and Kubiak–Šostak's fuzzy topology by "*L*-fuzzy topology" where *L* is any an appropriate lattice.

Intuitionistic fuzzy sets [8] have been found to be highly profitable to deal with vagueness. While fuzzy sets only give a membership degree to each element of the universe, and the non-membership degree equals one minus the membership degree, in intuitionistic fuzzy set theory the two degrees are more or less independent.

Intuitionistic fuzzy sets followed the same steps of fuzzy sets in topology. Çoker and his colleague [9–11] defined intuitionistic fuzzy topology in Chang's sense and in Kubiak- Šostak's sense. Thereafter came the definition of intuitionistic fuzzy gradation of openness by Samanta and Mondal [12,13].

Working under the term "intuitionistic" discontinuous due to some doubts around it. In 2005, Dubois et al. [14] showed

* Tel.: +201014725551.
E-mail address: nasserfuzt@hotmail.com, a.ghareeb@sci.svu.edu.eg
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that there are terminological difficulties in intuitionistic fuzzy set theory. In the same year, Atanassov [15] responded to Dubois questions defending his concept.

Gutiérrez Garcia and Rodabaugh [16] ended these controversy. They proved that this term is not suitable in mathematics and also its applications. Their conclusion was to work under the term "double". The notions studied under the term "intuitionistic" were given new names. Double fuzzy topology replaced intuitionistic fuzzy gradation of openness.

Our motivation in this paper is to define new functions in L-double fuzzy topological spaces based on the concepts (r, s)-fuzzy semi-preopen and (r, s)-fuzzy semi-preclosed subsets and investigate some of their properties.

2. Preliminaries

Throughout this paper $(L, \leq, \bigwedge, \bigvee, ')$ is a complete DeMorgan algebra, X is a nonempty set. L^X is the set of all L-subsets on X. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively. A complete lattice L is a complete Heyting algebra if it satisfies the following infinite distributive law: For all $a \in L$ and all $B \subset L$,

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}.$$

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is prime [17]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq sup\, D$ always implies the existence of $d \in D$ with $a \leq d$ [18]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. A set $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [15,19], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap J(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

An L-fuzzy point in L^X is an L-subset x_λ, where $\lambda \in L_\perp = L - \{\perp\}$, such that $x_\lambda(y) = \lambda$ when $y = x$ and $\perp$ otherwise. For L-subsets $U, V \in L^X$, we write UqV to mean that U is quasi-coincident (q-coincident, for short) with U, i.e., there exists at least one point $x \in X$ such that $U(x) \not\leq U(x)'$. Negation of such a statement is denoted as $U\neg qV$. Let f: $X \to Y$ be a crisp mapping. Then an L-fuzzy mapping $f_L^{\rightarrow} : L^X \to L^Y$ is induced by f as usual, i.e., $f_L^{\rightarrow}(U)(y) = \bigvee_{x \in X,\ f(x)=y} U(x)$ and $f_L^{\leftarrow}(V)(x) = V(f(x))$.

An L-topological space (or L-space, for short) is a pair (X, τ), where τ is a subfamily of L^X which contains $\underline{\perp}$; $\underline{\top}$ and is closed for any suprema and finite infima. τ is called an L-topology on X. Members of τ are called open L-subsets and their complements are called closed L-subsets.

Definition 2.1 [12,13,16]**.** The pair of function $\mathcal{T}, \mathcal{T}^* : L^X \longrightarrow L$ is called an L-double fuzzy topology on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\underline{\perp}) = \mathcal{T}(\underline{\top}) = \top$ and $\mathcal{T}^*(\underline{\perp}) = \mathcal{T}^*(\underline{\top}) = \perp$.
(O2) $\mathcal{T}(U) \leq \mathcal{T}^*(U)'$.
(O3) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$ and $\mathcal{T}^*(U \wedge V) \leq \mathcal{T}^*(U) \vee \mathcal{T}^*(V)$ for each $U, V \in L^X$.
(O4) $\mathcal{T}(\bigvee_{i\in\Gamma} U_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(U_i)$ and $\mathcal{T}^*(\bigvee_{i\in\Gamma} U_i) \leq \bigvee_{i\in\Gamma} \mathcal{T}^*(U_i)$ for any $\{U_i\}_{i\in\Gamma} \subset L^X$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called an L-double fuzzy topological spaces. $\mathcal{T}(U)$ and $\mathcal{T}^*(U)$ can be interpreted as the degree to which U is an open L-subset and to which U is a closed L-subset. A function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is said to be continuous with respect to L-double fuzzy topologies $(\mathcal{T}_1, \mathcal{T}_1^*)$ and $(\mathcal{T}_2, \mathcal{T}_2^*)$ if $\mathcal{T}_1(f_L^{\leftarrow}(V)) \geq \mathcal{T}_2(V)$ and $\mathcal{T}_1^*(f_L^{\leftarrow}(V)) \leq \mathcal{T}_2^*(V)$ holds for all $V \in L^Y$.

