Journal of the Egyptian Mathematical Society (2016) **24**, 555–561

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Representations for parameter derivatives of some Koornwinder polynomials in two variables

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Received 15 August 2015; revised 18 October 2015; accepted 12 January 2016
Available online 18 February 2016

Keywords

Orthogonal polynomials; Jacobi polynomials; Laguerre polynomials; Koornwinder polynomials; Parameter derivatives

Abstract In this paper, we give the parameter derivative representations in the form of

$$\frac{\partial P_{n,k}(\lambda; x, y)}{\partial \lambda} = \sum_{m=0}^{n-1} \sum_{j=0}^{m} d_{n,j,m} P_{m,j}(\lambda; x, y) + \sum_{j=0}^{k} e_{n,j,k} P_{n,j}(\lambda; x, y)$$

for some Koornwinder polynomials where λ is a parameter and $0 \le k \le n; n = 0, 1, 2, \ldots$ and present orthogonality properties of the parametric derivatives of these polynomials.

2010 MSC: Primary 42C05; 33C45

1. Introduction

Recently, many authors [1–14] have studied the representations for the parameter derivatives of the classical orthogonal polynomials and various special functions which have many applications in applied mathematics, mathematical and theoretical physics and many branches of mathematics. In [9,10] , the derivative of the Legendre function of the first kind, with respect to its degree ν, $[\partial P_\nu(z)/\partial \nu]_{\nu=n} (n \in \mathbb{N})$, and its some representations have been examined by Szmytkowski, which are seen in some engineering and physical problems such as in the general theory of relativity and in solving some boundary value problems of potential theory, of electromagnetism and of heat conduction in solids. In [11], explicit expressions of second-order derivative $[\partial^2 P_\nu(z)/\partial \nu^2]_{\nu=0}$ and of third-order derivative $[\partial^3 P_\nu(z)/\partial \nu^3]_{\nu=0}$ have been derived. In [12–14], the author has presented the derivatives of the associated Legendre function of the first kind with respect to its order and its degree and also a relationship between these derivatives. Such derivatives of the associated Legendre function are met in solutions of various

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2016.01.004

problems of theoretical acoustics, heat conduction and other branches of theoretical physics. In [3–8,15] , the representations of parametric derivatives in the form

$$\frac{\partial P_n(\lambda; x)}{\partial \lambda} = \sum_{k=0}^{n} c_{n,k}(\lambda) P_k(\lambda; x) \tag{1}$$

for orthogonal polynomials in one variable, λ being a parameter, have been studied. For instance, the representations of parametric derivatives have been obtained by Wulkow [15] for discrete Laguerre polynomials, by Froehlich [3] for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by Koepf [4] for generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ and Gegenbauer polynomials $C_n^{(\lambda)}(x)$, by Koepf and Schmersau [5] for all the continuous and discrete classical orthogonal polynomials. In [8], Szmytkowski has derived again the expansions in the form of (1) for Jacobi polynomials, Gegenbauer polynomials and the generalized Laguerre polynomials by means of a method which is different from the methods given by Froehlich [3] and Koepf [4]. In [7], Ronveaux et al. have presented the recurrence relations for coefficients in the expansion

$$\frac{\partial^m P_n(\lambda; x)}{\partial \lambda^m} = \sum_{k=0}^{n} a_{n,k}(m, \lambda) P_k(\lambda; x) \qquad (m \in \mathbb{N})$$

which is more general than the expansion form of (1). Moreover, Lewanowicz [6] has given a method to obtain iteratively explicit parameter derivative representations of order $m = 1, 2, \ldots$ for almost all the classical orthogonal polynomial families, i.e., continuous classical orthogonal polynomials, classical orthogonal polynomials of a discrete variable or q-classical orthogonal polynomials of the Hahn's class.

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}$$

and they satisfy the following orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{n,m} = d_n^{(\alpha,\beta)}\delta_{n,m} \tag{2}$$

where $\delta_{n,m}$ denotes Kronecker's delta [16]. The generalized Laguerre polynomials defined by

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^{x}}{n!}\frac{d^n}{dx^n}\{e^{-x}x^{n+\alpha}\}$$

hold

$\int_0^{\infty} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) e^{-x} x^{\alpha} dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{n,m}$.

