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Original Article

Statistical measures approximations for the Gaussian part of the stochastic nonlinear damped Duffing oscillator solution process under the application of Wiener Hermite expansion linked by the multi-step differential transformed method

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Abstract In this paper, the stochastic Wiener Hermite expansion (WHE) is used to find the statistical measures (mean and variance) of the first order stochastic approximation (Gaussian part) of the stochastic solution processes related to the nonlinear damped Duffing oscillator model which is excited randomly by white noise process. Under the application of WHE, a deterministic model is generated to simulate the statistical measures. In next stages, smi-analytical treatments are performed under applying multi-step differential transformed method (Ms-DTM) and some cases study are illustrated related to the statistical properties using Mathematica10 software.

2010 Mathematics Subject Classification: 65C50; 60H10; 34K50; 58J65

1. Introduction

Stochastic non-linear differential equations are interested mathematical models simulate the probabilistic behavior related to the applied phenomena in different scientific branches. The main motivations related to the study of these models due to some random variations which effect on the solution behavior of models. In this case the solution of the models will be become a function in random parameters or random processes. The next step, some items must be simulated to find the statistical

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behavior of the unknown stochastic solution processes. The study of random solutions of partial differential equations was initiated by Kampe de Feriet in 1956 [1].

In the recent years, Wiener Hermite expansion (WHE) has an interest research area to analyze some stochastic system linked by perturbation and Homotoy perturbation methods to find analytical treatments [2–10]. In WHE approach, there is no randomness directly involved in the computations. One does not have to rely on pseudo random number generators, and there is no need to solve the SPDEs repeatedly for many realizations.

In recent years, the analysis of the nonlinear oscillator subjected to random excitation has been studied by many investigators. This subject has become important in the study of a wide variety of applied problems, for example, the vibrational studies of mechanical and electrical systems, earthquake disturbances, wind load in structural analysis, noise-corrupted signals in communication theory, and the motion of the sea or ground roughness in vehicle dynamics and economical systems.

Recently, piecewise semi-analytical methods, which do not require perturbation or linearization, are introduced for finding solutions of nonlinear problems. Multi-step Differential Transform Method (Ms-DTM) is one of the most effective, convenient and accurate methods for both weakly and strongly nonlinear problems. Ms-DTM does not require analytical integration or symbolic computations as other peer piecewise semi analytical-numerical method. The natural of Ms-DTM algorithm plays an important role to find a rapid approximation for the nonlinear initial value problem but in the boundary value problem, the applying of Ms-DTM requires transforming the problem into an initial value problem.

The items of this paper simulate the stochastic solution process approximate for a probabilistic model subject to deterministic initial conditions. This model is described by the nonlinear damped Duffing oscillator under the effect of an external stochastic excitation and its mathematical form is described as follow:

$$\ddot{x}(t) + \alpha\,\dot{x}(t) + \beta\,x(t) + \gamma\,x^3(t) = \lambda\,n(t;\omega), \quad t \ge 0$$
$$x(0) = a, \quad \dot{x}(0) = b \tag{1}$$

where $n(t;\omega)$ is the white noise process, whose intensity is given by parameter λ and $\alpha, \beta, \gamma, a, b$ are deterministic parameters. The next items of the paper is summered in the following points:

- Section 2 presents a brief for the basics of WHE.
- Section 3 simulates the mathematical analysis of (DTM) and (Ms-DTM).
- Section 4 presents the results the stochastic approximation analysis for the model due to WHE application.
- Section 5 presents the smi-analytical treatments due to DTM simulation for the deterministic system.
- Section 6 discusses Ms-DTM results and some cases studies.
- Section 7 is a general conclusion for the paper work done.

2. The stochastic Wiener–Hermite expansion (WHE)

The stochastic Wiener–Hermite expansion (WHE) plays an important role to find an approximation for any stochastic process $x(t)$. For further details we recommend [11]. This expansion consists of two different quantities, the first is a deterministic and the other is a probabilistic. The probabilistic type contains stochastic processes take the symbolic formula $H^{(i)}(t_1, t_2, \ldots, t_i)$ which is called stochastic Wiener – Hermite (WH) polynomials and subject to the recurrence relation

$$H^{(i)}(t_1, t_2, \ldots, t_i) = H^{(i-1)}(t_1, t_2, \ldots, t_{i-1})H^{(1)}(t_i) - \sum_{m=1}^{i-1} H^{(i-2)}(t_1, t_2, \ldots, t_{i-2})\delta(t_{i-m} - t_i), \quad i \ge 2, \tag{2}$$

where $H^{(0)} = 1$, $H^{(1)}(t) = n(t)$ is the stochastic white noise process and $\delta(.)$ is the Dirac-delta function (see Appendix A) and (WH) polynomials are elements of a complete set of statistically orthogonal random functions, i.e.

$$E[H^{(i)}(t_1, t_2, \ldots, t_i)H^{(j)}(t_1, t_2, \ldots, t_j)] = 0, \quad \forall\ i \ne j, \tag{3}$$

where $E[.]$ denotes the expectation operator.

As a consequence of the completeness of WHPs set, the general description for the WHE of any arbitrary stochastic process $x(t;\omega)$ can be presented in the form,

$$x(t;\omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t, t_1)H^{(1)}(t_1)dt_1 + \iint_{-\infty}^{\infty} x^{(2)}(t, t_1, t_2)H^{(2)}(t_1, t_2)dt_1dt_2 + \cdots \tag{4}$$

where $x^{(0)}(t)$, $x^{(i)}(t_1, \ldots, t_i)$, $i \ge 1$ are called the (unknown deterministic) kernels of the WHE. The first two terms of the right-hand side (1^{st} order term) define the Gaussian part of the stochastic process, while the second-order and higher terms correspond to the non-Gaussian part.

Under making some sequenced expectations linked by the statistical properties of WHps set (see Appendix B), the mean and variance for the Gaussian part of WHE can be expressed as follows:

$$E[x(t;\omega)] = x^{(0)}(t), \quad Var[x(t;\omega)] = \int_{-\infty}^{\infty} \left[x^{(1)}(t, t_1)\right]^2 dt_1 \tag{5}$$

3. The differential transformation method (DTM) and multi-step (DTM)

The differential transform method (DTM) is a numerical as well as analytical method for solving integral equations, ordinary, partial differential equations and differential equation systems. The method provides the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou [12] and its main application concerns with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in [13–23].