For $a, b \in L$ and the functions $\mathcal{T}, \mathcal{T}^* : L^X \to L$, we use the following notation:

$$\mathcal{T}_{[a,b]} = \{A \in L^X \mid \mathcal{T}(A) \geq a,\ \mathcal{T}^*(A) \leq b\}.$$

The proof of the following theorem is straightforward and so omitted.

Theorem 2.1. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space. Then the following conditions are equivalent:*

(1) *$(\mathcal{T}, \mathcal{T}^*)$ is an L-double fuzzy topology on X;*
(2) *$\mathcal{T}_{[a,b]}$ is an L-topology on X, for each $a \in J(L)$ and $b \in P(L)$.*

Theorem 2.2 [11,20]**.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space. Then for each $r \in L_\perp$, $s \in L_\top$ and $U \in L^X$, we define an operator $C_{\mathcal{T},\mathcal{T}^*} : L^X \times L_\perp \times L_\top \to L^X$ as follows:*

$$C_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigwedge \{V \in L^X \mid U \leq V,\ \mathcal{T}(V') \geq r,\ \mathcal{T}^*(V') \leq s\}.$$

For $U, V \in L^X$, $r, r_1 \in L_\perp$ and $s, s_1 \in L_\top$, the operator $C_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(C1) $C_{\mathcal{T},\mathcal{T}^*}(\underline{\perp}, r, s) = \underline{\perp}$.
(C2) $U \leq C_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.
(C3) $C_{\mathcal{T},\mathcal{T}^*}(U, r, s) \vee C_{\mathcal{T},\mathcal{T}^*}(V, r, s) = C_{\mathcal{T},\mathcal{T}^*}(U \vee V, r, s)$.
(C4) $C_{\mathcal{T},\mathcal{T}^*}(U, r, s) \leq C_{\mathcal{T},\mathcal{T}^*}(U, r_1, s_1)$ *if* $r \leq r_1$ *and* $s \geq s_1$.
(C5) $C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s) = C_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.

Theorem 2.3 [11,20]**.** *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space. Then for each $r \in L_\top$, $s \in L_\perp$ and $U \in L^X$, we define an operator $\mathcal{I}_{\mathcal{T},\mathcal{T}^*} : L^X \times L_\perp \times L_\top \to L^X$ as follows:*

$$\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigvee \{V \in L^X \mid V \leq U,\ \mathcal{T}(V) \geq r,\ \mathcal{T}^*(V) \leq s\}.$$

For $U, V \in L^X$ and $r, r_1 \in L_\top$ and $s, s_1 \in L_\perp$, the operator $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(I1) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U', r, s) = C_{\mathcal{T},\mathcal{T}^*}(U, r, s)'$.
(I2) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\underline{\top}, r, s) = \underline{\top}$.
(I3) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s) \leq U$.
(I4) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s) \wedge \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(V, r, s) = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U \wedge V, r, s)$.
(I5) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r_1, s_1) \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s)$ *if* $r \leq r_1$ *and* $s \leq s_1$.
(I6) $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s) = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.
(I7) *If* $\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s) = U$, *then* $C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U', r, s), r, s) = U'$.

Definition 2.2. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space. For $U \in L^X$, $r \in L_\perp$ and $s \in L_\perp$, U is called:

(1) An (r, s)-fuzzy preopen (resp. (r, s)-fuzzy preclosed) [21] if $U \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s)$ (resp. $C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s) \leq U$).
(2) An (r, s)-fuzzy regular open (resp. (r, s)-fuzzy regular closed) [22] if $U = \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s)$ (resp. $U = C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s)$).

(3) An (r, s)-fuzzy α-open (resp. (r, s)-fuzzy α-closed) [22] if $U \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s), r, s)$(resp. $C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s), r, s) \leq U$).

Definition 2.3. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space. For $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$, U is called an (r, s)-fuzzy semi-preopen subset (resp. (r, s)-fuzzy semi-preclosed) if

$$U \leq C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s), r, s)$$
$$(\text{res. } \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s), r, s) \leq U).$$

The semi-preinterior $sp\mathcal{I}_{\mathcal{T},\mathcal{T}^*}$ and semi-preclosur $spC_{\mathcal{T},\mathcal{T}^*}$ operators in L-double fuzzy topological space $(X, \mathcal{T})$ defined as follows:

$$sp\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigvee\{V \in L^X \mid V \leq U \quad \text{and}$$
$$V \text{ is } (r, s) - \text{fuzzy semi-preopen}\},$$
$$spC_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigwedge\{V \in L^X \mid U \leq V \quad \text{and}$$
$$V \text{ is } (r, s) - \text{fuzzy semi-preclosed}\}.$$