The representations of parametric derivatives obtained for the Jacobi polynomials $P_n^{(\alpha,\beta)}$ ([3]) and generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ ([4]) are as follows

$$\frac{\partial P_n^{(\alpha,\beta)}(x)}{\partial \alpha} = \sum_{k=0}^{n-1}\frac{1}{n+k+\alpha+\beta+1}P_n^{(\alpha,\beta)}(x) + \frac{(\beta+1)_n}{(\alpha+\beta+1)_n} \times \sum_{k=0}^{n-1}\frac{(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(n-k)(n+k+\alpha+\beta+1)(\beta+1)_k}P_k^{(\alpha,\beta)}(x) \tag{3}$$

and

$$\frac{\partial P_n^{(\alpha,\beta)}(x)}{\partial \beta} = \sum_{k=0}^{n-1}\frac{1}{n+k+\alpha+\beta+1}P_n^{(\alpha,\beta)}(x) + \frac{(\alpha+1)_n}{(\alpha+\beta+1)_n} \times \sum_{k=0}^{n-1}\frac{(-1)^{n+k}(2k+\alpha+\beta+1)(\alpha+\beta+1)_k}{(n-k)(n+k+\alpha+\beta+1)(\alpha+1)_k}P_k^{(\alpha,\beta)}(x) \tag{4}$$

for $\alpha, \beta > -1$ and

$$\frac{\partial L_n^{(\alpha)}(x)}{\partial \alpha} = \sum_{k=0}^{n-1}\frac{1}{n-k}L_k^{(\alpha)}(x) \tag{5}$$

for $\alpha > -1$ where the Pochhammer symbol is defined by

$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1), k = 1, 2, \ldots$

With motivation from the expansion (1) for orthogonal polynomials in one variable, we consider similar expansion in the form of

$$\frac{\partial P_{n,k}(\lambda; x, y)}{\partial \lambda} = \sum_{m=0}^{n-1}\sum_{j=0}^{m} d_{n,j,m}P_{m,j}(\lambda; x, y) + \sum_{j=0}^{k} e_{n,j,k}P_{n,j}(\lambda; x, y) \tag{6}$$

for orthogonal polynomials of variables x and y, with λ being a parameter and $0 \le k \le n; n = 0, 1, 2, \ldots$. In the recent papers [1,2], parametric derivative representations in the form of (6) for Jacobi polynomials on the triangle and a family of orthogonal polynomials with two variables on the unit disc have been studied. The present paper is devoted to obtain parametric derivatives for the polynomials on the parabolic biangle, on the square and some new examples of Koornwinder polynomials introduced in [17] (see also [18]). Although the parameter derivatives of these polynomials with respect to their some parameters are in the form of (6), there exist some other parameters such that derivatives with respect to them are not in the form of (6). Because, some of the coefficients $d_{n,j,m}$ and $e_{n,j,k}$ depend on the variable x. The set up of this paper is summarized as follows. In Section 2, we remind the method given by Koornwinder [19] and some examples of this method. Section 3 contains parametric derivatives of Koornwinder polynomials on the parabolic biangle, orthogonal polynomials on the square, Laguerre–Jacobi Koornwinder polynomials and Laguerre–Laguerre Koornwinder polynomials. In Section 4, some orthogonality relations for the derivatives of these polynomials are studied.

2. Preliminaries

First we recall some basic properties of orthogonal polynomials in two variables [20]. Let Π be the set of all polynomials in two variables and let Π_n denote the linear space of polynomials in two variables of total degree at most n. A polynomial $p \in \Pi_n$ is called an orthogonal polynomial with respect to the weight function $\omega(x, y)$ if

$$\left\langle p, q\right\rangle := \int_{\Omega} p(x, y)q(x, y)\omega(x, y)dxdy = 0$$

for all $q \in \Pi_{n-1}$. Let V_n denote the space of orthogonal polynomials of degree n with respect to $\langle , \rangle$. In 1975, Koornwinder [19] constructed the following method to derive orthogonal polynomials in two variables from orthogonal polynomials in one variable. Let $\omega_1(x)$ and $\omega_2(y)$ be univariate weight functions defined on the intervals (a, b) and (c, d), respectively. Let $\rho(x)$ be a positive function on (a, b) which is either a polynomial of degree r, $(r = 0, 1, \ldots)$ or the square root of a polynomial of degree $2r$ $\quad (r = \frac{1}{2}, 1, \frac{3}{2}, \ldots)$. If $\rho(x)$ is not a polynomial, $c = -d < 0$ and $\omega_2(y)$ is an even function on $(-d, d)$. For $k \geq 0$, let $p_n(x; k)$, $n = 0, 1, \ldots$ be orthogonal polynomial respect to the weight function $\rho^{2k+1}(x)\omega_1(x)$ and let $q_n(y)$, $n \geq 0$ be orthogonal polynomial with respect to the weight function $\omega_2(y)$. Then, the family of polynomials

$$P_{n,k}(x, y) = p_{n-k}(x; k)\rho^k(x)q_k\left(\frac{y}{\rho(x)}\right), \quad 0 \leq k \leq n$$

are orthogonal with respect to the Koornwinder weight function $\omega(x, y) = \omega_1(x)\omega_2(\frac{y}{\rho(x)})$ over the domain

$$\Omega = \{(x, y) : a \leq x \leq b, c\rho(x) \leq y \leq d\rho(x)\}$$

with respect to the inner product

$$\left\langle f, g\right\rangle := \int_\Omega f(x, y)g(x, y)\omega(x, y)dxdy.$$