However, the DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series

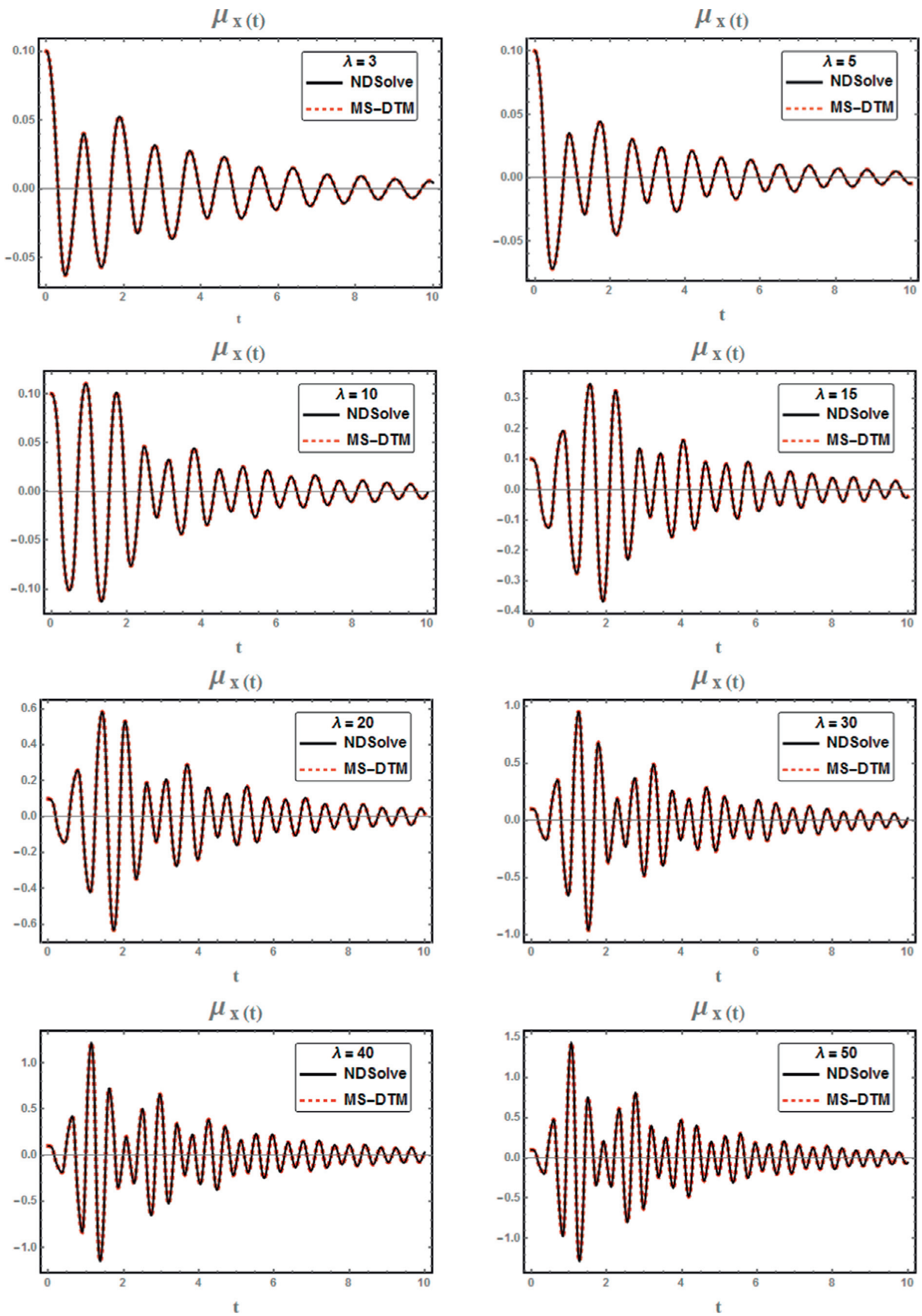

Fig. 1 The comparisons of the results of Ms-DTM and NDSolve package for $\mu_{x(t)}$ for different values of λ at $\alpha = 0.5$, $\beta = 25$, $\gamma = 25$, $a = 0.1$, $b = 0$.

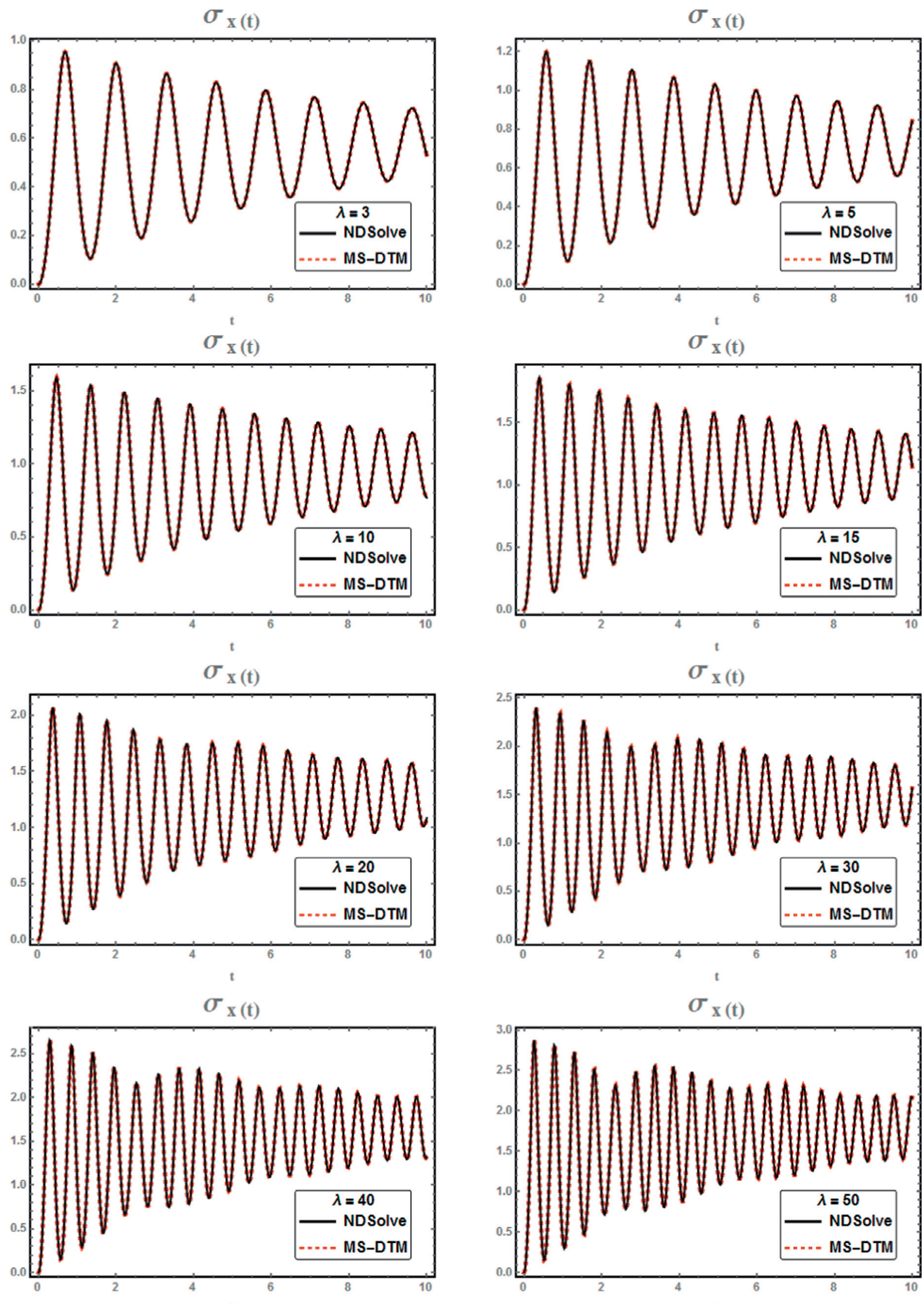

Fig. 2 The comparisons of the results of Ms-DTM and NDSolve package for $\sigma_{x(t)}$ for different values of λ at $\alpha = 0.5, \quad \beta = 25, \quad \gamma = 25, \quad a = 0.1, \quad b = 0$.

