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Original Article

Deformation of a long thermoelastic rod of rectangular normal cross-section under mixed boundary conditions by boundary integrals

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Singular behavior

Abstract Using a well-known solution for steady temperature distribution in a rectangle, a boundary integral method is used to obtain an approximate solution for a plane problem of uncoupled thermoelasticity with mixed mechanical boundary conditions. The unknown functions in the cross-section are obtained in the form of series in Cartesian harmonics, enriched with harmonic functions that have a singular behavior at the transition points. The results are discussed and the functions of practical interest are represented on the boundary and also inside the domain. The locations where possible debonding may take place are noted.

2010 Mathematics Subject Classifications: 74B05; 74G70; 65L60

1. Introduction

Thermoelasticity has many applications in Technology and elsewhere. The subject has been covered in many monographs (cf. [1]). The methods used to treat problems of thermoelasticity have a wide spectrum. Shanker and Dhaliwal [2] investigate asymmetric thermoelasticity using integral representations. Singh and Dhaliwal [3] consider mixed boundary-value problems of thermoelastostatics and electrostatics. Abou-Dina and Ghaleb [4] present a boundary integral method for the solution of plane strain problems of uncoupled thermoelasticity in stresses using real functions for long cylinders of a homogeneous isotropic medium with simply connected normal cross-section. They consider different thermal and mechanical boundary conditions, but the case of mixed conditions was not dealt with. Computational aspects of this method are considered in [5], with reference to the ellipse [6]. An approach by complex

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analysis may be found in [7], where the general solutions of two-dimensional problems under uniform heat flux and under point source are considered. Şeremet, and Şeremet and Bonnet in a series of papers [8–15] present integral representations for thermoelastic Green's functions for Poisson's equation with numerous examples. Meleshko [16–18] addresses the problem of determining thermal stresses in a rectangle.

Thermoelastostatics relies on results from harmonic functions. Different aspects of this theory, as well as applications, may be found in [19,20]. A huge number of problems of thermostatics and elastostatics have been investigated. The method of fundamental solutions was used in numerous publications [21]. Abou-Dina and Ghaleb [22] investigate the approximate solutions to some regular and singular boundary-value problems for Laplace's operator in rectangular regions by a boundary Fourier expansion. Read [23] uses analytic series to find solutions to Laplacian problems with mixed boundary conditions. Problems with mixed boundary conditions are also treated in [24,25]. El-Dhaba et al. [26] investigate the deformation of a rectangle by finite Fourier transform.

Boundary integral formulations are popular because they rely on the well-developed theory of Fredholm integral equations, and also for less computational effort. The use of integral equation methods in potential theory and in elastostatics is presented in [19]. Altiero and Gavazza [27] propose a unified boundary integral method for linear elastostatics. Heise [28,29] applies boundary integral equations to treat problems of elastostatics with discontinuous boundary conditions. Koizumia et al. [30] present a boundary integral equation analysis for thermoelastostatics using thermoelastic potential.

Mixed boundary conditions are treated in [3]. Helsing [31] proposes an integral equation method to solve Laplace's equation under mixed Dirichlet and Neumann conditions on contiguous parts of the boundary, and the problem of elastostatics under mixed conditions. Boundary-value problems of mixed type with applications are considered by Khuri [32]. Gjam et al. [33] consider the ellipse with mixed conditions and use a harmonic function with logarithmic behavior at the boundary.

Corner boundary points introduce singular behavior of the solution. An extensive treatment of singularities exists in the literature for the Laplacian and for the elastic problems. Williams [34] discusses stress singularities in plates. An algorithm for plane potential solving problems with mixed boundary conditions involving extraction of singularities is treated in [35]. Gusenkova and Pleshchinskii [36] construct complex potentials with logarithmic singularities for elastic bodies with defect along a smooth arc. Abou-Dina and Ghaleb [22] introduce logarithmic singularities on the boundary of rectangular domains for approximate solutions to Laplacian boundary-value problems with mixed boundary conditions. Kotousov and Lew [37] study stress singularities under various boundary conditions at corners of plates. El-Seadawy et al. [38] solve 2D problems with mixed geometry including parts of ellipse and circle. The corners are smoothed locally by polynomial functions. Helsing and Ojala [39] treat corner singularities for elliptic problems by boundary integral equation methods on domains having a large number of corners and branching points. Helsing [40] presents a fast and stable algorithm for treating singular integral equations on piecewise smooth curves. Mixed-type boundary conditions at corners are treated in [22,41,42]. Gillman et al. [43] present techniques for discretizing the boundary integral equations in 2D domains with corners.