Some examples of Koornwinder's method are as follows:

(i) Orthogonal polynomials on the parabolic biangle: For $\alpha, \beta > -1$, Koornwinder polynomials on the parabolic biangle $\Omega = \{(x, y) : y^2 \leq x \leq 1\}$ correspond with

$$\omega_1(x) = (1 - x)^\alpha x^\beta, \quad 0 \leq x \leq 1,$$

$$\omega_2(y) = (1 - y^2)^\beta, \quad -1 \leq y \leq 1,$$

$$\rho(x) = \sqrt{x}.$$

These polynomials can be defined as

$$P_{n,k}^{(\alpha,\beta)}(x, y) = P_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x - 1)x^{k/2}P_k^{(\beta,\beta)}(\frac{y}{\sqrt{x}}), \quad 0 \leq k \leq n \tag{7}$$

and they are orthogonal with respect to the weight function

$$\omega(x, y) = (1 - x)^\alpha (x - y^2)^\beta.$$

In fact,

$$\begin{aligned}\left\langle P_{n,k}^{(\alpha,\beta)}(x, y), P_{m,j}^{(\alpha,\beta)}(x, y)\right\rangle &:= \int_\Omega P_{n,k}^{(\alpha,\beta)}(x, y)P_{m,j}^{(\alpha,\beta)}(x, y)(1 - x)^\alpha (x - y^2)^\beta dxdy \\ &= h_{n,k}^{(\alpha,\beta)}\delta_{n,m}\delta_{k,j}\end{aligned} \tag{8}$$

where

$$h_{n,k}^{(\alpha,\beta)} = \frac{2^{(2\beta+1)}\Gamma^2(\beta + k + 1)\Gamma(\alpha + n - k + 1)\Gamma\left(\beta + n + \frac{3}{2}\right)}{(n - k)!k!(2\beta + 2k + 1)\Gamma(2\beta + k + 1)\Gamma\left(\alpha + \beta + n + \frac{3}{2}\right)\left(\alpha + \beta + 2n - k + \frac{3}{2}\right)}. \tag{9}$$

(ii) Orthogonal polynomials on the square: For $\alpha, \beta, \gamma, \delta > -1$, the polynomials defined by

$$P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x, y) = P_{n-k}^{(\alpha,\beta)}(x)P_k^{(\gamma,\delta)}(y), \quad 0 \leq k \leq n \tag{10}$$

are orthogonal with respect to the weight function $\omega(x, y) = (1 - x)^\alpha (1 + x)^\beta (1 - y)^\gamma (1 + y)^\delta$ on the square $\Omega = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. In fact,

$$\begin{aligned}\left\langle P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x, y), P_{m,j}^{(\alpha,\beta,\gamma,\delta)}(x, y)\right\rangle &:= \int_\Omega P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x, y)P_{m,j}^{(\alpha,\beta,\gamma,\delta)} \\ &\times (x, y)(1 - x)^\alpha (1 + x)^\beta (1 - y)^\gamma (1 + y)^\delta dxdy \\ &= d_{n-k}^{(\alpha,\beta)}d_k^{(\gamma,\delta)}\delta_{n,m}\delta_{k,j}\end{aligned} \tag{11}$$

where $d_n^{(\alpha,\beta)}$ is given by (2).

Some new examples of Koornwinder polynomials were introduced in [17] by using Koornwinder construction. These cases are as follows:

(iii) Laguerre–Jacobi Koornwinder polynomials: The case of

$$\omega_1(x) = x^\alpha e^{-x}, \quad 0 \leq x < \infty,$$

$$\omega_2(y) = (1 - y)^\beta, \quad -1 \leq y \leq 1,$$

$$\rho(x) = x$$

leads to the polynomials

$$P_{n,k}^{(\alpha,\beta)}(x, y) = L_{n-k}^{(\alpha+2k+1)}(x)x^k P_k^{(\beta,0)}(\frac{y}{x}), \quad 0 \leq k \leq n \tag{12}$$

which are orthogonal with respect to the weight function $\omega(x, y) = x^{\alpha-\beta}e^{-x}(x - y)^\beta$, $(\alpha, \beta > -1)$ over the domain $\Omega = \{(x, y) : -x < y < x, x > 0\}$. The following relation holds

$$\begin{aligned}&\left\langle P_{n,k}^{(\alpha,\beta)}(x, y), P_{m,j}^{(\alpha,\beta)}(x, y)\right\rangle \\ &:= \int_\Omega P_{n,k}^{(\alpha,\beta)}(x, y)P_{m,j}^{(\alpha,\beta)}(x, y)x^{\alpha-\beta}e^{-x}(x - y)^\beta dxdy \\ &= s_{n,k}^{(\alpha,\beta)}\delta_{n,m}\delta_{k,j}\end{aligned}$$

where

$$s_{n,k}^{(\alpha,\beta)} = \frac{2^{\beta+1}\Gamma(\alpha + n + k + 2)}{(n - k)!(\beta + 2k + 1)}. \tag{13}$$