Definition 2.4. Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be a function from an L-double fuzzy topological space $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ into an L-double fuzzy topological space $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$. The function $f_L^\to$ is called:

(1) An L-double fuzzy semi-preclosed (resp. L-double fuzzy semi-preopen)if $f_L^\to(U)$ is (r, s)-fuzzy semi-preclosed (resp. (r, s)-fuzzy semi-preopen) subset in L^Y for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$ (resp. $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$),
(2) An L-double fuzzy almost open, if $\mathcal{T}_2(f_L^\to(U)) \geq r$ and $\mathcal{T}_2^*(f_L^\to(U)) \leq s$ for each (r, s)-fuzzy regular open subset $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$,
(3) An L-double fuzzy strongly continuous, if $f_L^\to(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)) \leq f_L^\to(U)$ for every $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$,
(4) An L-double fuzzy weakly open (resp. L-double fuzzy weakly closed) [22], if

$$f_L^\to(U) \leq \mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^\to(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s)$$

(resp. $C_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^\to(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s) \leq f_L^\to(U)$) for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$ (resp. $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$).

Definition 2.5. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space, $U \in L^X$, $x_\lambda \in J(L^X)$, $r \in L_\perp$ and $s \in L_\top$. U is called an (r, s)-fuzzy Q-neighborhood of x_λ if $\mathcal{T}(U) \geq r$, $\mathcal{T}^*(U) \leq s$ and $x_\lambda qU$.

We will denote the set of all (r, s)-fuzzy open Q-neighborhood of x_λ by $\mathcal{Q}_{\mathcal{T},\mathcal{T}^*}(x_\lambda, r, s)$.

Definition 2.6. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space, $U \in L^X$, $x_\lambda \in J(L^X)$, $r \in L_\perp$ and $s \in L_\top$. x_λ is called an (r, s)-fuzzy θ-cluster point of U if for each $V \in \mathcal{Q}_{\mathcal{T},\mathcal{T}^*}(x_\lambda, r, s)$, we have $C_{\mathcal{T},\mathcal{T}^*}(V, r, s)qU$.

We denote $D_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigvee\{x_\lambda \in J(L^X) | x_\lambda$ is $(r, s) -$ fuzzy $\theta -$ cluster point of $U\}$. Where $D_{\mathcal{T},\mathcal{T}^*}(U, r, s)$ is called (r, s)-fuzzy θ-closure of U.

The following theorem can be easily extended it from the case $L = [0, 1]$.

Theorem 2.4. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ an L-double fuzzy topological space. For $U, V \in L^X$ and $r \in L_\perp$, $s \in L_\top$, we have the following:*

(1) $D_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigwedge\{V \in L^X | \ U \leq \mathcal{I}_{\mathcal{T},\mathcal{T}^*}(V, r, s), \ \mathcal{T}(V') \geq r, \ \mathcal{T}^*(V') \leq s\}$.
(2) *x_λ is (r, s)-fuzzy θ-cluster point of U iff $x_\lambda \in D_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.*
(3) $C_{\mathcal{T},\mathcal{T}^*}(U, r, s) \leq D_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.
(4) *If $\mathcal{T}(U) \geq r$ and $\mathcal{T}^*(U) \leq s$, then $C_{\mathcal{T},\mathcal{T}^*}(U, r, s) = D_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.*
(5) *If U is (r, s)-fuzzy preopen, then $C_{\mathcal{T},\mathcal{T}^*}(U, r, s) = D_{\mathcal{T},\mathcal{T}^*}(U, r, s)$.*
(6) *If U is (r, s)-fuzzy preopen and $\lambda = C_{\mathcal{T},\mathcal{T}^*}(\mathcal{I}_{\mathcal{T},\mathcal{T}^*}(U, r, s), r, s)$, then $D_{\mathcal{T},\mathcal{T}^*}(U, r, s) = U$.*

The complement of (r, s)-fuzzy θ-closed set is called (r, s)-fuzzy θ-open and the (r, s)-fuzzy θ-interior operator denoted by $T_{\mathcal{T},\mathcal{T}^*}(U, r, s)$ is defined by

$$T_{\mathcal{T},\mathcal{T}^*}(U, r, s) = \bigvee\{V \in L^X | C_{\mathcal{T},\mathcal{T}^*}(V, r, s) \leq U, \ \mathcal{T}(V) \geq r, \ \mathcal{T}^*(V) \leq s\}.$$

3. *L*-Double fuzzy weakly semi-preopen “semi-preclosed” functions

Definition 3.1. A function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is said to be:

(a) An L-double fuzzy weakly semi-preopen function if $f_L^\to(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^\to(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s)$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$.
(b) An L-double fuzzy weakly semi-preclosed function if $spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^\to(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s) \leq f_L^\to(U)$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$.

