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Original Article

Estimation of parameters for the exponentiated Pareto distribution based on progressively type-II right censored data

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Abstract In this paper, we derive the best linear unbiased estimates (BLEUs) and the maximum likelihood estimates (MLEs) of the location and scale parameters from the Exponentiated Pareto distribution based on progressively Type-II right censored order statistics. In addition, we use Monte-Carlo simulation method to obtain the mean square error of the best linear unbiased estimates and the maximum likelihood estimates and make comparison between them. Finally, we present numerical example.

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1. Introduction

Progressive censoring is very important in life-testing experiments. Its allowance for the removal of live-units from the experiment at various strange is an attractive feature as it will potentially save a lot for experimenter in terms of cost and time. In a series of papers, [1–5] discussed the inference problems for a wide range of distributions under this progressive censoring sampling scheme. These developments have been summarized by Cohen and Whitten [6], and more by Cohen [7]. While most of the above mentioned works were on the maximum likelihood method. Mann, Thomas and Wilson, and Cacciari, and Montanari [8–10] have discussed some linear inferences for the case of progressive Type-II right censoring. Mahmoud et al. [11] have derived approximate moments of progressively Type-II right censored order statistics from the Weibull Gamma distribution and using these moments to derive the best linear unbiased estimates and maximum likelihood estimates. Viveros

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and Balakrishnan [12] have proposed a conditional method of inference based on progressive Type II censored sample when the life time distributions are Weibull and exponential distributions. Aggarwala and Balakrishnan [13] have established an independence result for general progressive Type-II censored samples from the standard uniform distribution by generalizing the algorithm given by Balakrishnan and Sandhu [14]. This result is used in order to obtain moments of general progressive Type-II censored order statistics from the standard uniform distribution. Finally, they have derived the best linear unbiased estimators (BLUEs) of the parameters of one- and two-parameter uniform distributions. Fernandez [15] has discussed the problem of estimation of parameters of exponential distribution, on the basis of the general progressive Type-II censored sample. Balakrishnan and Sandhu [16] have derived the best linear unbiased estimators for the parameters of one- and two-parameters exponential distributions based on general progressive Type-II censored samples and the maximum likelihood estimators. Wu [17] has obtained the maximum likelihood estimates of the shape and scale parameters based on progressively Type-II censored sample from the Weibull distribution.

Recently [18,19] introduced a new distribution called generalized exponential distribution which has been studied quite extensively. In (2001), they discussed also a different method of estimations of the parameters of a generalized exponential distribution. Gupta et al. [20] showed that the exponentiated Pareto distribution can be used quite effectively in analyzing many lifetime data. M. M. Ali et al. [21,22] studied several exponentiated distributions including exponentiated Pareto distribution and discussed their properties. They showed that the exponentiated Pareto distribution gives a good fit to the tail-distribution of NASDAQ data. In (2010) they derived the distribution of the ratio of two independent exponentiated Pareto random variables X and Y and study its properties. Shawky and Hanaa Abu-Zinadah [23] studied how the different estimators of the unknown parameters of exponentiated Pareto distribution can behave for different sample sizes and for different parameter values.

In this paper, we derive the best linear unbiased estimates (BLUEs) and maximum likelihood estimates (MLE) of the location and scale parameters of progressively Type-II right censored data from exponentiated Pareto distribution .In addition, we use Monte-Carlo simulation method to make comparison of the MSE of BLUEs and MLE .