(iv) Laguerre–Laguerre Koornwinder polynomials: From the Koornwinder construction with

$$\omega_1(x) = x^\alpha e^{-x}, \quad 0 \leq x < \infty, \quad \alpha > -1$$

$$\omega_2(y) = y^\beta e^{-y}, \quad 0 \leq y < \infty, \quad \beta > -1$$

$$\rho(x) = x, \quad \alpha - \beta > -1,$$

the Laguerre–Laguerre Koornwinder polynomials defined by

$$P_{n,k}^{(\alpha,\beta)}(x, y) = L_{n-k}^{(\alpha+2k+1)}(x)x^k L_k^{(\beta)}(\frac{y}{x}), \quad 0 \leq k \leq n \tag{14}$$

are orthogonal with respect to the weight function $\omega(x,y) = x^{\alpha-\beta}y^{\beta}e^{-(x+y/x)}$ over the domain

$$\Omega = \{(x,y) : 0 \le x < \infty, 0 \le y < \infty\}.$$

It follows that

$$\left\langle P_{n,k}^{(\alpha,\beta)}(x,y), P_{m,j}^{(\alpha,\beta)}(x,y)\right\rangle := \int_{\Omega} P_{n,k}^{(\alpha,\beta)}(x,y)P_{m,j}^{(\alpha,\beta)}(x,y)x^{\alpha-\beta}y^{\beta}e^{-(x+y/x)}dxdy = t_{n,k}^{(\alpha,\beta)}\delta_{n,m}\delta_{k,j}$$

where

$$t_{n,k}^{(\alpha,\beta)} = \frac{\Gamma(\beta+k+1)\Gamma(\alpha+n+k+2)}{k!(n-k)!}. \tag{15}$$

3. Parametric derivatives of some Koornwinder polynomials

In [1,2], parametric derivative representations in the form of (6) for Jacobi polynomials on the triangle and a family of orthogonal polynomials with two variables on the unit disc have been studied. In this section, we derive parameter derivatives of Koornwinder polynomials on the parabolic biangle, on the square and some new examples of Koornwinder polynomials introduced in [17] (see also [18]). Since variable x is included in some of the coefficients, there exist some parameter derivatives such that they are not in the form (6). Now, we consider such representations of parameter derivatives.

Theorem 1. *For the Koornwinder polynomials over the parabolic biangle $P_{n,k}^{(\alpha,\beta)}(x,y)$ defined by (7), the parameter derivative with respect to the parameter α is as follows*

$$\frac{\partial}{\partial\alpha}P_{n,k}^{(\alpha,\beta)}(x,y) = \sum_{s=0}^{n-k-1}\frac{1}{\alpha+\beta+n+s+\frac{3}{2}}P_{n,k}^{(\alpha,\beta)}(x,y) + \sum_{s=0}^{n-k-1} \times \frac{\left(\alpha+\beta+2n-k-2s-\frac{1}{2}\right)\left(\beta+n-s+\frac{1}{2}\right)_{s+1}}{(s+1)\left(\alpha+\beta+2n-k-s+\frac{1}{2}\right)\left(\alpha+\beta+n-s+\frac{1}{2}\right)_{s+1}}P_{n-s-1,k}^{(\alpha,\beta)}(x,y) \tag{16}$$

for $n \ge k+1$, $k \ge 0$ and $\frac{\partial}{\partial\alpha}P_{n,n}^{(\alpha,\beta)}(x,y) = 0$ for $n = k \ge 0$.

Proof. If we differentiate the both side of (7) with respect to the parameter α, we get

$$\frac{\partial}{\partial\alpha}P_{n,k}^{(\alpha,\beta)}(x,y) = x^{k/2}P_k^{(\beta,\beta)}\left(\frac{y}{\sqrt{x}}\right)\frac{\partial}{\partial\alpha}P_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x-1)$$

By using (3), it concludes that for $n \ge k+1$, $k \ge 0$

$$\frac{\partial}{\partial\alpha}P_{n,k}^{(\alpha,\beta)}(x,y) = \sum_{s=0}^{n-k-1}\frac{1}{\alpha+\beta+n+s+\frac{3}{2}}P_{n,k}^{(\alpha,\beta)}(x,y) + \sum_{s=0}^{n-k-1} \times \frac{\left(\alpha+\beta+2n-k-2s-\frac{1}{2}\right)\left(\beta+n-s+\frac{1}{2}\right)_{s+1}}{(s+1)\left(\alpha+\beta+2n-k-s+\frac{1}{2}\right)\left(\alpha+\beta+n-s+\frac{1}{2}\right)_{s+1}}P_{n-s-1,k}^{(\alpha,\beta)}(x,y).$$