Remark.

1. Every L-double fuzzy weakly open function is L-double fuzzy weakly semi-preopen function and every L-double fuzzy semi-preopen function is also L-double fuzzy weakly semi-preopen function. But the converse need not be true in general.
2. Every L-double fuzzy weakly closed function is L-double fuzzy weakly semi-preclosed but the converse need not be true in general.

Counter Example 3.1.

(1) Let $L = [0, 1]$, $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. The fuzzy subsets U, V and W are defined as:

$$U(a) = 0.5, \quad U(b) = 0.3, \quad U(c) = 0.2;$$
$$V(x) = 0.9, \quad V(y) = 1, \quad V(z) = 0.7;$$
$$W(x) = 0.2, \quad W(y) = 0.9, \quad W(z) = 0.3.$$

Let $\mathcal{T}_1, \mathcal{T}_1^* : I^X \to I$ and $\mathcal{T}_2, \mathcal{T}_2^* : I^Y \to I$ defined as follows:

$$\mathcal{T}_1(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_1^*(A) = \begin{cases} 0, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{T}_2(A) = \begin{cases} 1, & \text{if } B = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } B = V; \\ \frac{1}{4}, & \text{if } B = W; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_2^*(A) = \begin{cases} 0, & \text{if } B = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } B = V; \\ \frac{1}{4}, & \text{if } B = W; \\ 1, & \text{otherwise.} \end{cases}$$

Then the function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ defined by

$$f(a) = z, \quad f(b) = x, \quad f(c) = y,$$

is I-double fuzzy weakly semi-preopen but not I-double fuzzy weakly open.

(2) Let $L = I = [0, 1]$ and $X = \{a, b, c\}$. The fuzzy subsets U, V, W and H are defined as follows:

$$\begin{aligned} &U(a) = 0.4, \quad U(b) = 0.7, \quad A(c) = 0.2; \\ &V(a) = 0.3, \quad V(b) = 0.1, \quad V(c) = 0.6; \\ &W(a) = 0.5, \quad W(b) = 0.8, \quad W(c) = 0.3; \\ &H(a) = 0.4, \quad H(b) = 0.2, \quad H(c) = 0.7. \end{aligned}$$

Let $\mathcal{T}_1, \mathcal{T}_1^* : I^X \to I$ and $\mathcal{T}_2, \mathcal{T}_2^* : I^X \to I$ defined as follows:

$$\mathcal{T}_1(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ \frac{1}{4}, & \text{if } A = V; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_1^*(A) = \begin{cases} 0, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ \frac{1}{4}, & \text{if } A = V; \\ 1, & \text{otherwise.} \end{cases}$$

$$\mathcal{T}_2(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = W; \\ \frac{1}{4}, & \text{if } A = H; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_2^*(A) = \begin{cases} 0, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = W; \\ \frac{1}{4}, & \text{if } A = H; \\ 1, & \text{otherwise.} \end{cases}$$

Then the identity function $i : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (X, \mathcal{T}_2, \mathcal{T}_2^*)$ is I-double fuzzy weakly semi-preopen function but not I-double fuzzy semi-preopen.

(3) Let $L = [0, 1]$, $X = \{a, b\}$ and $Y = \{x, y\}$. The fuzzy subsets U, V are defined as follows:

$$\begin{aligned} &U(a) = 0.5, \quad U(b) = 0.6; \\ &V(x) = 0.4, \quad V(y) = 0.3. \end{aligned}$$

Let $\mathcal{T}_1, \mathcal{T}_1^* : I^X \to I$ and $\mathcal{T}_2, \mathcal{T}_2^* : I^Y \to I$ defined as follows:

$$\mathcal{T}_1(A) = \begin{cases} 1, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_1^*(A) = \begin{cases} 0, & \text{if } A = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } A = U; \\ 1, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{T}_2(B) = \begin{cases} 1, & \text{if } B = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } B = V; \\ 0, & \text{otherwise.} \end{cases},$$

$$\mathcal{T}_2^*(B) = \begin{cases} 0, & \text{if } B = \underline{0} \text{ or } \underline{1}; \\ \frac{1}{2}, & \text{if } B = V; \\ 1, & \text{otherwise.} \end{cases}$$

The function $f\colon (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ defined by

$$f(a) = x, \quad f(b) = y,$$

is I-double fuzzy weakly semi-preclosed but not I-double fuzzy weakly closed.