Let $X_1, X_2, \ldots, X_n$ denote a random sample from the Exponentiated Pareto distribution $EP(\theta, \alpha)$ with probability density function (pdf)

$$f(x_i) = \alpha\theta[1-(1+x_i)^{-\alpha}]^{\theta-1}(1+x_i)^{-\alpha-1}, \quad x_i \geq 0, \quad \alpha, \theta > 0, \tag{1.1}$$

and cumulative distribution function (cdf)

$$F(x_i) = [1-(1+x_i)^{-\alpha}]^{\theta}, \quad x_i \geq 0, \quad \alpha, \theta > 0. \tag{1.2}$$

2. Maximum likelihood estimation (MLE)

A maximum likelihood estimation is often the most feasible method to use when doing statistical inference, as the only information required to obtain MLEs is the joint distribution of the observed values. We are already well acquainted with the likelihood function to be maximized when a general progressively Type II censored sample based on n independent units with identical lifetime distributions from an arbitrary continuous distribution $F(x)$ is observed. Recall the likelihood function to be maximized will be

$$L(\mu, \sigma) = A(n, m-1)\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i}, \tag{2.1}$$

where

$$A(n, m-1) = n(n-R_1-1)(n-R_1-R_2-2) \ldots (n-R_1-\cdots-R_{m-1}-m+1).$$

Let $X_{1:m:n}$, $X_{2:m:n}, \ldots, X_{m:m:n}$ be a progressively Type II right censored sample from exponentiated Pareto distribution, with censoring scheme $(R_1, R_2, \ldots, R_m)$ whose pdf. and cdf. given in Eqs. (1.1) and (1.2)

Then the likelihood function to be maximized for estimators of μ and σ (which we will denote by $\hat{\mu}$ and $\hat{\sigma}$)

$$L(\mu, \sigma) = (const.)(\alpha\theta)^m \prod_{i=1}^{m}\left[1-\left(1+\frac{x_i-\mu}{\sigma}\right)^{-\alpha}\right]^{\theta-1} \times\left(1+\frac{x_i-\mu}{\sigma}\right)^{-\alpha-1}\left[1-\left[1-\left(1+\frac{x_i-\mu}{\sigma}\right)^{-\alpha}\right]^{\theta}\right]^{R_i}.$$

For simplicity of notation, we will use X_i instead of $X_{i:m:n}$. The log-likelihood function may be then written as

$$\begin{aligned}\ln L(\mu, \sigma) = {} & const. + m\ln\alpha\theta \\ & +(\theta-1)\sum_{i=1}^{m}\ln\left[1-\left(1+\frac{x_i-\mu}{\sigma}\right)^{-\alpha}\right] \\ & -(\alpha+1)\sum_{i=1}^{m}\ln\left(1+\frac{x_i-\mu}{\sigma}\right) \\ & +\sum_{i=1}^{m}R_i\ln\left[1-\left[1-\left(1+\frac{x_i-\mu}{\sigma}\right)^{-\alpha}\right]^{\theta}\right],\end{aligned}$$

and hence we have the likelihood equations for μ and σ to be

$$\begin{aligned}&\frac{\alpha(\theta-1)}{\hat{\sigma}}\sum_{i=1}^{m}\frac{1}{\left[\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{\alpha+1}-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)\right]} \\ &+\frac{(\alpha+1)}{\hat{\sigma}}\sum_{i=1}^{m}\frac{1}{\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)}-\frac{\alpha\theta}{\hat{\sigma}}\sum_{i=1}^{m}R_i \\ &\times\frac{\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha-1}}{\left\{\left[1-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha}\right]^{1-\theta}-\left[1-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha}\right]\right\}}=0,\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\frac{\alpha(\theta-1)}{\hat{\sigma}^2}\sum_{i=1}^{m}\frac{x_i-\hat{\mu}}{\left[\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{\alpha+1}-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)\right]} \\ &+\frac{(\alpha+1)}{\hat{\sigma}^2}\sum_{i=1}^{m}\frac{x_i-\hat{\mu}}{\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)}-\frac{\alpha\theta}{\hat{\sigma}^2}\sum_{i=1}^{m}R_i \\ &\times\frac{(x_i-\hat{\mu})\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha-1}}{\left\{\left[1-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha}\right]^{1-\theta}-\left[1-\left(1+\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)^{-\alpha}\right]\right\}}=0.\end{aligned} \tag{2.3}$$

The MLEs $\hat{\mu}$ and $\hat{\sigma}$ can be obtained by solving the likelihood equations. Since Eqs. (2.2) and (2.3) cannot be solved analytically, so we use Matlap 6.1 to solve them.