In the present paper, a problem of uncoupled thermoelasticity is solved in a rectangular domain. The heat problem has a known solution in closed form. The mechanical boundary conditions are of mixed type: a variable pressure on half of the boundary, the other half is fixed. A semi-analytical scheme presented in [33] for the purely elastic problem is applied here: the problem is replaced with two subproblems of uncoupled thermoelasticity having common solution. One subproblem has given stresses on the boundary, while the other subproblem has given displacements on the boundary. Each of these two subproblems has the prescribed entries on part of the boundary, while the other part carries unknown values, to be determined as part of the solution. These two subproblems yield a system of boundary integral equations following the framework proposed in [5]. A simple discretization procedure finally reduces the system of integral equations to a rectangular system of linear algebraic equations which is solved by Least Squares. The obtained results clearly show a singular behavior of the stress components at the two separation boundary points. For the solution inside the domain, proper expansions of the two basic harmonic functions are proposed in terms of Cartesian harmonics. To take account of the singularities, the stress function is enriched with a harmonic function having second order singularities at the two separation points. After truncation of the expansions, the coefficients are determined by the Boundary Collocation Method using the previously obtained values. Boundary plots and 3D plots in the domain of the normal cross-section are provided for the functions of practical interest. The results and the efficiency of the used scheme are discussed. All figures were produced using Mathematica 9.0 Software.

The problem under consideration models a long elastic pad support and thereby is of practical importance. The presence of corner points and mixed boundary conditions is challenging from the computational point of view and clearly indicates the efficiency of the proposed method.

2. Problem description

The uncoupled, plane theory of thermoelasticity for cylinders made from an isotropic, homogeneous, elastic material is treated by a boundary integral method. The normal cross-section D of the cylinder is simply connected and bounded by a sufficiently smooth contour C. The governing equations, boundary conditions and other closure relations are formulated in an orthogonal system of Cartesian coordinates (x, y) with origin O inside the domain. The lateral surface of the cylinder is acted upon by forces in the plane of the cross-section. No body forces or heat sources are considered. The parametric equations of contour C in terms of the usual polar angle θ is:

$$x = x(\theta), \quad y = y(\theta). \tag{1}$$

The vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$ denote the unit vector tangent to C at any arbitrary point on the contour, and the unit outwards normal at this point respectively. These two vectors form a basis that is similar to the basis of the orthogonal Cartesian system of coordinates, and may be easily calculated from the derivatives $\dot{x}$ and $\dot{y}$ with respect to angle θ.

3. Basic equations

The governing equations are listed below without proof, in accordance with [4,5]. The exact solution for temperature is given in closed form elsewhere (cf. [44]).

3.1. Equations of equilibrium

In the approach by stresses, the identically non-vanishing components of stress in the cross-section plane are derived from a stress function U by:

$$\sigma_{xx}=\frac{\partial^2 U}{\partial y^2},\quad \sigma_{xy}=-\frac{\partial^2 U}{\partial x\,\partial y},\quad \sigma_{yy}=\frac{\partial^2 U}{\partial x^2}. \tag{2}$$

and this function satisfies the biharmonic equation in virtue of the compatibility condition:

$$\nabla^4 U=0. \tag{3}$$

The generalized Hooke's law reads:

$$\begin{aligned}\sigma_{xx}&=\frac{\nu E}{(1+\nu)\,(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{E}{(1+\nu)}\frac{\partial u}{\partial x}-\frac{\alpha E}{(1-2\nu)}T\\ \sigma_{xy}&=\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\ \sigma_{yy}&=\frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{E}{(1+\nu)}\frac{\partial v}{\partial y}-\frac{\alpha E}{(1-2\nu)}T\end{aligned} \tag{4}$$

where E, ν and α denote Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion, respectively, and u, v denote the displacement components.

The stress function U solving Eq. (3) is represented through two harmonic functions as:

$$U=x\,\phi+y\,\phi^c+\psi \tag{5}$$

where the superscript 'c' denotes the harmonic conjugate.

The stress components are expressed in terms of ϕ and ψ as:

$$\begin{aligned}\sigma_{xx}&=x\,\frac{\partial^2\phi}{\partial y^2}+2\,\frac{\partial\phi^c}{\partial y}+y\,\frac{\partial^2\phi^c}{\partial y^2}+\frac{\partial^2\psi}{\partial y^2}\\ \sigma_{xy}&=-x\,\frac{\partial^2\phi}{\partial x\,\partial y}-y\,\frac{\partial^2\phi^c}{\partial x\,\partial y}-\frac{\partial^2\psi}{\partial x\,\partial y}\\ \sigma_{yy}&=x\,\frac{\partial^2\phi}{\partial x^2}+2\,\frac{\partial\phi}{\partial x}+y\,\frac{\partial^2\phi^c}{\partial x^2}+\frac{\partial^2\psi}{\partial x^2}\end{aligned} \tag{6}$$