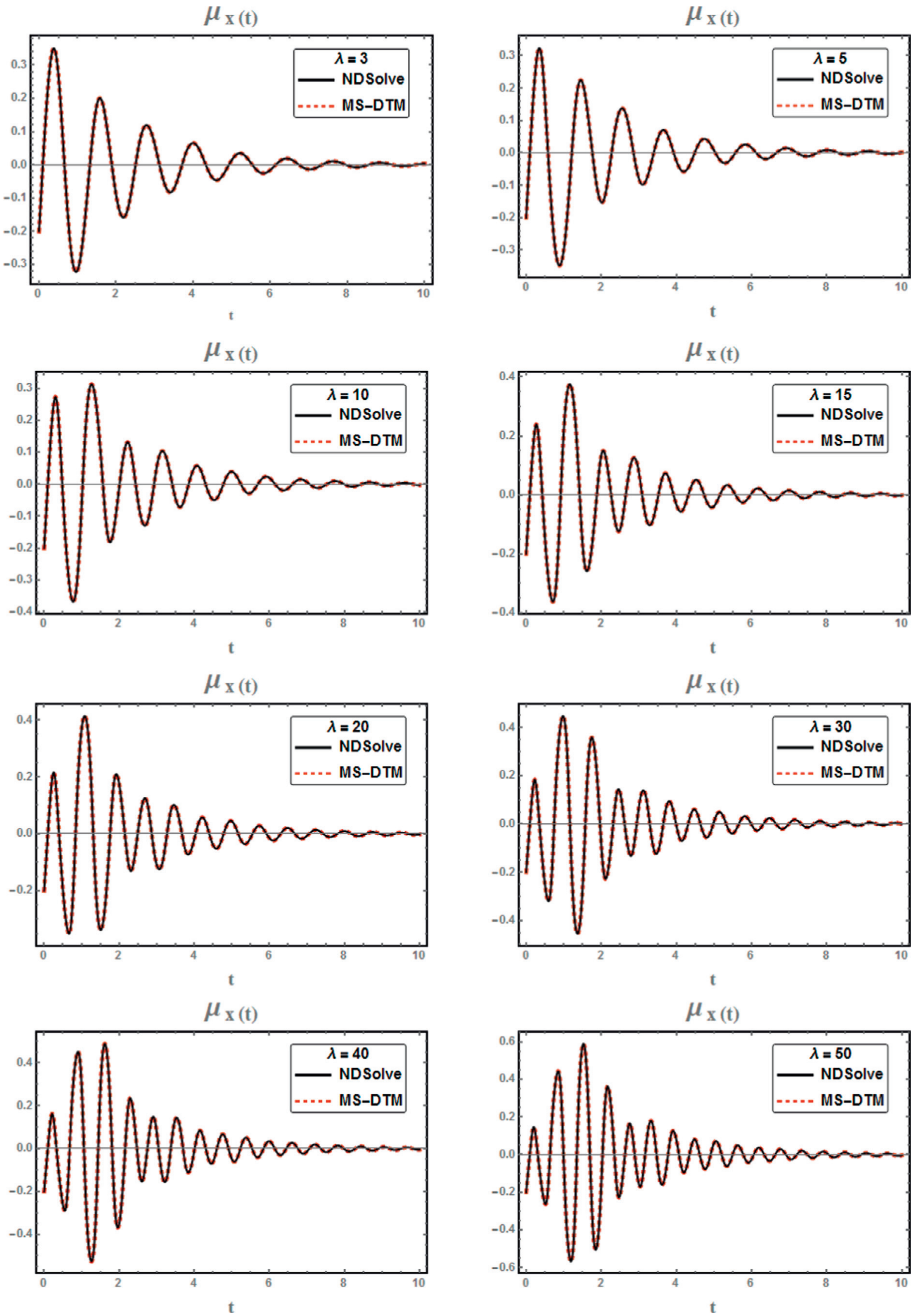

Fig. 3 The comparisons of the results of Ms-DTM and NDSolve package for $\mu_{x(t)}$ for different values of λ at $\alpha = 1,\ \ \beta = 20,\ \ \gamma = 2,\ \ a = -0.2,\ \ b = 2.$

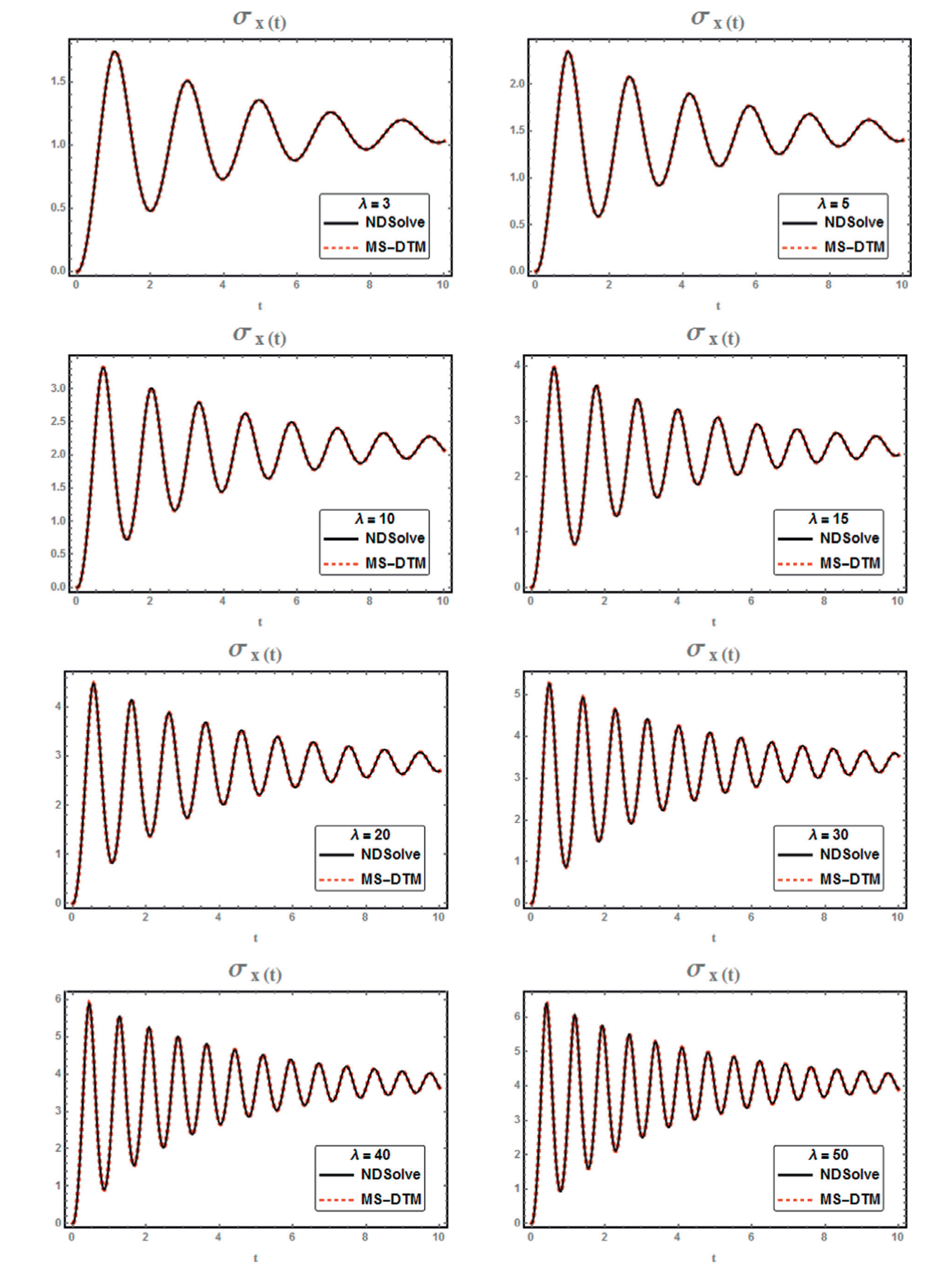

Fig. 4 The comparisons of the results of Ms-DTM and NDSolve package for $\sigma_{x(t)}$ for different values of λ at $\alpha = 1, \;\; \beta = 20, \;\; \gamma = 2, \;\; a = -0.2, \;\; b = 2.$

Table 1 The series coefficients for piecewise solution by Ms-DTM for $\mu_{x(t)}$ at $\lambda = 10$ $\alpha = 0.5$, $\beta = 25$, $\gamma = 25$, $a = 0.1$, $b = 0$.