Theorem 3.1. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ are L-double fuzzy topological spaces. The function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is L-double fuzzy weakly semi-preopen (resp. L-double fuzzy weakly semi-preclosed) iff $f : (X, \mathcal{T}_{1[a,b]}) \to (Y, \mathcal{T}_{2[a,b]})$ is fuzzy weakly semi-preopen (resp. fuzzy weakly semi-preclosed) function for each $a \in J(L)$ and $b \in P(L)$.*

Proof. Straightforward. □

Theorem 3.2. *For a function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$, the following conditions are equivalent:*

(1) *$f_L^{\to}$ is L-double fuzzy weakly semi-preopen function;*
(2) *$f_L^{\to}(T_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)) \le sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\to}(U), r, s)$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$;*
(3) *$T_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V), r, s) \le f_L^{\leftarrow}(sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(V, r, s))$ for each $V \in L^Y$, $r \in L_\perp$ and $s \in L_\top$;*
(4) *$f_L^{\leftarrow}(spC_{\mathcal{T}_2,\mathcal{T}_2^*}(V, r, s)) \le D_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V), r, s)$ for each $V \in L^Y$, $r \in L_\perp$ and $s \in L_\top$;*
(5) *For each $x_\lambda \in J(L^X)$ and $U \in L^X$ such that $\mathcal{T}_1(\lambda) \ge r$, $\mathcal{T}_1^*(\lambda) \le s$ and $x_\lambda \le U$ there exists an (r, s)-fuzzy semi-preopen subset V such that $f_L^{\to}(x_\lambda) \le V$ and $V \le f_L^{\to}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s))$.*

Proof.

(1)⇒(2): Let $x_\lambda \le T_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)$. Then $f_L^{\to}(x_\lambda) \le f_L^{\to}(U)$. Since $f_L^{\to}$ is L-double fuzzy weakly semi-preopen function, then

$$\begin{aligned} f_L^{\to}(x_\lambda) &\le f_L^{\to}(U) \\ &\le sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\to}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s). \end{aligned}$$

Therefore, $x_\lambda \le f_L^{\leftarrow}(sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\to}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s))$. Thus

$$T_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s) \le f_L^{\leftarrow}(sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\to}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s)),$$

i.e.,

$$f_L^{\to}(T_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)) \le sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\to}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)), r, s).$$

(2)⇒(1): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \ge r$ and $\mathcal{T}_1^*(U) \le s$. Since $U \le T_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s), r, s)$ and by using (2), we have

$$f_L^{\rightarrow}(U) \leq f_L^{\rightarrow}(T_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s).$$

Hence $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen function.

(2)⇒(3): Let $V \in L^Y$. By using (2), we have $f_L^{\rightarrow}(T_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V),r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(V,r,s)$. Therefore, $T_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V),r,s) \leq f_L^{\leftarrow}(sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(V,r,s))$.

(3)⇒(2): It is trivial and omitted.

(3)⇒(4): Let $V \in L^Y$, $r \in L_\perp$ and $s \in L_\top$. By using (3), we have

$$D_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V),r,s)' = T_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V)',r,s) = T_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V'),r,s) \leq f_L^{\leftarrow}(sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(V',r,s)) = f_L^{\leftarrow}(spC_{\mathcal{T}_2,\mathcal{T}_2^*}(V,r,s)') = (f_L^{\leftarrow}(spC_{\mathcal{T}_2,\mathcal{T}_2^*}(V,r,s)))'.$$

Therefore, we obtain $f_L^{\leftarrow}(spC_{\mathcal{T}_2,\mathcal{T}_2^*}(V,r,s)) \leq D_{\mathcal{T}_1,\mathcal{T}_1^*}(f_L^{\leftarrow}(V),r,s)$.

(4)⇒(3): It is trivial and omitted.

(1)⇒(5): Let $x_\lambda \in J(L^X)$ and $U \in L^X$ such that $\mathcal{T}_1(U) \geq r$, $\mathcal{T}_1^*(U) \leq s$ and $x_\lambda \leq U$. Since $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen, $f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$. Let $V = sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$. Then $V \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ with $f_L^{\rightarrow}(x_\lambda) \leq V$.

(5)⇒(1): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$, $\mathcal{T}_1^*(U) \leq s$ and let $y_\beta \leq f_L^{\rightarrow}(U)$. By using (2), we have $V \leq f_L^{\rightarrow}(C_{\mathcal{T}_2}(U,r,s))$ for some (r, s)-fuzzy semi-preopen subset $V \in L^Y$ and $y_\beta \leq V$. Hence we have, $y_\beta \leq V \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$. This shows that $f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$. □

Theorem 3.3. *For a function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$, the following conditions are equivalent:*

1. *$f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen function;*
2. *$f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_l^{\rightarrow}(U),r,s)$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$;*
3. *$f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$ for each $U \in L^X$, $r \in L$ and $s \in L$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$;*
4. *$f_L^{\rightarrow}(U) \leq sp\mathcal{I}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$ for each (r, s)-fuzzy preopen subset $U \in L^X$;*
5. *$f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$ for each (r, s)-fuzzy α-open subset $U \in L^X$.*

Proof.