It is obvious from (7) that for $n = k \ge 0$, $\frac{\partial}{\partial\alpha}P_{n,n}^{(\alpha,\beta)}(x,y) = 0$. □

Theorem 2. *For $\alpha, \beta, \gamma, \delta > -1$, the polynomials on the square defined by (10) satisfy*

$$\frac{\partial P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\alpha} = \sum_{s=0}^{n-k-1}\frac{1}{n-k+s+\alpha+\beta+1}P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) + \sum_{s=0}^{n-k-1}\frac{(\alpha+\beta+2n-2k-2s-1)(\beta+n-k-s)_{s+1}}{(s+1)(\alpha+\beta+2n-2k-s)(\alpha+\beta+n-k-s)_{s+1}} \times P_{n-s-1,k}^{(\alpha,\beta,\gamma,\delta)}(x,y), \tag{17}$$

$$\frac{\partial P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\beta} = \sum_{s=0}^{n-k-1}\frac{1}{n-k+s+\alpha+\beta+1}P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) + \sum_{s=0}^{n-k-1}\frac{(-1)^{n-k-s}(\alpha+\beta+2n-2k-2s-1)(\alpha+n-k-s)_{s+1}}{(s+1)(\alpha+\beta+2n-2k-s)(\alpha+\beta+n-k-s)_{s+1}} \times P_{n-s-1,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) \tag{18}$$

for $n \ge k+1$, $k \ge 0$ and

$$\frac{\partial P_{n,n}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\alpha} = \frac{\partial P_{n,n}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\beta} = 0$$

for $n = k \ge 0$. Also,

$$\frac{\partial P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\gamma} = \sum_{s=0}^{k-1}\frac{1}{\gamma+\delta+k+s+1}P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) + \sum_{s=0}^{k-1}\frac{(\gamma+\delta+2k-2s-1)(\delta+k-s)_{s+1}}{(s+1)(\gamma+\delta+2k-s)(\gamma+\delta+k-s)_{s+1}}P_{n-s-1,k-s-1}^{(\alpha,\beta,\gamma,\delta)}(x,y), \tag{19}$$

and

$$\frac{\partial P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\delta} = \sum_{s=0}^{k-1}\frac{1}{\gamma+\delta+k+s+1}P_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) + \sum_{s=0}^{k-1}\frac{(-1)^{k-s}(\gamma+\delta+2k-2s-1)(\gamma+k-s)_{s+1}}{(s+1)(\gamma+\delta+2k-s)(\gamma+\delta+k-s)_{s+1}}P_{n-s-1,k-s-1}^{(\alpha,\beta,\gamma,\delta)}(x,y), \tag{20}$$

for $n \ge k \ge 1$ and

$$\frac{\partial P_{n,0}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\gamma} = \frac{\partial P_{n,0}^{(\alpha,\beta,\gamma,\delta)}(x,y)}{\partial\delta} = 0$$

for $n \ge 0$, $k = 0$.

Proof. In view of equalities (3) and (4), the proof is clear. □

Now, we can get similar results for Laguerre–Jacobi Koornwinder and Laguerre–Laguerre Koornwinder polynomials.

Theorem 3. *The representations of parameter derivatives with respect to the parameters α and β for Laguerre–Jacobi Koornwinder polynomials $P_{n,k}^{(\alpha,\beta)}(x,y)$ defined by (12) are given by*

$$\frac{\partial}{\partial\alpha}P_{n,k}^{(\alpha,\beta)}(x,y) = \sum_{s=0}^{n-k-1}\frac{1}{n-k-s}P_{k+s,k}^{(\alpha,\beta)}(x,y) \tag{21}$$

for $n \ge k+1$, $k \ge 0$ and $\frac{\partial}{\partial\alpha}P_{n,n}^{(\alpha,\beta)}(x,y) = 0$ for $n = k \ge 0$. Similarly,

$$\frac{\partial}{\partial\beta}P_{n,k}^{(\alpha,\beta)}(x,y) = \sum_{s=0}^{k-1}\frac{1}{\beta+k+s+1}P_{n,k}^{(\alpha,\beta)}(x,y) + \sum_{s=0}^{k-1}\frac{(\beta+2k-2s-1)(k-s)_{s+1}}{(s+1)(\beta+2k-s)(\beta+k-s)_{s+1}}x^{s+1}P_{n-s-1,k-s-1}^{(\alpha+2s+2,\beta)}(x,y)$$

for $n \geq k \geq 1$ *and* $\frac{\partial}{\partial \beta} P_{n,0}^{(\alpha,\beta)}(x, y) = 0$ *for* $n \geq 0, k = 0$. *It is seen that the parametric derivative with respect to the parameter* β *is not in the form of (6) since the coefficients include variable* x.