3. Best linear unbiased estimations (BLUEs)

The joint probability density function of $X_{1:m:n}$, $X_{2:m:n}$, . . ., $X_{m:m:n}$, a progressively Type II right censored sample from exponentiated Pareto distribution, with censoring scheme (R_1, R_2 , . . ., R_m) is as follows

$$f_{X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n}}(x_1, x_2, \dots, x_m)$$
$$= A(n, m-1) \prod_{i=1}^{m} \alpha\theta[1-(1+x_i)^{-\alpha}]^{\theta-1}(1+x_i)^{-\alpha-1} \times [1-[1-(1+x_i)^{-\alpha}]^{\theta}]^{R_i}, \tag{3.1}$$

where $A(n, m-1)$ is given in Eq. (2.1).

It is difficult to obtain the moments using Eq. (3.1), so we use the relation between the Exponentiated Pareto distribution and uniform distribution to obtain this moments. The relation is

$$u = [1-(1+x_i)^{-\alpha}]^{\theta} \backsim U(0, 1).$$

Here, $U_{1:m:n}$, $\imath = 1, 2, 3, \dots, m$, denoted the progressively Type II right censored sample from the $U(0, 1)$ distribution from a sample of size n with the censoring scheme (R_1, R_2, ..., R_m).

We use this relation to find the approximate mean, variance and covariance of the Exponentiated Pareto distribution based on the resulting of [24,25].

For this purpose, let us use these notations

$$E(U) = \mu, \quad D^2(x) = \sigma^2, \quad X = \varphi(u) = (1-u^{1/\theta})^{-1/\alpha} - 1,$$

U 's are progressively Type-II right censored order statistics from the uniform (0, 1), and $X_{1:m:n}$, $\imath = 1, 2, \dots, m$ are progressively Type II right censored order statistics from exponentiated Pareto distribution.Then we get the following expressions

$$E(X_{i:m:n}) \simeq [(1-\mu^{1/\theta})^{-1/\alpha} - 1] + \frac{1}{2}\frac{\sigma^2}{\alpha\theta}\left[\frac{1}{\theta}\left(\frac{1}{\alpha}+1\right)(1-\mu^{1/\theta})^{-1/\alpha-2}\mu^{2/\theta-2} \left(\frac{1}{\theta}-1\right)(1-\mu^{1/\theta})^{-1/\alpha-1}\mu^{1/\theta-2}\right],$$

$$D^2(X_{i:m:n}) \simeq \left(\frac{\sigma}{\alpha\theta}\right)^2 [(1-\mu^{1/\theta})^{-1/\alpha-1}\mu^{1/\theta-1}]^2,$$

$$E(Z) = [(1-\mu^{1/\theta})^{-1/\alpha} - 1][(1-\upsilon^{1/\theta})^{-1/\alpha} - 1] + \frac{1}{2}\sigma^2 S + \rho\sigma\tau T + \frac{1}{2}\tau^2 Y,$$

and

$$Cov\big(X_{i:m:n}, X_{j:m:n}\big) \simeq E(Z) - E(X_{i:m:n})E(X_{j:m:n}),$$

where

$$S = \frac{1}{\alpha\theta}[(1-\upsilon^{1/\theta})^{-1/\alpha} - 1] \times\left[\frac{1}{\theta}\left(\left(\frac{1}{\alpha}\right)+1\right)(1-\mu^{1/\theta})^{-(1/\alpha+2)}\mu^{2((1/\theta)-1)} + ((1/\theta)-1)\big(1-\mu^{1/\theta}\big)^{-((1/\alpha)+1)}\mu^{1/\theta-2}\right],$$

$$Y = \frac{1}{\alpha\theta}[(1-\mu^{1/\theta})^{-1/\alpha} - 1] \times\left[\frac{1}{\theta}\left(\left(\frac{1}{\alpha}\right)+1\right)(1-\upsilon^{1/\theta})^{-(1/\alpha+2)}\upsilon^{2((1/\theta)-1)} + ((1/\theta)-1)\big(1-\upsilon^{1/\theta}\big)^{-((1/\alpha)+1)}\upsilon^{1/\theta-2}\right],$$

and

$$T = \left(\frac{1}{\alpha\theta}\right)^2 (1-\upsilon^{1/\theta})^{-1/\alpha-1}\upsilon^{(1/\theta)-1}(1-\mu^{1/\theta})^{-1/\alpha-1}\mu^{(1/\theta)-1}.$$