(1)⇒(2): Let $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$, then $\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s) = \mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)$. By using (1),

$$f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) = f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) = sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s).$$

for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$.

(2)⇒(3): It is trivial and omitted.

(3)⇒(4): Let $U \in L^X$ be an (r, s)-fuzzy preopen subset, then $U \leq \mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)$. By using (3), we have

$$f_L^{\rightarrow}(U) \leq f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s).$$

(4)⇒(5): It is trivial and omitted.

(5)⇒(1): It is trivial and omitted. □

Theorem 3.4. *If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is L-double fuzzy weakly semi-preopen and L-double fuzzy strongly continuous, then $f_L^{\rightarrow}$ is L-double fuzzy semi-preopen.*

Proof. Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$. Since $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen

$$f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s).$$

However, since $f_L^{\rightarrow}$ is L-fuzzy strongly continuous, $f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s)$ and therefore $f_L^{\rightarrow}(U)$ is (r, s)-fuzzy semi-preopen subset. □

Theorem 3.5. *If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is L-double fuzzy almost open function, then $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen.*

Proof. Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$. Since $f_L^{\rightarrow}$ is L-double fuzzy almost open and $\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)$ is (r, s)-fuzzy regular open, then

$$\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(U,r,s),r,s)),r,s) = f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(U,r,s),r,s))$$

and hence

$$f_L^{\rightarrow}(\lambda) \leq f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s) \leq \mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s).$$

This shows that $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen. □

Theorem 3.6. *For the function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$, the following conditions are equivalent:*

1. *$f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preclosed;*
2. *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$;*
3. *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s)) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each an (r, s)-fuzzy regular open subset $U \in L^X$;*
4. *For each $V \in L^Y$, $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$, $\mathcal{T}_1^*(U) \leq s$ and $f_L^{\leftarrow}(V) \leq U$, there exists an (r, s)-fuzzy semi-preopen subset $W \in L^Y$ with $V \leq W$ and $f_L^{\leftarrow}(V) \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)$;*
5. *For each $x_\lambda \in J(L^Y)$, $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $f_L^{\leftarrow}(x_\lambda) \leq U$, there exists an (r, s)-fuzzy semi-preopen subset $V \in L^Y$ with $x_\lambda \leq V$ and $f_L^{\leftarrow}(V) \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)$.*
6. *$spC_{\mathcal{T}_2}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)),r,s) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$;*
7. *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(D_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)),r,s) \leq f_L^{\rightarrow}(D_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$;*
8. *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each (r, s)-fuzzy Semi-Preopen subset $U \in L^X$.*

Proof.

(1)⇒(2): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$. Then

$$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s) = spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))) \leq spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)),r,s) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)).$$

(2)⇒(1): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$. Then

$$\begin{aligned} &spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \\ &\leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)) \\ &\leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) \\ &= f_L^{\rightarrow}(U). \end{aligned}$$

(3)⇒(4): Let $U \in L^X$, $V \in L^Y$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$, $\mathcal{T}_1^*(U) \leq s$ and $f_L^{\leftarrow}(V) \leq U$. Then $f_L^{\leftarrow}(V)$ $\neg q C_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)',r,s)$. This implies to V $\neg q f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)',r,s))$. Since $C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)'$ is (r,s)-fuzzy regular open subset, V $\neg q spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)'),r,s)$. Let $W = spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)'),r,s)$. Then W is (r,s)-fuzzy semi-preopen subset with $V \leq W$ and

$$\begin{aligned} f_L^{\leftarrow}(W) &= f_L^{\leftarrow}(spC_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)',r,s))' \\ &\leq f_L^{\leftarrow}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)')') \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s). \end{aligned}$$

(5)⇒(1): Let $V \in L^Y$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_2(V') \geq r$, $\mathcal{T}_2^*(V') \leq s$ and $y_\beta \leq f_L^{\rightarrow}(V)'$. Since $f_L^{\leftarrow}(y_\beta) \leq V'$, there exists (r,s)-fuzzy semi-preopen subset $W \in L^Y$ such that $y_\beta \leq W$ and $f_L^{\leftarrow}(W) \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(V',r,s) = \mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(V,r,s)'$. Therefore W $\neg q f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(V,r,s))$. Then $y_\beta \leq spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(V,r,s)),r,s)'$. □

Theorem 3.7. *If the function $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is a bijective function. Then the following conditions are equivalent:*

(1) *$f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen function;*
(2) *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s))$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$;*
(3) *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(V,r,s)),r,s) \leq f_L^{\rightarrow}(V)$ for each $V \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(V') \geq r$ and $\mathcal{T}_1^*(V') \leq s$.*

Proof.