Theorem 4. *For Laguerre–Laguerre Koornwinder polynomials defined by (14), we have*

$$\frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)}(x, y) = \sum_{s=0}^{n-k-1} \frac{1}{n-k-s} P_{k+s,k}^{(\alpha,\beta)}(x, y) \tag{22}$$

for $n \geq k+1, k \geq 0$ *and* $\frac{\partial}{\partial \alpha} P_{n,n}^{(\alpha,\beta)}(x, y) = 0$ *for* $n = k \geq 0$. *Similarly, for* $n \geq k \geq 1$

$$\frac{\partial}{\partial \beta} P_{n,k}^{(\alpha,\beta)}(x, y) = \sum_{s=0}^{k-1} \frac{1}{s+1} x^{s+1} P_{n-s-1,k-s-1}^{(\alpha+2s+2,\beta)}(x, y)$$

which is different from the form of (6) since the coefficients include variable x. *Also,* $\frac{\partial}{\partial \beta} P_{n,0}^{(\alpha,\beta)}(x, y) = 0$ *for* $n \geq 0, k = 0$.

4. Orthogonality properties of parametric derivatives

Now, we consider orthogonality properties for the parametric derivatives of the polynomials on the parabolic biangle, the polynomials on the square, Laguerre–Jacobi and Laguerre–Laguerre Koornwinder polynomials.

Theorem 5. *For Koornwinder polynomials on the parabolic biangle given by (7) and their derivative with respect to the parameter* α, *we have for* $n \geq k+1, k \geq 0, 0 \leq j \leq m; n, m \in \mathbb{N}_0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle = \begin{cases} 0, & k \neq j \\ 0, & k = j, m > n \\ A_{n,k,m}^{(\alpha,\beta)}, & n > m \geq k = j \\ B_{n,k}^{(\alpha,\beta)}, & k = j, n = m \end{cases}$$

and for $n = k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,n}^{(\alpha,\beta)} \right\rangle = 0$$

where

$$A_{n,k,m}^{(\alpha,\beta)} = \frac{\left(\alpha+\beta-k+2m+\frac{3}{2}\right)\left(\beta+m+\frac{3}{2}\right)_{n-m}}{(n-m)\left(\alpha+\beta+n-k+m+\frac{3}{2}\right)\left(\alpha+\beta+m+\frac{3}{2}\right)_{n-m}} h_{m,k}^{(\alpha,\beta)}$$

and

$$B_{n,k}^{(\alpha,\beta)} = \sum_{s=0}^{n-k-1} \frac{1}{\alpha+\beta+n+s+\frac{3}{2}} h_{n,k}^{(\alpha,\beta)}$$

where $h_{n,k}^{(\alpha,\beta)}$ *is given by (9).*

Proof. We will divide the proof into two cases.

Case 1. We consider the case $n = k \geq 0$. It is seen that

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,n}^{(\alpha,\beta)} \right\rangle = 0$$

since $\frac{\partial}{\partial \alpha} P_{n,n}^{(\alpha,\alpha)}(x, y) = 0$ for $n = k \geq 0$.

Case 2. We assume that $n \geq k+1, k \geq 0$. From (16), we can write

$$\begin{aligned} \left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle &= \sum_{s=0}^{n-k-1} \frac{1}{\alpha+\beta+n+s+\frac{3}{2}} \left\langle P_{m,j}^{(\alpha,\beta)}, P_{n,k}^{(\alpha,\beta)} \right\rangle \\ &+ \sum_{s=0}^{n-k-1} \frac{\left(\alpha+\beta+2n-k-2s-\frac{1}{2}\right)\left(\beta+n-s+\frac{1}{2}\right)_{s+1}}{(s+1)\left(\alpha+\beta+2n-k-s+\frac{1}{2}\right)\left(\alpha+\beta+n-s+\frac{1}{2}\right)_{s+1}} \\ &\times \left\langle P_{m,j}^{(\alpha,\beta)}, P_{n-s-1,k}^{(\alpha,\beta)} \right\rangle. \end{aligned} \tag{23}$$

For this case, we consider three subcases.

Case 2.1. Let consider the case $k \neq j$ or $k = j, m > n$. It follows from (8) that

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle = 0.$$

Case 2.2. Assume that $k = j, n > m$. Since the first inner product in the right-hand side of the equality (23) from (8) is zero, we get

$$\begin{aligned} \left\langle P_{m,k}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle &= \sum_{s=0}^{n-k-1} \\ &\times \frac{\left(\alpha+\beta+2n-k-2s-\frac{1}{2}\right)\left(\beta+n-s+\frac{1}{2}\right)_{s+1}}{(s+1)\left(\alpha+\beta+2n-k-s+\frac{1}{2}\right)\left(\alpha+\beta+n-s+\frac{1}{2}\right)_{s+1}} \\ &\times h_{n-s-1,k}^{(\alpha,\beta)} \delta_{n-s-1,m}, \end{aligned}$$

which contains only one non-vanishing term with $s = n-m-1$ for $m \geq k$. One may deduce that

$$A_{n,k,m}^{(\alpha,\beta)} = \frac{\left(\alpha+\beta-k+2m+\frac{3}{2}\right)\left(\beta+m+\frac{3}{2}\right)_{n-m}}{(n-m)\left(\alpha+\beta+n-k+m+\frac{3}{2}\right)\left(\alpha+\beta+m+\frac{3}{2}\right)_{n-m}} h_{m,k}^{(\alpha,\beta)}$$

where $h_{n,k}^{(\alpha,\beta)}$ is given by (9).