The Cartesian displacement components u and v are given as:

$$\begin{aligned}\frac{E}{(1+\nu)}u&=-\frac{\partial U}{\partial x}+4(1-\nu)\phi+\frac{E}{1+\nu}u_T,\\ \frac{E}{(1+\nu)}v&=-\frac{\partial U}{\partial y}+4(1-\nu)\phi^c+\frac{E}{1+\nu}v_T\end{aligned}$$

where

$$\begin{aligned}u_T&=\alpha\,(1+\nu)\int_{M_0}^{M}(T\,dx-T^c\,dy),\\ v_T&=\alpha\,(1+\nu)\int_{M_0}^{M}(T^c\,dx+T\,dy)\end{aligned} \tag{7}$$

are the temperature displacements. The integrals in (7) are noted in ([1, p. 323]). Point $M\in D$ is the general point where the displacements are calculated, while the initial point M_0 is adequately chosen in the cross-sectional domain or on the boundary C. Relations (7) yield:

$$\begin{aligned}2\mu\,u&=(3-4\,\nu)\,\phi-x\frac{\partial\phi}{\partial x}-y\,\frac{\partial\phi^c}{\partial x}-\frac{\partial\psi}{\partial x}+2\mu\,u_T,\\ 2\mu\,v&=(3-4\,\nu)\phi^c-x\,\frac{\partial\phi}{\partial y}-y\,\frac{\partial\phi^c}{\partial y}-\frac{\partial\psi}{\partial y}+2\mu\,v_T\end{aligned} \tag{8}$$

where $\mu=\frac{E}{2(1+\nu)}$ is the modulus of rigidity of the elastic material.

Thus, in the absence of heat sources, the only contribution of temperature to the elastic solution is confined to the additional displacements u_T and v_T in the expressions for the displacement.

4. Accompanying conditions

For a unique solution to the considered problem, the basic field equations and boundary conditions are complemented by conditions for removal of rigid body motion, and by other conditions which have no physical insight. Details may be found in [45].

4.1. Boundary conditions

The considered problem involves mixed mechanical boundary conditions.

- *The first fundamental problem of elasticity*
  Assuming that the density of the given distribution of the total external surface forces is:

  $$\boldsymbol{f}=f_x\boldsymbol{i}+f_y\boldsymbol{j}=\sigma_{nx}\boldsymbol{i}+\sigma_{ny}\boldsymbol{j},$$

  the boundary conditions take the form:

  $$\begin{aligned}f_x&=(x\phi_{yy}+2\phi^c_y+y\phi^c_{yy}+\psi_{yy})\,\frac{\dot y}{\omega}+(x\phi_{xy}+y\phi^c_{xy}+\psi_{xy})\,\frac{\dot x}{\omega},\\ f_y&=-(x\phi_{xy}+y\phi^c_{xy}+\psi_{xy})\,\frac{\dot y}{\omega}-(x\phi_{xx}+2\phi^c_x+y\phi^c_{xx}+\psi_{xx})\,\frac{\dot x}{\omega}.\end{aligned} \tag{9}$$

- *The second fundamental problem of elasticity*
  Assuming that the displacement vector is

  $$\boldsymbol{d}=d_x\boldsymbol{i}+d_y\,\boldsymbol{j}=d_n\boldsymbol{n}+d_\tau\boldsymbol{\tau},$$

  the boundary conditions take the form:

  $$\begin{aligned}2\mu\,d_x&=(3-4\,\nu)\,\phi-x\,\phi_x-y\,\phi^c_x-\psi_x+2\mu\,u_T,\\ 2\mu\,d_y&=(3-4\,\nu)\,\phi^c-x\,\phi_y-y\,\phi^c_y-\psi_y+2\mu\,v_T.\end{aligned}$$

4.2. Elimination of rigid body motion

The present boundary conditions prohibit any rigid body motion in the elastic solution. In setting any of the above accompanying conditions, one needs the first two derivatives of any harmonic function f with respect to x and y on the boundary. These may be calculated as explained in [45].

4.3. Additional simplifying conditions

The following supplementary purely mathematical conditions are adopted for simplicity at the point of the boundary where $\theta = 0$:

$$\begin{aligned} x(0)\,\phi(0) + y(0)\,\phi^c(0) + \psi(0) &= 0 \\ x(0)\,\phi^c(0) - y(0)\,\phi(0) + \psi^c(0) &= 0 \\ x(0)\,\phi_x(0) + \phi(0) + y(0)\phi_x^c(0) + \psi_x(0) &= 0 \\ x(0)\,\phi_y(0) + \phi^c(0) + y(0)\,\phi_y^c(0) + \psi_y(0) &= 0 \\ T^c(0,0) &= 0 \end{aligned} \quad (10)$$

The above mentioned equations and conditions can be transformed into boundary integral equations using the boundary integral representation of the harmonic functions ϕ and ψ (and their conjugates) and the Cauchy–Riemann relations. Details may be found in [4,5].