Itervals	$X_0^{(0)}$	$X_1^{(0)}$	$X_2^{(0)}$	$X_3^{(0)}$	$X_4^{(0)}$	$X_5^{(0)}$	$X_6^{(0)}$	$X_7^{(0)}$	$X_8^{(0)}$	$X_9^{(0)}$	$X_{10}^{(0)}$
$0 \le t \le 0.01$	0.1	0	−1.263	0.2104	2.683	−0.5392	−627.66	194.07	5919.09	−2449.63	−20174.9
$0.01 \le t \le 0.02$	0.09987	−0.02518	−1.255	0.3047	1.725	−37.46	−597.74	656.53	5618.52	−4012.64	−7225.62
$0.02 \le t \le 0.03$	0.0995	−0.05017	−1.245	0.3246	−1.018	−71.64	−536.4	1091.5	5268.16	−3219.82	25821.1
$0.03 \le t \le 0.04$	0.09887	−0.07498	−1.237	0.2019	−5.362	−101.2	−445.44	1505.8	5158.79	1531.35	70135.1
$0.04 \le t \le 0.05$	0.098	−0.09968	−1.235	−0.1222	−11.04	−124.5	−325.29	1933.8	5679.68	10762.8	112979.
$0.05 \le t \le 0.06$	0.09688	−0.1245	−1.246	−0.6939	−17.68	−139.6	−172.87	2441.4	7203.7	23515.5	137261.
$0.06 \le t \le 0.07$	0.09551	−0.1497	−1.279	−1.543	−24.83	−144.4	20.464	3118.8	9931.78	36818.5	119310.
$0.07 \le t \le 0.08$	0.09388	−0.1758	−1.342	−2.679	−31.9	−136.1	269.91	4058.2	13677.2	44862.1	25489.8
$0.08 \le t \le 0.09$	0.09198	−0.2036	−1.443	−4.084	−38.15	−110.5	596.03	5311.8	17561.8	37903.6	−188863.
$0.09 \le t \le 0.1$	0.0898	−0.2339	−1.589	−5.707	−42.58	−62.58	1019.7	6820.8	19620.5	1500.26	−569979.
$0.1 \le t \le 0.11$	0.0873	−0.2675	−1.786	−7.45	−43.93	14.01	1550.8	8312.1	16410.2	−82226.1	−1.13176×10^(6)
$0.11 \le t \le 0.12$	0.08443	−0.3057	−2.036	−9.159	−40.6	125.3	2169.	9173.6	2939.51	−228043.	−1.782×10^(6)
$0.12 \le t \le 0.13$	0.08117	−0.3493	−2.333	−10.61	−30.76	274.5	2796.3	8357.1	−26415.1	−431595.	−2.21037×10^(6)
$0.13 \le t \le 0.14$	0.07743	−0.3992	−2.667	−11.51	−12.57	457.8	3266.4	4426.6	−75042.8	−642398.	−1.79845×10^(6)
$0.14 \le t \le 0.15$	0.07316	−0.4561	−3.015	−11.49	15.31	658.	3309.	−4061.2	−138598.	−735492.	278351.
$0.15 \le t \le 0.16$	0.06828	−0.5197	−3.343	−10.15	52.93	839.4	2576.9	−17656.	−197985.	−513630.	4.50105×10^(6)
$0.16 \le t \le 0.17$	0.06274	−0.5893	−3.607	−7.152	98.	945.3	755.25	−34642.	−215861.	207818.	9.93066×10^(6)
$0.17 \le t \le 0.18$	0.05648	−0.6632	−3.753	−2.286	145.	906.3	−2233.8	−49834.	−145865.	1.40452×10^(6)	1.32547×10^(7)
$0.18 \le t \le 0.19$	0.04947	−0.7383	−3.726	4.359	185.2	661.6	−5985.1	−54885.	38118.9	2.62422×10^(6)	9.61511×10^(6)
$0.19 \le t \le 0.2$	0.04173	−0.8108	−3.479	12.29	207.4	192.8	−9483.9	−41546.	301285.	3.00527×10^(6)	−3.42133×10^(6)
$0.2 \le t \le 0.21$	0.03328	−0.8758	−2.985	20.58	201.6	−442.9	−11311.	−7558.4	530082.	1.78586×10^(6)	−2.07918×10^(7)
$0.21 \le t \le 0.22$	0.02425	−0.9286	−2.253	27.97	162.6	−1106.	−10255.	38375.	577834.	−883826.	−3.01451×10^(7)
$0.22 \le t \le 0.23$	0.01477	−0.9647	−1.329	33.18	93.68	−1610.	−6087.4	77889.	368455.	−3.6218×10^(6)	−2.13703×10^(7)
$0.23 \le t \le 0.24$	0.005022	−0.981	−0.2941	35.23	6.967	−1796.	55.962	92385.	−21743.7	−4.65462×10^(6)	2.00527×10^(6)
$0.24 \le t \le 0.25$	−0.004782	−0.9764	0.7492	33.74	−79.61	−1606.	6085.7	74869.	−395287.	−3.28142×10^(6)	2.36325×10^(7)
$0.25 \le t \le 0.26$	−0.01444	−0.9517	1.699	29.1	−148.5	−1109.	9998.6	34603.	−568975.	−489296.	2.89745×10^(7)
$0.26 \le t \le 0.27$	−0.02376	−0.9096	2.473	22.26	−188.1	−468.4	10845.	−9421.8	−494814.	1.9463×10^(6)	1.77272×10^(7)
$0.27 \le t \le 0.28$	−0.03259	−0.8542	3.025	14.48	−195.9	137.6	8993.3	−40376.	−265086.	2.87874×10^(6)	1.12152×10^(6)
$0.28 \le t \le 0.29$	−0.04081	−0.7902	3.344	6.943	−177.1	581.2	5663.	−51493.	−20754.3	2.36602×10^(6)	−9.91919×10^(6)
$0.29 \le t \le 0.3$	−0.04838	−0.7219	3.453	0.535	−141.3	814.4	2176.2	−45967.	141166.	1.19609×10^(6)	−1.2147×10^(7)
$0.3 \le t \le 0.31$	−0.05525	−0.6532	3.392	−4.276	−98.86	857.5	−570.26	−31752.	198132.	130237.	−8.60123×10^(6)
$0.31 \le t \le 0.32$	−0.06145	−0.587	3.213	−7.394	−57.81	767.7	−2243.2	−16315.	178629.	−481089.	−3.6928×10^(6)
$0.32 \le t \le 0.33$	−0.067	−0.5251	2.964	−8.989	−23.24	608.2	−2931.5	−4074.9	124827.	−654334.	−91000.2
$0.33 \le t \le 0.34$	−0.07197	−0.4686	2.686	−9.369	2.706	429.9	−2921.4	3611.7	68736.3	−564051.	1.60395×10^(6)
$0.34 \le t \le 0.35$	−0.0764	−0.4177	2.411	−8.888	19.99	265.4	−2519.8	7290.9	26026.7	−380323.	1.90029×10^(6)
$0.35 \le t \le 0.36$	−0.08034	−0.372	2.158	−7.871	29.75	130.5	−1964.7	8224.	−81.3655	−206403.	1.51604×10^(6)
$0.36 \le t \le 0.37$	−0.08385	−0.3311	1.941	−6.587	33.61	29.7	−1403.6	7640.	−12660.6	−82406.9	963818.
$0.37 \le t \le 0.38$	−0.08697	−0.2941	1.764	−5.238	33.25	−39.27	−909.35	6430.8	−16496.9	−10684.9	494291.
$0.38 \le t \le 0.39$	−0.08974	−0.2603	1.626	−3.964	30.13	−81.25	−505.4	5121.	−15765.3	21702.9	179603.
$0.39 \le t \le 0.4$	−0.09219	−0.2288	1.524	−2.848	25.48	−101.7	−188.96	3953.1	−13310.7	29960.4	5891.93
$0.4 \le t \le 0.41$	−0.09433	−0.1991	1.453	−1.933	20.24	−105.4	52.976	2993.9	−10728.	26221.1	−67534.4
$0.41 \le t \le 0.42$	−0.09617	−0.1706	1.406	−1.226	15.15	−96.52	234.56	2221.2	−8708.3	18447.8	−80799.8
$0.42 \le t \le 0.43$	−0.09774	−0.1428	1.377	−0.7115	10.75	−78.25	367.04	1581.5	−7394.35	11021.4	−64933.3
$0.43 \le t \le 0.44$	−0.09903	−0.1154	1.362	−0.3519	7.437	−53.3	457.85	1022.6	−6659.36	5707.25	−41424.7
$0.44 \le t \le 0.45$	−0.1	−0.08822	1.355	−0.09824	5.49	−24.05	511.18	506.02	−6301.92	2529.97	−23899.8
$0.45 \le t \le 0.46$	−0.1008	−0.06112	1.355	0.1077	5.068	7.331	529.12	8.3706	−6169.69	480.717	−19588.4
$0.46 \le t \le 0.47$	−0.1013	−0.03396	1.362	0.3283	6.224	38.75	512.43	−486.01	−6225.11	−1889.4	−30156.
$0.47 \le t \le 0.48$	−0.1015	−0.006592	1.376	0.626	8.909	68.12	460.75	−995.02	−6560.64	−5934.55	−52074.4
$0.48 \le t \le 0.49$	−0.1014	0.02115	1.401	1.059	12.97	93.3	372.11	−1548.2	−7367.02	−12397.9	−76611.6
$0.49 \le t \le 0.5$	−0.101	0.04954	1.441	1.678	18.13	111.9	241.89	−2192.	−8854.25	−20868.7	−89271

solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. To overcome the shortcoming, the Ms-DTM was presented in [24, 25]. On the other hand, the Ms-DTM has also some drawbacks. By using the DTM, the interval $[0, T]$ is divided into M sub-intervals and the series solutions are obtained in $t \in [t_{i-1}, t_i]$, $i =, 1, \ldots, M$. In some problems, interval $[0, T]$ can be required a very small sub-division of intervals. In this case, both the solution time lengths and series solutions are obtained for a great number of sub-intervals.

Consider a general equation of n-th order ordinary differential equation

$$f\big(t, y, y', \ldots, y^{(n)}\big) = 0, \tag{6}$$

subject to the initial conditions

$$y^{(k)} = d_k,\ k = 0, 1, \ldots, n-1 \tag{7}$$

To illustrate the differential transformation method (DTM) for solving differential equations, the basic definitions differential transformations are introduced as follows. Let $y(t)$ be analytic in a domain D and let $t = t_0$ represents any point in D. The function $y(t)$ is then represented by one power series whose center is located at t_0. The differential transformation of the k-th derivative of a function $y(t)$ is defined as follows:

$$Y(k) = \frac{1}{k\,!}\left[\frac{d^k\, y(t)}{dt^k}\right]_{t=t_0}, \quad \forall\, t \in D. \tag{8}$$

In (8), $y(t)$ is the original function and $Y(k)$ is the transformed function. As in [23, 24] the differential inverse transformation of $Y(k)$ is defined as follows:

$$y(t) = \sum_{k=0}^{\infty} Y(k)(t-t_0)^k, \quad \forall\, t \in D. \tag{9}$$

In real applications, the function $y(t)$ is expressed by a finite series and (9) can be written as

$$y(t) = \sum_{k=0}^{N} Y(k)(t-t_0)^k, \quad \forall\, t \in D, \tag{10}$$

where $\sum_{k=N+1}^{\infty} Y(k)(t-t_0)^k$ is negligibly small.

Table 2 The comparisons of the results between Ms-DTM and NDSolve package for $\mu_{x(t)}$ and $\sigma_{x(t)}$ for points sample of t at $\lambda = 50$, $\alpha = 0.5$, $\beta = 25$, $\gamma = 25$, $a = 0.1$, $b = 0$.

t	$\mu_{x(t)}$ *Ms − DTM*	$\mu_{x(t)}$ *NDsolve*	$\sigma_{x(t)}$ *Ms − DTM*	$\sigma_{x(t)}$ *NDsolve*
0.	0.1	0.1	0.	0.
0.25	−0.142203	−0.142203	0.195718	0.195719
0.5	0.29943	0.29943	0.409234	0.409235
0.75	−0.641721	−0.641721	0.630488	0.630489
1.	1.12851	1.12851	0.964617	0.964616
1.25	−1.21146	−1.21146	1.39109	1.39108
1.5	0.749735	0.749734	1.54132	1.54131
1.75	−0.308498	−0.308498	1.44372	1.44371
2.	−0.0263493	−0.0263473	1.29191	1.2919
2.25	0.353061	0.353057	1.19463	1.19463
2.5	−0.668741	−0.668739	1.20183	1.20183
2.75	0.794344	0.794347	1.24709	1.24709
3.	−0.608734	−0.608741	1.21549	1.21549
3.25	0.259973	0.259981	1.17176	1.17176
3.5	0.0923136	0.0923057	1.23837	1.23838
3.75	−0.359891	−0.359888	1.4204	1.42042
4.	0.440994	0.441	1.62153	1.62155
4.25	−0.290168	−0.290182	1.79077	1.79079
4.5	0.0246758	0.0246901	1.97747	1.97748
4.75	0.168016	0.16801	2.14407	2.14407
5.	−0.165895	−0.165901	2.16343	2.16343
5.25	−0.00534259	−0.0053304	2.04233	2.04231
5.5	0.193403	0.193395	1.88864	1.88863
5.75	−0.246274	−0.246276	1.7333	1.73329
6.	0.130878	0.130888	1.59406	1.59405
6.25	0.0595993	0.0595891	1.53851	1.53852
6.5	−0.193045	−0.193042	1.58656	1.58657
6.75	0.183252	0.183259	1.69101	1.69103
7.	−0.0522873	−0.0522986	1.81923	1.81924
7.25	−0.085253	−0.0852462	1.94545	1.94546
7.5	0.118119	0.118122	2.00768	2.00768
7.75	−0.0323549	−0.0323637	1.97849	1.97848
8	−0.0819873	−0.081981	1.89606	1.89605

Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (6) and (7). In actual applications of the DTM, the approximate solution of the initial value problem (6) and (7) can be expressed by the finite series,

$$y(t) = \sum_{k=0}^{N} b_k\, t^k, \quad \forall\, t \in [0, T]. \tag{11}$$

Assume that the interval $[0, T]$ is divided into M subintervals $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, M$ of equal step size $h = \frac{T}{M}$ by using the nodes $t_i = i\,h$. The main idea of the multi-steps DTM is follows [24, 25]. First we apply the DTM to the basic problem over the interval $[0, t_1]$, we will obtain the following approximate solution,

$$y_1(t) = \sum_{k=0}^{N} b_{1k}\, t^k, \quad \forall\, t \in [0, t_1], \tag{12}$$

using the initial conditions $y_1^{(k)}(0) = d_k$. For $i \geq 2$, at each subintervals $[t_{i-1}, t_i]$ we will use the initial conditions $y_i^{(k)}(t_{i-1}) = y_{i-1}^{(k)}(t_{i-1})$ and apply the DTM to Eqs. (6) and (7) over the interval $[t_{i-1}, t_i]$ where t_0 in Eqs. (6) and (7) is replaced by t_{i-1}. The process is repeated and generates a sequence of approximated solutions $y_i(t)$, $i = 1, 2, \ldots, M$ for the solution $y(t)$,

$$y_i(t) = \sum_{k=0}^{N} b_{ik}\, t^k, \quad \forall\, t \in [t_{i-1,} t_i], \tag{13}$$

and the final form of $y(t)$ can be written as follow:

$$y(t) = \begin{cases} y_1(t)\,, & t \in [0, t_1] \\ y_2(t), & t \in [t_1 t_2] \\ \vdots & \\ y_M(t) & t \in [t_{M-1,} t_M] \end{cases}. \tag{14}$$

4. Application of the WHE to find the stochastic approximation for problem

In this section the WHE will be applied to analyze the stochastic response of the nonlinear model (1). The study of this response is limited to find the statistical behavior of the Gaussian part of the stochastic solution process $x(t;\omega)$ which can be putted in the following form

$$x(t;\omega) = x^{(0)}(t) + \int_0^{\infty} x^{(1)}(t, t_1) H^{(1)}(t_1) dt_1 \tag{15}$$

Table 3 The series coefficients for piecewise solution by Ms-DTM for $\mu_{x(t)}$ at $\lambda = 5,\ \ \alpha = 1,\ \ \beta = 20,\ \ \gamma = 2,\ \ a = -0.2,\ \ b = 2.$

Itervals	$X_0^{(0)}$	$X_1^{(0)}$	$X_2^{(0)}$	$X_3^{(0)}$	$X_4^{(0)}$	$X_5^{(0)}$	$X_6^{(0)}$	$X_7^{(0)}$	$X_8^{(0)}$	$X_9^{(0)}$	$X_{10}^{(0)}$
$0 \le t \le 0.01$	−0.2	2	1.008	−7.0827	0.47051	6.5155	21.698	−193.31	1.1246	801.87	−407.43
$0.01 \le t \le 0.02$	−0.1799	2.018	0.79587	−7.057	0.82207	7.4124	8.2354	−190.39	71.243	753.96	−549.
$0.02 \le t \le 0.03$	−0.1597	2.0318	0.58473	−7.0166	1.1984	7.5117	−4.8303	−182.05	136.44	692.74	−670.6
$0.03 \le t \le 0.04$	−0.1393	2.0414	0.37503	−6.9613	1.5605	6.8481	−17.135	−168.72	195.62	621.07	−755.41
$0.04 \le t \le 0.05$	−0.1188	2.0469	0.16719	−6.8924	1.8715	5.4774	−28.347	−150.93	248.04	543.28	−791.79
$0.05 \le t \le 0.06$	−0.09836	2.0481	−0.03841	−6.8127	2.0977	3.4742	−38.173	−129.22	293.38	464.51	−774.42
$0.06 \le t \le 0.07$	−0.07789	2.0453	−0.2415	−6.7261	2.2098	0.9294	−46.36	−104.17	331.79	390.14	−704.86
$0.07 \le t \le 0.08$	−0.05747	2.0385	−0.44196	−6.6378	2.1834	−2.0519	−52.692	−76.305	363.88	325.01	−591.5
$0.08 \le t \le 0.09$	−0.03713	2.0277	−0.63981	−6.5536	1.9993	−5.3529	−56.988	−46.091	390.68	272.81	−449.17
$0.09 \le t \le 0.1$	−0.01693	2.0129	−0.83528	−6.4801	1.6449	−8.8467	−59.098	−13.904	413.44	235.46	−298.41
$0.1 \le t \le 0.11$	0.003113	1.9943	−1.0288	−6.4243	1.1137	−12.398	−58.895	19.987	433.5	212.57	−164.46
$0.11 \le t \le 0.12$	0.02295	1.9718	−1.221	−6.3933	0.40642	−15.866	−56.264	55.416	452.04	201.05	−76.209
$0.12 \le t \le 0.13$	0.04254	1.9454	−1.4127	−6.3941	−0.469	−19.099	−51.103	92.295	469.85	194.77	−64.876
$0.13 \le t \le 0.14$	0.06184	1.9153	−1.605	−6.4329	−1.4971	−21.945	−43.31	130.57	486.98	184.44	−162.42
$0.14 \le t \le 0.15$	0.08083	1.8812	−1.7991	−6.5156	−2.6544	−24.242	−32.792	170.17	502.56	157.63	−399.45
$0.15 \le t \le 0.16$	0.09945	1.8433	−1.9964	−6.6466	−3.9094	−25.824	−19.46	210.89	514.41	99.013	−802.37
$0.16 \le t \le 0.17$	0.1177	1.8013	−2.1984	−6.8291	−5.222	−26.52	−3.2514	252.28	518.9	−8.9949	−1389.4
$0.17 \le t \le 0.18$	0.1355	1.7553	−2.4067	−7.0644	−6.5437	−26.156	15.857	293.57	510.74	−185.17	−2165.
$0.18 \le t \le 0.19$	0.1528	1.705	−2.6228	−7.3519	−7.8171	−24.56	37.817	333.48	482.97	−447.78	−3113.2
$0.19 \le t \le 0.2$	0.1696	1.6503	−2.8483	−7.6883	−8.9763	−21.564	62.468	370.1	427.08	−812.11	−4189.1
$0.2 \le t \le 0.21$	0.1858	1.591	−3.0845	−8.0675	−9.9476	−17.016	89.494	400.81	333.46	−1287.2	−5310.2
$0.21 \le t \le 0.22$	0.2014	1.5269	−3.3327	−8.4805	−10.65	−10.788	118.36	422.18	192.1	−1871.4	−6348.5
$0.22 \le t \le 0.23$	0.2163	1.4576	−3.5936	−8.9148	−10.997	−2.7911	148.28	430.02	−6.1945	−2548.2	−7125.7
$0.23 \le t \le 0.24$	0.2305	1.383	−3.8676	−9.3543	−10.899	7.0052	178.14	419.48	−268.26	−3280.4	−7415.8
$0.24 \le t \le 0.25$	0.2439	1.3028	−4.1547	−9.7796	−10.267	18.555	206.46	385.33	−596.51	−4006.4	−6957.7
$0.25 \le t \le 0.26$	0.2565	1.2168	−4.454	−10.167	−9.0166	31.713	231.41	322.39	−986.61	−4638.	−5483.
$0.26 \le t \le 0.27$	0.2682	1.1246	−4.7641	−10.492	−7.0733	46.214	250.82	226.17	−1425.1	−5061.3	−2760.6
$0.27 \le t \le 0.28$	0.279	1.0261	−5.0826	−10.723	−4.3795	61.652	262.23	93.721	−1887.4	−5143.9	1341.7
$0.28 \le t \le 0.29$	0.2887	0.92126	−5.4063	−10.832	−0.90171	77.47	263.08	−75.479	−2336.7	−4748.1	6783.5
$0.29 \le t \le 0.3$	0.2974	0.80988	−5.731	−10.785	3.3621	92.96	250.87	−278.5	−2724.	−3751.8	13279.
$0.3 \le t \le 0.31$	0.3049	0.69204	−6.0515	−10.553	8.3747	107.27	223.46	−508.13	−2991.5	−2076.2	20246.
$0.31 \le t \le 0.32$	0.3112	0.56789	−6.362	−10.106	14.053	119.44	179.39	−752.29	−3077.1	283.71	26792.
$0.32 \le t \le 0.33$	0.3163	0.43768	−6.6555	−9.4212	20.266	128.45	118.19	−994.06	−2922.7	3230.5	31780.
$0.33 \le t \le 0.34$	0.32	0.30183	−6.9247	−8.4801	26.829	133.3	40.767	−1212.3	−2484.4	6546.1	33964.
$0.34 \le t \le 0.35$	0.3223	0.16091	−7.1616	−7.2732	33.511	133.07	−50.432	−1383.5	−1743.4	9891.8	32217.
$0.35 \le t \le 0.36$	0.3232	0.015631	−7.3584	−5.8012	40.04	127.05	−151.26	−1483.6	−715.99	12833.	25808.
$0.36 \le t \le 0.37$	0.3226	−0.13311	−7.5072	−4.0761	46.113	114.84	−255.99	−1491.9	540.14	14895.	14676.
$0.37 \le t \le 0.38$	0.3205	−0.28429	−7.6007	−2.1224	51.419	96.396	−357.63	−1393.7	1925.4	15637.	−382.79
$0.38 \le t \le 0.39$	0.3169	−0.43673	−7.6326	0.023142	55.655	72.139	−448.49	−1183.9	3305.5	14743.	−17675.
$0.39 \le t \le 0.4$	0.3118	−0.58915	−7.5979	2.3121	58.551	42.946	−520.92	−869.08	4526.6	12107.	−34794.
$0.4 \le t \le 0.41$	0.3051	−0.74018	−7.493	4.6864	59.89	10.133	−568.14	−468.	5436.8	7881.3	−49008.
$0.41 \le t \le 0.42$	0.297	−0.88839	−7.3165	7.0806	59.531	−24.625	−585.13	−10.891	5909.9	2487.2	−57793.
$0.42 \le t \le 0.43$	0.2874	−1.0324	−7.0687	9.4256	57.426	−59.423	−569.26	463.81	5868.4	−3436.5	−59394.
$0.43 \le t \le 0.44$	0.2763	−1.1707	−6.7521	11.652	53.622	−92.282	−520.77	913.9	5298.2	−9132.	−53251.
$0.44 \le t \le 0.45$	0.264	−1.302	−6.3714	13.695	48.262	−121.32	−442.84	1298.8	4254.	−13854.	−40176.
$0.45 \le t \le 0.46$	0.2503	−1.4251	−5.9329	15.495	41.58	−144.95	−341.25	1585.	2851.9	−17003.	−22213.
$0.46 \le t \le 0.47$	0.2355	−1.539	−5.4446	17.007	33.878	−161.96	−223.78	1749.8	1251.5	−18229.	−2211.2
$0.47 \le t \le 0.48$	0.2196	−1.6427	−4.9157	18.197	25.506	−171.66	−99.317	1784.6	−369.71	−17480.	16768.
$0.48 \le t \le 0.49$	0.2027	−1.7354	−4.3562	19.044	16.836	−173.91	23.147	1694.7	−1842.7	−14999.	32067.
$0.49 \le t \le 0.5$	0.1849	−1.8168	−3.7765	19.544	8.2328	−169.09	135.43	1497.4	−3031.3	−11250.	41901.