(1)⇒(3): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$. Then

$$\begin{aligned} (f_L^{\rightarrow}(U))' &= f_L^{\rightarrow}(U') \\ &\leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U',r,s)),r,s) \end{aligned}$$

and so

$$(f_L^{\rightarrow}(U))' \leq (spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s))'.$$

Hence,

$$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \leq f_L^{\rightarrow}(U).$$

(3)⇒(2): Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$. Since $\mathcal{T}_1(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)') \geq r$ and $U \leq \mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)$ and by using (3), we have

$$\begin{aligned} &spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(U),r,s) \\ &\leq spC_{\mathcal{T}_2}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s),r,s)),r,s) \\ &\leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)). \end{aligned}$$

(2)⇒(3): It is trivial and omitted.
(3)⇒(1): It is trivial and omitted. □

Theorem 3.8. *For the function $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$, the following conditions are equivalent:*

(1) *$f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preclosed;*
(2) *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \leq f_L^{\rightarrow}(U)$ for (r,s)-fuzzy semi-preclosed subset $U \in L^X$;*
(3) *$spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \leq f_L^{\rightarrow}(U)$ for each (r,s)-fuzzy α-closed subset $U \in L^X$.*

Proof. Straightforward. □

Theorem 3.9. *If the function $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is L-double fuzzy weakly semi-preopen and L-double fuzzy strongly continuous, then $f_L^{\rightarrow}$ is L-double fuzzy semi-preopen function.*

Proof. Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$. Since $f_L^{\rightarrow}$ is L-double fuzzy strongly continuous, $f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) \leq f_L^{\rightarrow}(U)$ for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$. But $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen, then

$$f_L^{\rightarrow}(U) \leq sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) \leq f_L^{\rightarrow}(U).$$

This is implies to that $f_L^{\rightarrow}(U)$ is an (r,s)-fuzzy semi-preopen subset. Therefore, $f_L^{\rightarrow}$ is L-double fuzzy semi-preopen function. □

Definition 3.2. The function $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is said to be:

(1) An L-double fuzzy contra-semi-preclosed function if $f_L^{\rightarrow}(U)$ is an (r,s)-fuzzy semi-preopen subset, for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$.
(2) An L-double fuzzy contra-semi-preopen if an (r,s)-fuzzy semi-preclosed subset, for each $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$.

Theorem 3.10. *If the function $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is L-double fuzzy contra-semi-preclosed (resp. L-double fuzzy contra-semi-preopen), then it is an L-double fuzzy weakly semi-preopen (resp. L-double fuzzy weakly semi-preclosed).*

Proof. Let $U \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$ and $\mathcal{T}_1^*(U) \leq s$ (resp. $\mathcal{T}_1(U') \geq r$ and $\mathcal{T}_1^*(U') \leq s$). Then, we have $f_L^{\rightarrow}(U) \leq f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) = sp\mathcal{I}_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s)$ (resp. $spC_{\mathcal{T}_2,\mathcal{T}_2^*}(f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)),r,s) = f_L^{\rightarrow}(\mathcal{I}_{\mathcal{T}_1,\mathcal{T}_1^*}(U,r,s)) \leq f_L^{\rightarrow}(U)$). □

4. Some applications of L-double fuzzy weakly semi-preopen "semi-preclosed" functions

In this section, we will present some applications for this kind of functions in separation and connectedness in L-double fuzzy topology.

Definition 4.1. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an L-double fuzzy topological space. An L-subsets $U, V \in L^X$ are (r,s)-fuzzy strongly separated if there exist $H, N \in L^X$ such that $\mathcal{T}(H) \geq r$, $\mathcal{T}^*(H) \leq s$, $\mathcal{T}(N) \geq r$ and $\mathcal{T}^*(N) \leq s$ with $U \leq H$, $V \leq N$ and

$$C_{\mathcal{T},\mathcal{T}^*}(H,r,s) \neg q\, C_{\mathcal{T},\mathcal{T}^*}(N,r,s).$$

Definition 4.2. An L-double fuzzy topological space $(X,\mathcal{T},\mathcal{T}^*)$ is called (r,s)-semi pre T_2 if for each x_{λ_1}, x_{λ_2} with different supports there exist (r,s)-fuzzy semi-preopen sets $U, V \in L^X$ such that $x_{\lambda_1} \leq U \leq x'_{\lambda_2}$, $x_{\lambda_2} \leq V \leq x'_{\lambda_1}$ and $U \neg q V$.