5. Calculation of the harmonic functions inside the domain

For the case under consideration, the analytical formulae allowing to calculate the unknown functions inside the cross-sectional domain are taken as expansions in terms of Cartesian harmonics, with coefficients to be determined by Boundary Collocation Method after truncation:

$$\begin{aligned} \phi(x,y) = {} & A + a_0\,x + b_0\,y + c_0\,xy + d_0(x^2 - y^2) \\ & + \sum_{n=1}^{\infty} a_n \cos nx \cosh ny + \sum_{n=1}^{\infty} b_n \cos nx \sinh ny \\ & + \sum_{n=1}^{\infty} c_n \sin nx \cosh ny + \sum_{n=1}^{\infty} d_n \sin nx \sinh ny, \end{aligned} \quad (11)$$

$$\begin{aligned} \phi^c(x,y) = {} & B - b_0 x + a_0 y + 2d_0 xy - \frac{1}{2}c_0(x^2 - y^2) \\ & + \sum_{n=1}^{\infty} d_n \cos nx \cosh ny + \sum_{n=1}^{\infty} c_n \cos nx \sinh ny \\ & - \sum_{n=1}^{\infty} b_n \sin nx \cosh ny - \sum_{n=1}^{\infty} a_n \sin nx \sinh ny, \end{aligned}$$

$$\begin{aligned} \psi(x,y) = {} & C + f_0\,x + g_0\,y + h_0\,xy + k_0(x^2 - y^2) \\ & + \sum_{n=1}^{\infty} f_n \cos nx \cosh ny + \sum_{n=1}^{\infty} g_n \cos nx \sinh ny \\ & + \sum_{n=1}^{\infty} h_n \sin nx \cosh ny \\ & + \sum_{n=1}^{\infty} k_n \sin nx \sinh ny + Q\psi^S(x,y), \end{aligned} \quad (12)$$

$$\begin{aligned} \psi^c(x,y) = {} & G - g_0 x + f_0 y + 2k_0 xy - \frac{1}{2}h_0(x^2 - y^2) \\ & + \sum_{n=1}^{\infty} k_n \cos nx \cosh ny + \sum_{n=1}^{\infty} h_n \cos nx \sinh ny \\ & - \sum_{n=1}^{\infty} g_n \sin nx \cosh ny - \sum_{n=1}^{\infty} f_n \sin nx \sinh ny, \end{aligned}$$

where ψ^S is an adequately chosen harmonic function with singular behavior at the transition points of the mechanical boundary conditions. All the coefficients appearing in the above equations, as well as the form of the singular function ψ^S, will be determined in the process of the solution.

6. Numerical treatment

The numerical treatment proceeds in two stages:

- Having transformed all the basic equations and conditions into boundary integral equations by means of the boundary integral representation of harmonic functions, these equations are then discretized by dividing the complete angle 2π uniformly into a sufficiently large number of sections and placing the corresponding number of nodes on the boundary. As a consequence, the contour C is approximated to a broken closed contour with unequal side lengths. The transition points are excluded from the set of nodes. Any contour integration on D is approximated by a finite sum. Derivatives of functions along C are approximated in a proper way. This is crucial for an efficient application of the method. For any node, the first and the second derivatives of functions along the boundary are evaluated by taking into account the values of the function at an equal number of nodes to the left, and to the right of the considered node. Numerical experiments have shown that this method of calculation of the tangential derivatives smoothens any existing discontinuities of the derivatives, similarly to the behavior of Fourier series at jumps. The removable singularities in the boundary integrals are also taken care of. Details of the calculations may be found elsewhere (cf. [5,33,46]). After discretization of all the basic equations and conditions, a linear rectangular algebraic system of equations is obtained for the boundary values of the unknown functions. This is resolved by Least Squares. The resulting maximal error in the solution is noted.
- A boundary analysis is carried out in order to evaluate the type of behavior of the different functions at the transition points. On the basis of this, the type of singular function ψ^S to be added to the expression for ψ will be determined. Boundary collocation is then used to find the coefficients of the above expansions of the basic functions in the cross-sectional domain.

7. Numerical results

A system of orthogonal Cartesian coordinates is used, with origin O at the center of the rectangle, x-axis along the major axis of the rectangle. Let $2a$ and $2b$ be respectively the length and width of the rectangle, while θ denotes the polar angle of a general point on the rectangle. For dimension analysis purposes, the half-length is taken to be the characteristic length, i.e. $a = \frac{1}{2}$. Also, $b = \frac{0.7}{2}$ for concreteness. It is clear that the rectangular boundary belongs to the class C^0 and consequently does not satisfy the smoothness condition necessary for the validity of the present approach. The smoothness process aims to achieve a new boundary close to the original one and belonging to the class C^2 *at least*. Fig. 1 shows the original contour and the smoothed one for comparison. Smoothing was done on parts at the corners subtending an angle of $2\Delta > 0$.

The boundary of the domain is subjected to the following boundary conditions:

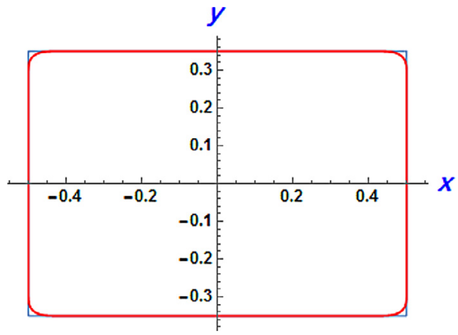

Fig. 1 Rectangle. Original and smoothed boundaries.