These expressions , will used to derive expressions for linear estimations for the parameters (μ, σ) of the exponentiated Pareto distribution.

Consider an arbitrary continuous distributions $F(x)$. Suppose now that we believe our progressively censored observations to be represented by the linear transformation $Y = \mu\ 1 + \sigma\ X$, where the vector X represents a vector of progressively Type II right censored order statistics from the standard distribution $F(x)$, then the best linear unbiased estimators of μ and σ will be obtained by minimizing the generalized variance $Q(\theta) = (Y - A\theta)^T \Sigma^{-1}(Y - A\theta)$ with respect to θ where $\theta = (\mu, \sigma)^T$, A is the $P \times 2$ matrix, 1 is the $P \times 1$ vector with components all $1's$ and μ is the mean vector of X and Σ is the variance-covariance matrix of X . the minimum occurs when

$$\mu^* = -\mu^T \Gamma Y = \sum_{i=1}^{m} A_i Y_{i:m:n}, \tag{3.2}$$

and

$$\sigma^* = 1^T \Gamma Y = \sum_{i=1}^{m} B_i Y_{i:m:n}, \tag{3.3}$$

where

$$\Gamma = \Sigma^{-1}(1\mu^T - \mu 1^T)\Sigma^{-1}/\Delta,$$

and

$$\Delta = (1^T\Sigma^{-1}1)(\mu^T\Sigma^{-1}\mu) - (1^T\Sigma^{-1}\mu)^2.$$

From these expressions, variance-covariance matrix of the estimators are readily obtained as

$$Var(\mu^*) = \sigma^2\mu^T\Sigma^{-1}\mu/\Delta,$$
$$Var(\sigma^*) = \sigma^2 1^T\Sigma^{-1}1/\Delta,$$
$$Cov(\mu^*, \sigma^*) = -\sigma^2\mu^T\Sigma^{-1}1/\Delta.$$

The coefficient A_i and B_i , $i = 1, 2, \dots, m$ satisfy the relation $\sum_{i=1}^{m} A_i = 1$ and $\sum_{i=1}^{m} B_i = 0$.

Using the results in Sections 2 and 3 and Eqs. (3.2) and (3.3) one may determine the (BLUE's) for location parameter μ and scale parameter σ .

Notice that we may not be able to easily write down explicit expressions for the BLUE's for μ and σ. So, a simulation study is considered to obtain the BLUEs of μ and σ.

Table 1 Different schemes of progressively censored.

n	m	Censoring scheme
10	4	[2 0 2 2]
15	6	[2 0 2 0 3 2]
20	8	[1 2 0 2 3 2 0 2]
25	10	[4 0 0 2 0 0 4 2 0 3]

Table 2 Coefficients of the BLUEs of μ and σ from the exponentiated Pareto distribution using the first scheme when $\mu = 0$ and $\sigma = 1$.

sch1		$\alpha = 1, \theta = 2$		$\alpha = 1.5, \theta = 2.5$		$\alpha = 2, \theta = 2$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
10	4	0.2041	0.0878	0.1955	0.1287	0.1863	0.2625
		0.4585	−0.1486	0.5121	−0.2807	0.5141	−0.5338
		1.0518	−0.6849	1.2913	−1.2478	1.3347	−2.4407
		−0.7144	0.7457	−0.9988	1.3998	−1.0351	2.7119
sum		≃1	≃0	≃1	≃0	≃1	≃0