Theorem 4.1. *If $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \rightarrow (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is L-double fuzzy weakly semi-preclosed surjective function and all fibers are (r,s)-fuzzy strongly separated, then $(Y,\mathcal{T}_1,\mathcal{T}_1^*)$ is (r,s)-semi pre-T_2.*

Proof. Let $y_{\beta_1}, y_{\beta_2} \in J(L^Y)$ and let $U, V \in L^X$, $r \in L_\perp$ and $s \in L_\top$ such that $\mathcal{T}_1(U) \geq r$, $\mathcal{T}_1^*(U) \leq s$, $\mathcal{T}_1(V) \geq r$, $\mathcal{T}_1^*(V) \leq s$, $f_L^{\leftarrow}(y_{\beta_1}) \leq U$ and $f_L^{\leftarrow}(y_{\beta_2}) \leq V$ respectively with

$$C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s) \neg q\, C_{\mathcal{T}_1,\mathcal{T}_1^*}(V, r, s).$$

By using Theorem 3.6, there are (r, s)-fuzzy semi-preopen sets $H, N \in L^Y$ such that $y_{\beta_1} \leq H$, $y_{\beta_2} \leq N$, $f_L^{\leftarrow}(U) \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s)$ and $f_L^{\leftarrow}(N) \leq C_{\mathcal{T}_1,\mathcal{T}_1^*}(V, r, s)$. Therefore $H \neg q N$, because

$$C_{\mathcal{T}_1,\mathcal{T}_1^*}(U, r, s) \neg q\, C_{\mathcal{T}_2,\mathcal{T}_2^*}(V, r, s)$$

and $f_L^{\rightarrow}$ is surjective. Thus $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is (r, s)-fuzzy semi-pre-T_2. □

Definition 4.3. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an L-double fuzzy topological space, $r \in L_\perp$ and $s \in L_\top$. The two L-subsets $U, V \in L^X$ are said to be (r, s)-fuzzy separated iff $U \neg q\, C_{\mathcal{T},\mathcal{T}^*}(V, r, s)$ and $V \neg q\, C_{\mathcal{T},\mathcal{T}^*}(U, r, s)$. An L-subset which cannot be expressed as the union of two (r, s)-fuzzy separated subsets is said to be (r, s)-fuzzy connected.

Definition 4.4. Let $(X, \mathcal{T}, \mathcal{T}^*)$ an L-double fuzzy topological space. For L-subsets $U, V \in L^X$ such that $U \neq \perp$ and $V \neq \perp$, are said to be (r, s)-fuzzy semi-preseparated if $U \neg q\, spC_{\mathcal{T},\mathcal{T}^*}(V, r, s)$ and $V \neg q\, spC_{\mathcal{T},\mathcal{T}^*}(U, r, s)$ or equivalently if there exist two (r, s)-fuzzy semi-preopen subsets H, N such that $U \leq H$, $V \leq N$, $U \neg q N$ and $V \neg q H$. An L-double fuzzy topological space which cannot be expressed as the union of two (r, s)-fuzzy semi-preseparated subsets is said to be (r, s)-fuzzy semi-preconnected space.

Theorem 4.2. *If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is injective L-double fuzzy weakly semi-preopen and L-double fuzzy strongly continuous function of space $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ onto an (r, s)-fuzzy semi-preconnected space $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$, then $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is (r, s)-fuzzy connected.*

Proof. Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ be not (r, s)-fuzzy connected. Then there exist (r, s)-fuzzy separated sets $U, V \in L^X$ such that $U \vee V = \top$. Since U and V are (r, s)-fuzzy separated, there exists $H, N \in L^X$ such that $\mathcal{T}_1(H) \geq r$, $\mathcal{T}_1(N) \geq r$ such that $U \leq H$, $V \leq N$, $U \neg q N$, $V \neg q H$. Hence we have $f_L^{\rightarrow}(U) \leq f_L^{\rightarrow}(H)$, $f_L^{\rightarrow}(V) \leq f_L^{\rightarrow}(N)$, $f_L^{\rightarrow}(U) \neg q\, f_L^{\rightarrow}(N)$ and $f_L^{\rightarrow}(V) \neg q\, f_L^{\rightarrow}(H)$. Since $f_L^{\rightarrow}$ is L-double fuzzy weakly semi-preopen and L-double fuzzy strongly continuous function, from Theorem 3.4 we have $f_L^{\rightarrow}(H)$ and $f_L^{\rightarrow}(N)$ are (r, s)-fuzzy semi-preopen. Therefore, $f_L^{\rightarrow}(U)$ and $f_L^{\rightarrow}(V)$ are (r, s)-fuzzy semi preseparated and

$$\top = f_L^{\rightarrow}(\top) = f_L^{\rightarrow}(U \vee V) = f_L^{\rightarrow}(U) \vee f_L^{\rightarrow}(V)$$

which is contradiction with $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is (r, s)-fuzzy semi preconnected. Thus $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is (r, s)-fuzzy connected. □

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