Thermal conditions

- Neumann type

$$\frac{\partial T}{\partial n}=0 \quad \text{for} \quad x=-a,\ -b\le y\le b \quad \text{and} \quad y=-b,\ -a\le x\le a,$$

- Dirichlet type

$$T=0 \quad \text{for} \quad x=a,\ -b\le y\le b,$$

- Robin type

$$\frac{\partial T}{\partial y}+Bi(T-1)=0 \quad \text{for} \quad y=b,\ -a\le x\le a$$

with $Bi=0.1$.

A steady temperature field establishes in the rectangle, due to heat inflow through the upper boundary, and heat outflow through the right boundary.

Mechanical conditions

- The right half of the boundary is subjected to a tension of intensity p given by:

$$p(\theta)=h_2\cos^8\theta, \quad 0\le\theta<\theta_1 \text{ and } \theta_2<\theta\le 2\pi, \tag{13}$$

and $h_2=0.1$. This choice makes the tension distribution tend to zero smoothly enough at both ends of its interval of definition. Stiffer choices for the applied tension is bound to increase the computational errors.
- The left half of the boundary is completely fixed,

$$u=0, \quad v=0, \quad \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}. \tag{14}$$

The exact solution of the thermal problem in the orthogonal Cartesian coordinates (x, y) is known (cf. [44]):

$$T(x,y)=2Bi\sum_{k=0}^{\infty}\frac{(-1)^k}{\mu_k^2\left(\frac{Bi\cosh(2b\mu_k)}{\mu_k}+\sinh(2b\mu_k)\right)} \times\cos[(a+x)\mu_k]\cosh[(b+y)\mu_k] \tag{15}$$

where

$$\mu_k=(2k+1)\frac{\pi}{2}$$

An analysis of this formula is presented in [22], where it is shown that the first and the second derivatives of the temperature function have different types of singularities at the upper right corner of the rectangle.

To calculate the temperature displacements u_T and v_T, the point M_0 was taken at the center of the rectangle, i.e. at the origin of coordinates. The integrations in (7) are then performed easily on paths formed by segments parallel to the coordinate axes. The resulting expressions have no symmetry with respect to the coordinate axes. The mechanical problem is replaced by two subproblems, one with given stresses, and the other with given (zero) displacements on the boundary, having a common solution (cf. [33]). For each of these two subproblems, the boundary conditions are given one part of the boundary and complemented with unknown values on the other part, to be determined as part of the solution. Following [45], the equations for each of these two subproblems are reduced to a system of boundary integral equations which are then discretized as explained above. The singular behavior of the stress components at the two separation boundary points is put in evidence and a singular solution is added to the basic harmonic function ψ to find the solution inside the cross-sectional domain in the form of expansions. The coefficients in these expansions are determined by the Boundary Collocation Method. Plots are given for the unknown functions on the boundary and in the bulk. The efficiency of the used numerical scheme is discussed. All figures were produced using Mathematica 9.0 Software.

Although there is symmetry with respect to the x-axis of the transition points and the type of mechanical boundary conditions, it is worth noting that the solutions of the basic unknown functions have no specific symmetry with respect to the axes of coordinates, due to the lack of symmetry of the temperature displacements u_T and v_T entering in the boundary conditions.

No analytical solution is available for comparison. The following figures show the optimal results obtained with 217 nodal points. Optimality in this context means less fluctuations and more regular curves. Many experimental experiments were carried out in order to find the best truncation of the expansions. It was found that 185 terms in the expansions for the harmonic functions ϕ, ϕ^c ψ and ψ^c functions yield optimal results. All systems of equations were solved by Least Squares.

Fig. 2 gives the boundary displacement due to temperature only. Such displacement is not bound to satisfy any boundary conditions.

The boundary analysis has clearly indicated some kind of discontinuities occurring in the stress components at the boundary separation points. Based on this observation, the expansion of the basic harmonic function ψ has been enriched with an additional term involving a harmonic function that has a singular boundary behavior at the separation points. The steps for building such a function are presented in Appendix. The plots in Figs. 3–7 show the values of the basic unknown functions as calculated from the boundary analysis (dotted curves), together with the values of these functions as obtained from the expansions (line curves). The maximum difference for the functions ϕ, ϕ^c, ψ and U does not exceed 0.0084. Turning next to the functions defining the boundary conditions, for the displacement functions u and v on the fixed part of the boundary, the maximum absolute values do not exceed 0.012 and 0.0079, respectively. For the normal stress σ_{nn} and the tangential stress $\sigma_{n\tau}$ on that part of the boundary where the stress is given, the

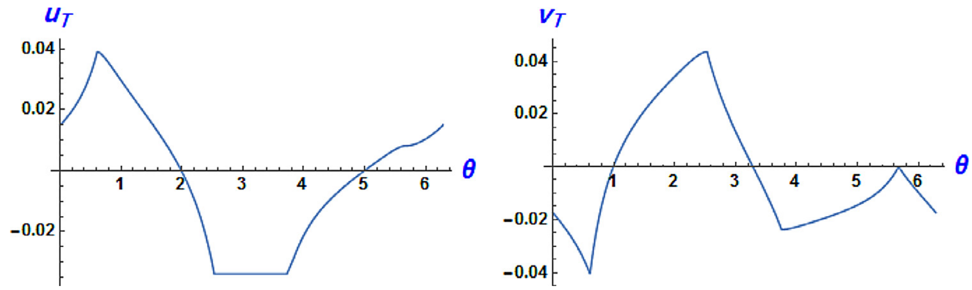

Fig. 2 Temperature displacements u_T and v_T on the boundary.