Table 3 Coefficients of the BLUEs of μ and σ from the exponentiated Pareto distribution using the second scheme when $\mu = 0$ and $\sigma = 1$.

sch2		$\alpha = 1, \theta = 2$		$\alpha = 1.5, \theta = 2.5$		$\alpha = 2, \theta = 2$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
15	6	−0.0061	0.8176	−0.0735	1.6150	−0.0916	0.6539
		0.1115	−0.7703	0.0912	−1.4655	0.0814	0.2506
		0.2549	−0.2494	0.2644	−0.4970	0.2658	−0.1918
		0.4389	−0.0630	0.5776	−0.0994	0.6021	−0.9914
		0.9679	0.0789	1.3588	0.1219	1.4518	−2.9650
		−0.7670	0.1863	−1.2184	0.3249	−1.3094	3.2437
sum		≃1	≃0	≃1	≃0	≃1	≃0

Table 4 Coefficients of the BLUEs of μ and σ from the exponentiated Pareto distribution using the third scheme when $\mu = 0$ and $\sigma = 1$.

sch3		$\alpha = 1, \theta = 2$		$\alpha = 1.5, \theta = 2.5$		$\alpha = 2, \theta = 2$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
20	8	−0.0696	0.1615	−0.1276	0.3060	−0.1535	0.6365
		−0.0021	0.1193	−0.0407	0.2140	−0.0601	0.4456
		0.0689	0.0663	0.0510	0.1095	0.0400	0.2283
		0.1368	0.0097	0.1424	0.0001	0.1411	0.0002
		0.2235	−0.0670	0.2633	−0.1480	0.2761	−0.3095
		0.3895	−0.2150	0.5022	−0.4398	0.5451	−0.9233
		1.1938	−0.8855	1.6785	−1.8198	1.8822	−3.8680
		−0.9409	0.8107	−1.4692	1.7781	−1.6708	3.7902
sum		≃1	≃0	≃1	≃0	≃1	≃0

Table 5 Coefficients of the BLUEs of μ and σ from the exponentiated Pareto distribution using the fourth scheme when $\mu = 0$ and $\sigma = 1$.

sch4		$\alpha = 1, \theta = 2$		$\alpha = 1.5, \theta = 2.5$		$\alpha = 2, \theta = 2$	
n	m	A_i	B_i	A_i	B_i	A_i	B_i
25	10	−0.3604	0.3700	−0.5200	0.7254	−0.6380	1.6276
		−0.2563	0.2987	−0.3709	0.5617	−0.4635	1.2608
		−0.1524	0.2196	−0.2296	0.4003	−0.2963	0.8985
		−0.0458	0.1341	−0.0858	0.2326	−0.1252	0.5221
		0.0858	0.0257	0.0918	0.0234	0.0866	0.0530
		0.2291	−0.0946	0.2905	−0.2124	0.3251	−0.4778
		0.3958	−0.2358	0.5292	−0.4962	0.6136	−1.1205
		0.7653	−0.5488	1.0704	−1.1388	1.2718	−2.5837
		2.5700	−2.0419	3.7793	−4.3139	4.5939	−9.8942
		−2.2312	1.8729	−3.55492	4.2179	−4.3680	9.7142
sum		≃1	≃0	≃1	≃0	≃1	≃0

4. Simulation study

By using the algorithm given in [24] the following steps are used to generate progressively Type-II right censored order statistics from Exponentiated Pareto distribution.

(1) Generate m independent uniform U(0,1), random variables W_1, W_2, ..., W_m.
(2) For given values of the progressive censoring scheme R_1, R_2, ..., R_m.
set $V_i = W_i^{1/(i+\sum_{j=m-i+1}^{m} R_j)}$ for $i = 1, 2, \ldots, m$.
(3) Set $U_i = 1 - V_m V_{m-1} \ldots V_{m-i+1}$ for $i = 1, 2, \ldots, m$. Then $U_{1:m:n}$, $U_{2:m:n}$, ..., $U_{m:m:n}$ is a progressively Type-II right censored sample of size m from $U(0, 1)$.
(4) Finally, for given values of the two parameters μ and σ, $X_i = (1 - u_i^{1/\theta})^{-1/\alpha} - 1$, $i = 1, 2, \ldots, m$ is a progressively Type-II right censored sample of size m from the exponentiated Pareto distribution.