The evaluation of unknown deterministic kernels $x^{(0)}(t)$ and $x^{(1)}(t, t_1)$ play an important role to reach the required statistical behavior. Substituting from (15) into (1) and taking the expectation for both of two sides before and after multiplying them by one dimension element of (WHPs) set. Under the statistical behavior of WHPs set which is presented in Appendix B Linked by Appendix A, a deterministic system is reduced in following form:

$$L\big[x^{(0)}(t)\big] + \gamma\left(\big[x^{(0)}(t)\big]^3 + 3x^{(0)}(t)\int_0^\infty \big[x^{(1)}(t;t_1)\big]^2 dt_1\right) = 0,$$

$$L\big[x^{(1)}(t;t_1)\big] + 3\,\gamma\, x^{(1)}(t;t_1) \times\left(\big[x^{(0)}(t)\big]^2 + \int_0^\infty \big[x^{(1)}(t;t_1)\big]^2 dt_1\right) = \lambda\,\delta(t-t_1),$$

$$x^{(0)}(0) = a,\quad \frac{dx^{(0)}}{dt}\bigg|_{t=0} = b,\quad x^{(1)}(0) = 0,\quad \frac{dx^{(1)}}{dt}\bigg|_{t=0} = 0, \tag{16}$$

where $L = [\frac{d^2}{dt^2} + \alpha\frac{d}{dt} + \beta]$.

5. Application of DTM to approximate the deterministic system

In order to perform a statistical analysis of the Gaussian part of the stochastic process solution of problem (1), we need to solve the nonlinear coupled deterministic problems (16). This section simulates the solution under the application of DTM and from the results of Section 2 linked by Appendix C, the system (16) is transformed into the following recurrence relations,

$$(k+2)(k+1)\,X^{(0)}(k+2) + \alpha(k+1)\,X^{(0)}(k+1) + \beta\,X^{(0)}(k) + \gamma\left(\sum_{s=0}^{k}\sum_{m=0}^{k-s} X^{(0)}(s)\Big[X^{(0)}(m)\,X^{(0)}(k-s-m) + 3\int_0^\infty X^{(1)}(m,t_1)\,X^{(1)}(k-s-m,t_1)dt_1\Big]\right) = 0 \tag{17}$$

$$\int_0^\infty \big[(k+2)(k+1)X^{(1)}(k+2,t_1) + \alpha(k+1)X^{(1)}(k+1,t_1) + \beta\,X^{(1)}(k,t_1)\big]\,dt_1$$