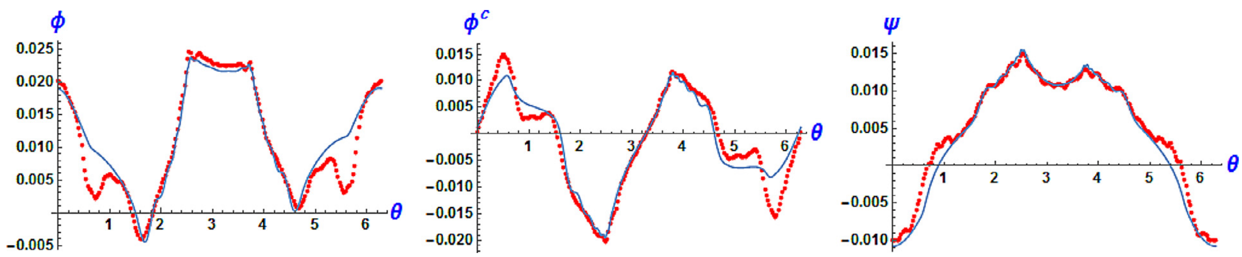

Fig. 3 The harmonic functions on the boundary.

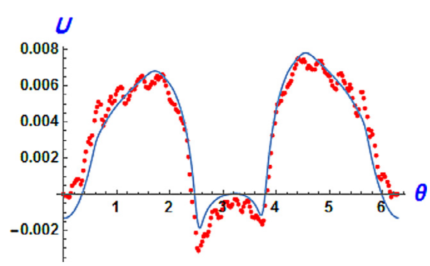

Fig. 4 Stress function on the boundary.

maximum values do not exceed 0.0082 and 0.0083, respectively. Globally, one can say that the presented series solution satisfies all the boundary conditions with absolute error less than 0.012.

The deformed contour showing the combined action of external mechanical and thermal factors is represented on the left in Fig. 8. One notices here the fulfillment of the partial fixing of the boundary. The right part of this same figure shows the boundary displacement due to temperature alone, i.e. the effect of the temperature displacements u_T and v_T.

The boundary distribution of the stress vector is represented in Fig. 9 in magnitude and direction. It is worth noting that this vector is directed outwards everywhere on the right half of the boundary as expected, while it is directed inwards on the left (fixed) half. There are two locations close to the separation points, and two other locations at the left corners, where the stress vector attains relatively large values. It is at these locations that a detachment of the boundary can potentially take place. The corresponding emplacements can be noticed on the curves

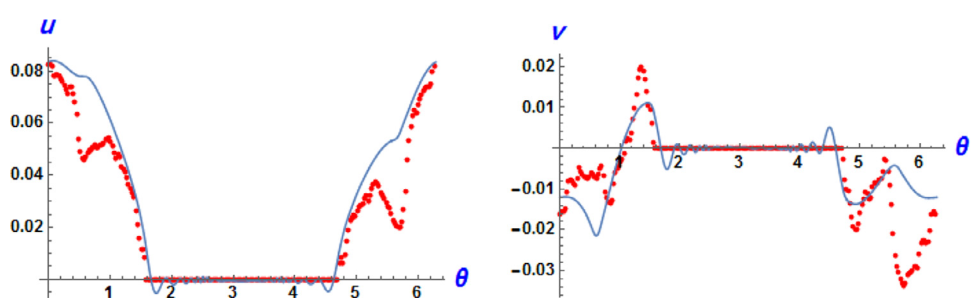

Fig. 5 Displacement on the boundary.

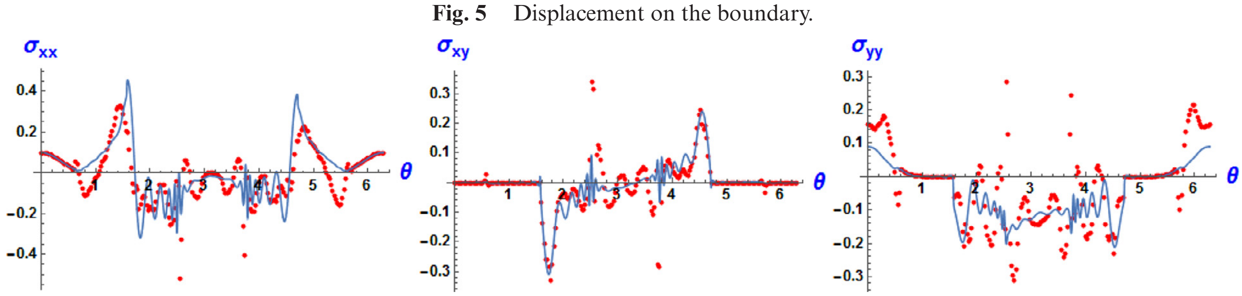

Fig. 6 Components of the stress tensor on the boundary.