We generate a progressively Type-II right censored samples from the Exponentiated Pareto distribution with location parameter $\mu = 0$ and scale parameter $\sigma = 1$. 5000 Monte Carlo runs are simulated based on different scheme given in Table 1.

By using the BLUEs presented in the previous section and Table 1, the coefficient of the BLUEs $A_i's$ and $B_i's$ from Eqs. (3.2) and (3.3) are calculated based on 5000 Monte Carlo simulation in Tables 2, 3, 4 and 5. The MSEs of the BLUEs and MLEs are calculated in Table 6.

Also, the variances and covariances of the estimators $\mu^*$and σ^* can be represented in the following table:

5. Discussion of the results

In Table 1, n denotes the sample size, m denotes the number of removable during the scheme and sch1, sch2 , . . . denote the different schemes considered in the Table.

Tables 2, 3, 4, 5 are given for selected values of m and n, the coefficient of the best linear unbiased estimators (BLUEs) of the progressive Type-II right censoring schemes A_i and B_i we can see that $\sum_{i=1}^{m} A_i = 1$ and $\sum_{i=1}^{m} B_i \simeq 0$. The MSE's of the BLUEs and MLEs are calculated in Table 7. From Table 7 and Figs. 1, 2, 3, 4, we see that as n increases, the mean square errors $MSE(\mu^*)$, $MSE(\sigma^*)$, $MSE(\hat{\mu})$,and $MSE(\hat{\sigma})$ decrease. The $MSEs$ of $\hat{\mu}$ and $\hat{\sigma}$ are better than $MSEs$ of $\hat{\mu}$ and σ^* for all schemes.

5.1. Numerical examples

A progressively type-II censored sample of size $m = 8$ from a sample of size $n = 20$ from the Exponentiated Pareto distribution with $\mu = 0$, $\sigma = 1$, $\alpha = 2$, $\theta = 2$ with scheme R_i = [1 2 0 2 3 2 0 2] , was simulated using MATLAB program. The simulated progressively type-II right censored sample is given by:

Table 6 MSE's of BLUEs and MLEs for μ and σ from the exponentiated Pareto distribution.

α	θ	n	m	sch	$MSE(\mu^*)$	$MSE(\sigma^*)$	$MSE(\hat{\mu})$	$MSE(\hat{\sigma})$
1	2	10	4	1	0.4451	0.6502	2.6138×10^{-10}	1.0089×10^{-8}
		15	6	2	0.2977	0.4768	2.7429×10^{-11}	2.8199×10^{-10}
		20	8	3	0.0574	0.2065	8.3851×10^{-12}	5.0945×10^{-11}
		25	10	4	0.0370	0.0505	1.4522×10^{-12}	3.5355×10^{-12}
1.5	2.5	10	4	1	0.6980	0.8415	3.1765×10^{-11}	3.1030×10^{-11}
		15	6	2	0.4575	0.6046	1.6990×10^{-12}	6.5727×10^{-12}
		20	8	3	0.3050	0.4465	6.9902×10^{-13}	6.8459×10^{-13}
		25	10	4	0.2590	0.3510	4.0624×10^{-14}	3.2334×10^{-14}
2	2	10	4	1	0.2714	0.7019	2.3803×10^{-25}	1.7593×10^{-14}
		15	6	2	0.1080	0.5135	8.5340×10^{-23}	1.1732×10^{-14}
		20	8	3	0.0706	0.2921	5.0704×10^{-17}	9.1356×10^{-14}
		25	10	4	0.0072	0.0744	8.0264×10^{-13}	1.1596×10^{-14}