Case 2.3 Let $k = j, m = n$. Then, from the relation (8) we have

$$\begin{aligned} \left\langle P_{n,k}^{(\alpha,\beta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle &= \sum_{s=0}^{n-k-1} \frac{1}{\alpha+\beta+n+s+\frac{3}{2}} \left\langle P_{n,k}^{(\alpha,\beta)}, P_{n,k}^{(\alpha,\beta)} \right\rangle \\ &= \sum_{s=0}^{n-k-1} \frac{1}{\alpha+\beta+n+s+\frac{3}{2}} h_{n,k}^{(\alpha,\beta)}, \end{aligned}$$

which completes the proof. □

By using the parameter derivatives given in Theorem 2 and the relation (11), the next theorem is readily verified.

Theorem 6. *For Koornwinder polynomials on the square defined by (10), we have for* $0 \leq j \leq m; n, m \in \mathbb{N}_0$, $n \geq k+1, k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta,\gamma,\delta)}, \frac{\partial}{\partial \alpha} P_{n,k}^{(\alpha,\beta,\gamma,\delta)} \right\rangle = \begin{cases} 0, & k \neq j \\ 0, & k = j, m > n \\ C_{n,k,m}^{(\alpha,\beta,\gamma,\delta)}, & n > m \geq k = j \\ D_{n,k}^{(\alpha,\beta,\gamma,\delta)}, & k = j, n = m \end{cases}$$

$$\left\langle P_{m,j}^{(\alpha,\beta,\gamma,\delta)}, \frac{\partial}{\partial\beta} P_{n,k}^{(\alpha,\beta,\gamma,\delta)} \right\rangle = \begin{cases} 0, & k \neq j \\ 0, & k = j, m > n \\ E_{n,k,m}^{(\alpha,\beta,\gamma,\delta)}, & n > m \geq k = j \\ D_{n,k}^{(\alpha,\beta,\gamma,\delta)}, & k = j, n = m \end{cases}$$

and for $n = k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta,\gamma,\delta)}, \frac{\partial}{\partial\alpha} P_{n,n}^{(\alpha,\beta,\gamma,\delta)} \right\rangle = \left\langle P_{m,j}^{(\alpha,\beta,\gamma,\delta)}, \frac{\partial}{\partial\beta} P_{n,n}^{(\alpha,\beta,\gamma,\delta)} \right\rangle = 0$$

where

$$C_{n,k,m}^{(\alpha,\beta,\gamma,\delta)} = \frac{(\alpha+\beta+2m-2k+1)(\beta-k+m+1)_{n-m}}{(n-m)(\alpha+\beta+n+m-2k+1)(\alpha+\beta-k+m+1)_{n-m}} \times d_{m-k}^{(\alpha,\beta)} d_k^{(\gamma,\delta)}$$

$$D_{n,k}^{(\alpha,\beta,\gamma,\delta)} = \sum_{s=0}^{n-k-1} \frac{1}{n-k+s+\alpha+\beta+1} d_{n-k}^{(\alpha,\beta)} d_k^{(\gamma,\delta)},$$

$$E_{n,k,m}^{(\alpha,\beta,\gamma,\delta)} = \frac{(-1)^{m-k+1}(\alpha+\beta+2m-2k+1)(\alpha-k+m+1)_{n-m}}{(n-m)(\alpha+\beta+n+m-2k+1)(\alpha+\beta-k+m+1)_{n-m}} \times d_{m-k}^{(\alpha,\beta)} d_k^{(\gamma,\delta)}$$

where $d_n^{(\alpha,\beta)}$ *is defined as in (2).*

Similarly, using the results in Theorems 3 and 4, one can easily obtain the next results.