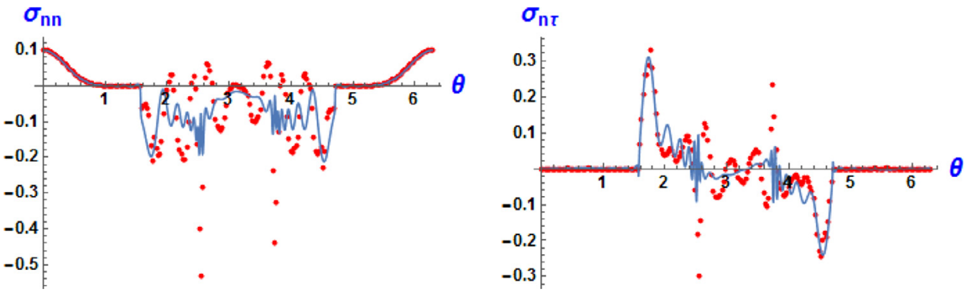

Fig. 7 Tangential and normal components of the stress tensor on C.

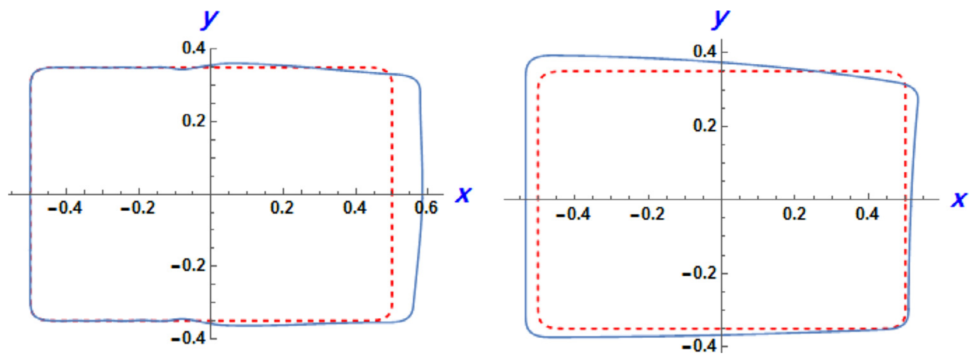

Fig. 8 Total displacement (left) and temperature displacement (right). The original boundary is shown for comparison (dashed curve).

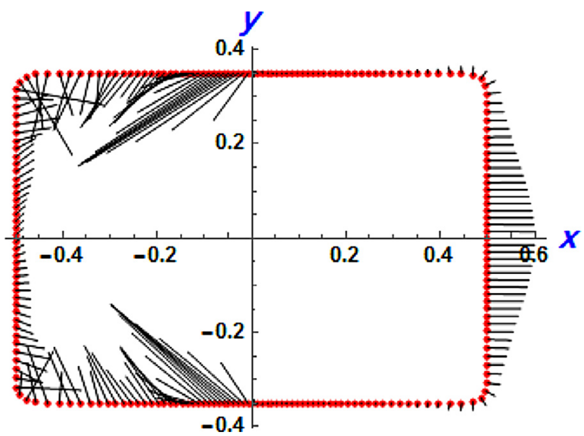

Fig. 9 Stress vector distribution on the boundary.

for the normal and the tangential components of stress obtained from boundary analysis (dotted curves) in Fig. 7.

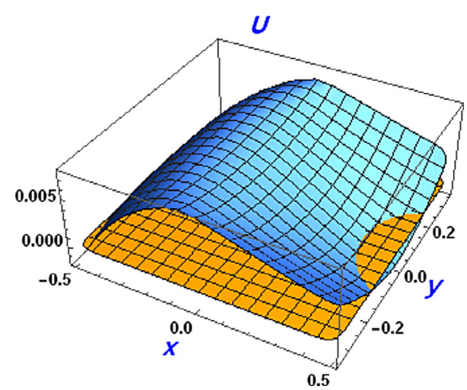

Fig. 11 $U(x, y)$.

The distributions of functions of practical interest inside the cross-sectional domain are shown in Figs. 10–12. The cross-sectional domain over which these functions are plotted is also shown in these figures for convenience.