Table 7 The variances and covariances of the estimators $\mu^*$and σ^*

α	θ	n	m	sch	$Var(\mu^*)$	$Var(\sigma^*)$	$Cov(\mu^*, \sigma^*)$
1	0.5	10	4	1	4.1511	0.8209	−1.2540
		15	6	2	6.9633	1.1625	−1.8854
		20	8	3	14.0915	1.4976	−2.5598
		25	10	4	16.0944	3.2464	−4.7911
1.5	2.5	10	4	1	1.9190	0.8307	−0.9546
		15	6	2	3.0805	1.2436	−1.4502
		20	8	3	5.3536	1.5881	−1.9046
		25	10	4	6.8064	3.6810	−3.7588
2	2	10	4	1	0.5499	0.8285	−0.5096
		15	6	2	0.8846	1.2872	−0.7928
		20	8	3	1.5089	1.6909	−1.0601
		25	10	4	2.0410	4.3159	−2.2896

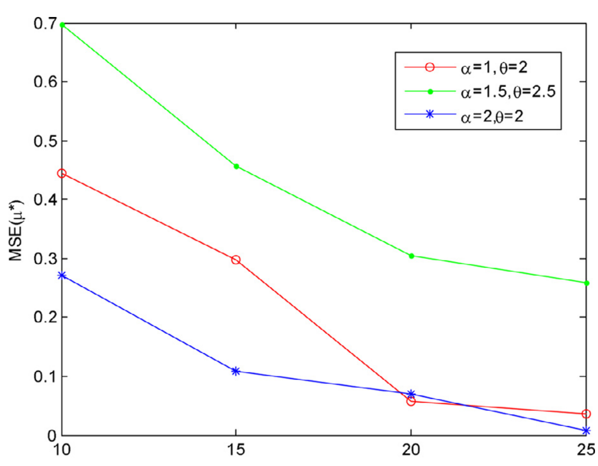

Fig. 1 The relation between the sample size n and the MSE's of μ^* based on the Exponentiated Pareto distribution.

$X_{i:8:20}$	0.1085	0.2453	0.3917	0.4891	0.6021	0.7275	1.0621	1.3771
R_i	1	2	0	2	3	2	0	2

By making use of Eqs. (3.1) and (3.2), and using the coefficients $A_i$and B_i given in Table 2 for $n = 20$ and $m = 8$, we get the *BLUEs* of the μ and σ as follows:

$$\mu^* = (-0.1535 \times 0.1085) + (-0.0601 \times 0.2453) + (0.0400 \times 0.3917) + (0.1411 \times 0.4891) + (0.2761 \times 0.6021) + (0.5451 \times 0.7275) + (1.8822 \times 1.0621) + (-1.6708 \times 1.3771) = 0.31430873$$

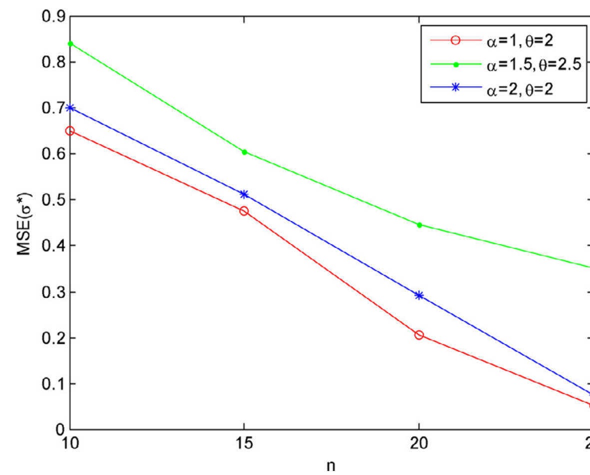

Fig. 2 The relation between the sample size n and the MSE's of σ^* based on the Exponentiated Pareto distribution.