Table 4 The comparisons of the results between Ms-DTM and NDSolve package for $\mu_{x(t)}$ and $\sigma_{x(t)}$ for points sample of t at $\lambda = 30,\ \alpha = 1,\ \beta = 20,\ \gamma = 2,\ a = -0.2,\ b = 2.$

t	$\mu_{x(t)}$ $Ms-DTM$	$\mu_{x(t)}$ $NDsolve$	$\sigma_{x(t)}$ $Ms-DTM$	$\sigma_{x(t)}$ $NDsolve$
0.	−0.2	−0.2	0.	0.
0.25	0.175821	0.175821	5.26089	5.26089
0.5	−0.249279	−0.249279	1.00487	1.00487
0.75	0.0200693	0.0200696	4.35541	4.35541
1.	0.443454	0.443453	2.25039	2.25039
1.25	−0.313804	−0.313804	3.09217	3.09217
1.5	−0.266822	−0.26682	3.63008	3.63009
1.75	0.358419	0.358418	2.37656	2.37657
2.	−0.0923918	−0.0923929	4.24106	4.24105
2.25	−0.1007	−0.100698	2.5507	2.55071
2.5	0.12777	0.127768	3.69016	3.69015
2.75	−0.124682	−0.124682	3.31887	3.31887
3.	0.0717273	0.071728	3.00559	3.00559
3.25	0.0554013	0.0554001	3.8264	3.8264
3.5	−0.118399	−0.118398	2.92891	2.92892
3.75	0.0802807	0.0802808	3.63021	3.6302
4.	−0.00609856	−0.00609949	3.31047	3.31047
4.25	−0.0369753	−0.0369744	3.23118	3.23118
4.5	0.0588117	0.0588112	3.61233	3.61233
4.75	−0.0488175	−0.0488175	3.1404	3.1404
5.	0.010175	0.0101755	3.53589	3.53589
5.25	0.0292435	0.0292429	3.33271	3.33271
5.5	−0.0392267	−0.0392263	3.31906	3.31906
5.75	0.0254048	0.0254049	3.50127	3.50127
6.	−0.0034975	−0.00349782	3.2538	3.25381
6.25	−0.015667	−0.0156666	3.47201	3.47201
6.5	0.0236955	0.0236953	3.35056	3.35056
6.75	−0.0166674	−0.0166674	3.35601	3.35601
7.	0.00131998	0.00132019	3.44368	3.44368
7.25	0.0106846	0.0106843	3.31438	3.31438
7.5	−0.0140269	−0.0140268	3.43356	3.43356
7.75	0.00942649	0.00942649	3.36264	3.36264
8	−0.000949285	−0.000949394	3.37208	3.37208

$$+3\gamma\left(\sum_{s=0}^{k}\sum_{m=0}^{k-s}\int_0^\infty X^{(1)}(s,t_1)dt_1\big[X^{(0)}(m)X^{(0)}(k-s-m) + \int_0^\infty X^{(1)}(m,t_1)\,X^{(1)}(k-s-m,t_1)dt_1\big]\right) = \lambda\,\delta_{k,0} \quad (18)$$

where $X^{(0)}(0) = a, X^{(0)}(1) = b, X^{(1)}(0,t_1) = 0, X^{(1)}(1,t_1) = 0$ and the final form for the solution by a finite number of terms N can be written as follow

$$x^{(0)}(t) = \sum_{k=0}^{N} X^{(0)}(k)t^k, \qquad x^{(1)}(t,t_1) = \sum_{k=0}^{N} X^{(1)}(k,t_1)t^k \quad (19)$$

6. Results of Ms-DTM application and discussion

The application of DTM reduces a sequence of algebraic equations generated after expanding the recurrence relations (17) and (18) using a simulated programing by Mathematica 10. The solution of theses algebraic equations determines the coefficients in (19) and the outputs are functions in the initial parameters related the problem. By repeating this process over sequenced steps by a certain range to reach the real behavior of the problem. For every step, a new initial value problem is considered and its conditions are estimated from the obtained solution at final range of the previous step. The mathematical computations related to this method is performed by a symbolic program was designed by Mathematica 10. By another one, a parallel program by the same version uses the NDSolve package to satisfy the result of the previous program.

Our problem is a generalized model with respect to the deterministic model in the reference [26]. This deterministic model was simulated in the absence of stochastic excited term ($\lambda = 0$) and its solution is approximated by DTM linked by Pade approximation methods. The application of Ms-DTM includes the case studies related to the spatial case [26] in the generalized problem (1). In these case studies, $\mu_{x(t)} = E[x(t;\omega)]$ and $\sigma_{x(t)} = \sqrt{Var[x(t;\omega)]}$ are simulated over two lines applications (Ms-DTM and NDSolve package) and the numerical results are displayed in Figs. 1–4 and Tables 1 and 3 simulate results of the analysis of Ms-DTM and the values of columns 2 and 3 already indicate to an initial solution for every interval and a final solution for every a previous interval. It is clear that, the comparison between Ms-DTM and NDSolve applications gives excellent agreements.

7. Conclusion

In this study, the combining between WHE and Ms-DTM was applied to determine the stochastic response related to the model which is described in (1). Due to applying WHE, a deterministic model was generated to describe the Gaussian part of the problem stochastic response. The next analysis included applying DTM to find approximations over multi steps points by the recurrence relations which were generated under the properties of differential transform. The results of Ms-DTM were obtained under Mathematica software 10 and were compared with NDSolve Mathematica 10 package which indicates the excellent accuracy of the solution (Tables 2 and 4). Some case studies related to a previous work were considered to simulate some statistical measures for the problem (mean and variance).

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Appendix A

The Dirac delta function [27] (δ-function) was introduced by Paul Dirac at the end of the 1920s in an effort to create the mathematical tools for the development of quantum field theory. It has since been used with great success in applied mathematics and mathematical physics. Some mathematical analyses in this paper need to apply some properties related to the Dirac delta function and they are simulated in the following items

- $\int_{-\infty}^{\infty} \delta(x)\ \ dx = 1$
- $\int_{-\infty}^{\infty} \delta(x - x_0)\, f(x)\, dx = f(x_0)$
- $\int_{a}^{b} \delta(x - x_0)\, f(x)\, dx = f(x_0)\quad \forall\, a \le x_0\ \le b$

Appendix B

The statistical properties of Wiener Hermite polynomials (WHPs) [11] which were used in this paper are simulated in the following items

- $E[\ H^{(1)}(t_1)H^{(1)}(t_2)] = \delta(t_2 - t_1)$
- $E[\ H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)] = 0$
- $E[\ H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)H^{(1)}(t_4)] = \delta(t_2 - t_1)\delta(t_3 - t_4) + \delta(t_1 - t_3)\delta(t_2 - t_4) + \delta(t_1 - t_4)\delta(t_2 - t_3)$

Appendix C

In this paper, the used properties [24, 25] related to the differential transformation $Y(k) = \frac{1}{k\,!}[\frac{d^k\ y(t)}{dt^k}]_{t=t_0}$ for a function $y(t)$ are stated in the following items

- $y(t) = mu(t) \pm n\, v(t)\ \Rightarrow\ Y(k) = m\, U(k) \pm n\, V(k)$
- $y(t) = u(t)\ v(t)\ \Rightarrow\ Y(k) = \sum_{l=0}^{k} U(l)\ V(k-l)$
- $y(t) = u(t)\ \ v(t)w(t)\ \ \Rightarrow\ \ Y(k) = \sum_{l=0}^{k}\sum_{s=0}^{k-l} U(l)V(s)W(k - l - s)$
- $y(t) = \frac{d^m u(t)}{dt^m}\ \Rightarrow\ Y(k) = \frac{(k+m)!}{k!}\ U(k+m)$
- $y(t) = t^m\ \Rightarrow\ Y(k) = \delta_{k,m}$ where $\delta_{k,m}$ is Kronecker's delta

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