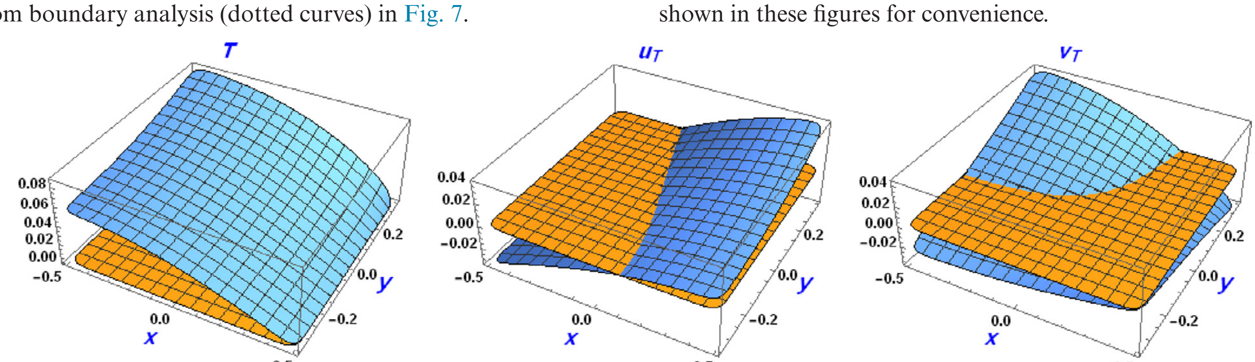

Fig. 10 $T(x, y)$, $u_T(x, y)$ and $v_T(x, y)$.

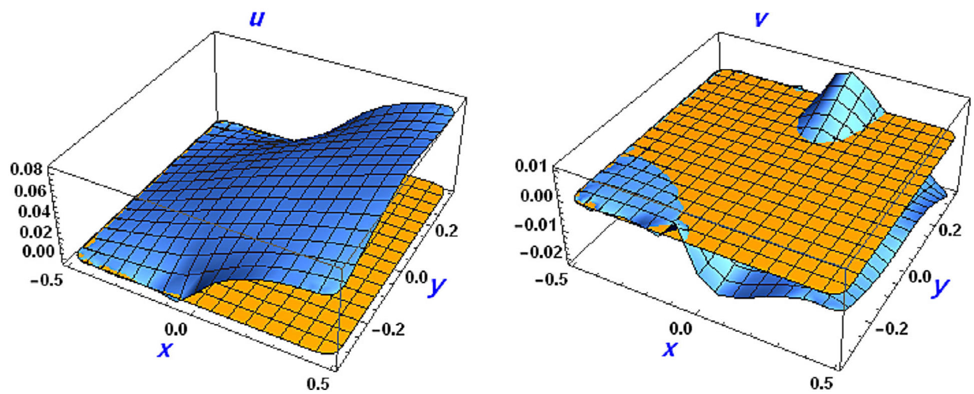

Fig. 12 $u(x, y)$ and $v(x, y)$.

8. Conclusions

A boundary integral method has been used to solve the plane problem of linear, uncoupled thermoelasticity for the rectangle under mixed mechanical conditions, with one part of the boundary fixed, the other subjected to a variable tension. A known exact solution for temperature is used in the solution. The unknown functions are obtained on the boundary and inside the domain of the cross-section. The weak singularities of the stress function arising at the transition points have been treated by introducing a harmonic function with singular behavior at these points. The boundary corner effects were removed by smoothing using polynomials (cf. [38]). The derivatives along the boundary were evaluated using 30 neighboring points, 15 from each side of the considered node.

For the present choices, the errors occurring within the boundary analysis do not exceed 1.2×10^{-2}. Inside the domain, the unknown functions were expanded in terms of harmonic functions. Boundary Collocation Method was used to find the coefficients. The deformations of the boundary due to heat effect alone, and due to the combined thermo-mechanical action are displayed. The results indicate that potential debonding of the fixed part of the boundary may occur near the transition points or at the fixed corners. The same method could be applied to other types of thermal or mechanical boundary conditions. The form of the singular function must be found separately for each case. The present investigation may be of interest in evaluating the stresses in long pad supports under mechanical loads and thermal action, when both factors are important.

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Appendix A. Treating the singularities

To simulate the singular behavior of stresses at the transition points, introduce a harmonic function in the upper half-plane $\{(x, y), y \geq 0\}$, with weak singularity at the origin of coordinates as:

$$f(x, y) = \frac{1}{2\pi} \int_0^{+\infty} \left[y - \frac{c_1^2}{2i} \frac{1}{\xi + c_1} + \frac{c_2^2}{2i} \frac{1}{\xi + c_2} \right] e^{-\xi} \, d\xi. \quad \text{(A.1)}$$

In terms of the integral exponential $E_1(z)$ of the complex argument z defined as ([47, p. 62]):

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} \, dt = e^{-z} \int_0^{\infty} \frac{e^{-t}}{t + z} \, dt = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \, n!}, \quad \text{(A.2)}$$

where $\gamma = 0.5772156649$ is the well-known Euler constant, one gets:

$$f(x, y) = \frac{1}{2\pi} \left[y + 2Re\left(\frac{i c_1^2}{2} e^{c_1} E_1(c_1) \right) \right].$$

The obtained function is centered at each of the two boundary separation points. The sum of the resulting two functions is then taken as ψS in the above expansions.

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