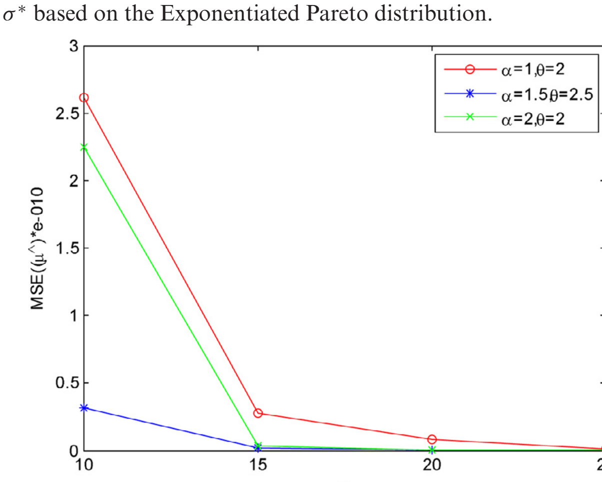

Fig. 3 The relation between the sample size n and the MSE's of $\hat{\mu}$ based on the Exponentiated Pareto distribution.

and

$$\sigma^* = (0.6365 \times 0.1085) + (0.4456 \times 0.2453) + (0.2283 \times 0.3917) + (0.0002 \times 0.4891) + (-0.3095 \times 0.6021) + (-0.9233 \times 0.7275)$$

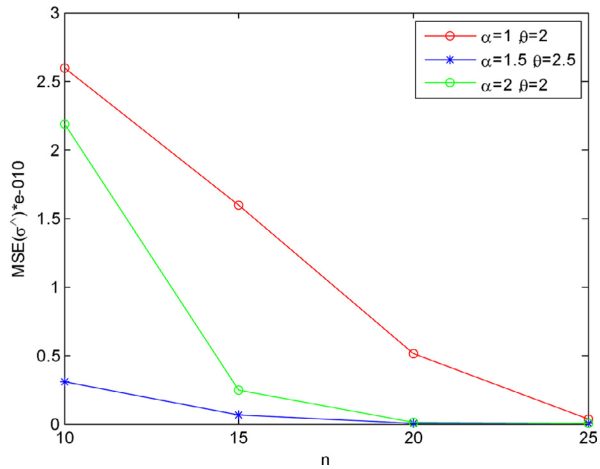

Fig. 4 The relation between the sample size n and the MSE's of $\hat{\sigma}$ based on the Exponentiated Pareto distribution.

$$+(-3.8680 \times 1.0621) + (3.7902 \times 1.3771)$$
$$= 0.52111978$$

The standard error of the estimates $\mu^*$and σ^* are

$$SE(\mu^*) = \sigma^*(Var(\mu^*))^{\frac{1}{2}} = 0.52111978 \times (1.5089)^{\frac{1}{2}} = 0.386088$$
$$SE(\sigma^*) = \sigma^*(Var(\sigma^*))^{\frac{1}{2}} = 0.52111978 \times (1.6909)^{\frac{1}{2}} = 0.677636$$

Using the same data,we can get by simulation the MLE of μ and σ as follows,

$$\widehat{\mu} = 0.0764$$
$$\widehat{\sigma} = 0.6035$$

5.2. *Illustrative example*

To illustrate the use of the estimation methods proposed in this paper, the following example is discussed.

Example [26]**.** presented data on the time to breakdown of an insulating fluid in an accelerated test conducted at various test voltages. In analyzing this data set, [26] considered the Exponentiated Pareto distribution (where correlation coefficient as high as 0.969). For the purposes of illustrating the methods discussed in this article, a progressively Type-II right censored sample of size m=8 was randomly selected from the n=19 observations recorded at 34 kv in Nelson's table, as given by Viveros and Balakrishnan [12]: The observations and censoring scheme are reported as follow:

i	1	2	3	4	5	6	7	8
$x_{i:m:n}$	.19	.78	.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

Using the formula described in Eqs. (2.2), (2.3), (3.1) and (3.2)we obtain the BlUEs and MlEs of μ and σ to be:

$$\overset{*}{\mu} = 0.45287, \quad \overset{*}{\sigma} = 0.7724,$$
$$\widehat{\mu} = 0.059 \quad \text{and} \quad \widehat{\sigma} = 1.1998.$$

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