Theorem 7. *For Laguerre–Jacobi Koornwinder polynomials* $P_{n,k}^{(\alpha,\beta)}(x,y)$ *defined by (12),* $n, m \in \mathbb{N}_0, 0 \leq j \leq m$, *we get for* $n \geq k+1, k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial\alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle = \begin{cases} 0, & k \neq j \\ 0, & k = j, m \geq n \\ G_{n,k,m}^{(\alpha,\beta)}, & n > m \geq k = j \end{cases}$$

and for $n = k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial\alpha} P_{n,n}^{(\alpha,\beta)} \right\rangle = 0$$

where

$$G_{n,k,m}^{(\alpha,\beta)} = \frac{1}{n-m} s_{m,k}^{(\alpha,\beta)}$$

where $s_{n,k}^{(\alpha,\beta)}$ *is given by (13).*

Theorem 8. *For Laguerre–Laguerre Koornwinder polynomials* $P_{n,k}^{(\alpha,\beta)}(x,y)$, $n, m \in \mathbb{N}_0, 0 \leq j \leq m$, *the following results hold for* $n \geq k+1, k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial\alpha} P_{n,k}^{(\alpha,\beta)} \right\rangle = \begin{cases} 0, & k \neq j \\ 0, & k = j, m \geq n \\ H_{n,k,m}^{(\alpha,\beta)}, & n > m \geq k = j \end{cases}$$

and for $n = k \geq 0$

$$\left\langle P_{m,j}^{(\alpha,\beta)}, \frac{\partial}{\partial\alpha} P_{n,n}^{(\alpha,\beta)} \right\rangle = 0$$

where

$$H_{n,k,m}^{(\alpha,\beta)} = \frac{1}{n-m} t_{m,k}^{(\alpha,\beta)}$$

where $t_{n,k}^{(\alpha,\beta)}$ *is defined by (15).*

Acknowledgment

The author is grateful to the referees for their valuable comments and suggestions which improve the quality and the clarity of the paper.

References

[1] R. Aktaş, A note on parameter derivatives of the Jacobi polynomials on the triangle, Appl. Math. Comput. 247 (2014) 368–372.
[2] R. Aktas, On parameter derivatives of a family of polynomials in two variables, Appl. Math. Comput. 256 (2015) 769–777.
[3] J. Froehlich, Parameter derivatives of the Jacoby polynomials and the gaussian hypergeometric function, Integral Transforms Spec. Funct. 2 (4) (1994) 253–266.
[4] W. Koepf, Identities for families of orthogonal polynomials and special functions, Integral Transforms Spec. Funct. 5 (1-2) (1997) 69–102.
[5] W. Koepf, D. Schmersau, Representations of orthogonal polynomials, J. Comput. Appl. Math. 90 (1998) 57–94.
[6] S. Lewanowicz, Representations for the parameter derivatives of the classical orthogonal polynomials, Rend. Circ. Mat. Palermo, Ser. II, Suppl. 68 (2002) 599–613.
[7] A. Ronveaux, A. Zarzo, I. Area, E. Godoy, Classical orthogonal polynomials: dependence on parameters, J.Comput. Appl. Math. 121 (2000) 95–112.
[8] R. Szmytkowski, A note on parameter derivatives of classical orthogonal polynomials. arXiv:0901.2639v3.
[9] R. Szmytkowski, On the derivative of the Legendre function of the first kind with respect to its degree, J. Phys. A 39 (2006) 15147–15172.[corrigendum: J. Phys. A, 40 (2007), 7819–7820]
[10] R. Szmytkowski, Addendum to 'On the derivative of the Legendre function of the first kind with respect to its degree', J. Phys. A 40 (2007) 14887–14891.
[11] R. Szmytkowski, The parameter derivatives $[\partial^2 p_\nu(z)/\partial\nu^2]_{\nu=0}$ and $[\partial^3 p_\nu(z)/\partial\nu^3]_{\nu=0}$, where $p_\nu(z)$ is the legendre function of the first kind. preprint arXiv:1301.6586.
[12] R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order (with applications to the construction of the associated legendre function of the second kind of integer degree and order), J. Math. Chem. 46 (2009) 231–260.
[13] R. Szmytkowski, On the derivative of the associated Legendre function of the first kind of integer order with respect to its degree (with applications to the construction of the associated legendre function of the second kind of integer degree and order), J. Math. Chem. 49 (2011) 1436–1477.
[14] R. Szmytkowski, On parameter derivatives of the associated Legendre function of the first kind (with applications to the construction of the associated legendre function of the second kind of integer degree and order), J. Math. Anal.Appl. 386 (2012) 332–342.
[15] M. Wulkow, Numerical Treatment of Countable Systems of Ordinary Differential Equations, Konrad-Zuse-Zentrum Berlin, 1990 Technical report tr 90-8.
[16] E.D. Rainville, Special Functions, The Macmillan Co., New York, 1960.

[17] L. Fernandez, T.E. Perez, M. Pinar, On koornwinder classical orthogonal polynomials in two variables, J. Comput. Appl. Math. 236 (2012) 3817–3826.

[18] F. Marcellan, M.E. Marriaga, T.E. Perez, M.A. Pinar, Matrix pearson equations satisfied by koornwinder weights in two variables, arXiv:1411.2268

[19] T.H. Koornwinder, Two–variable analogues of the classical orthogonal polynomials, in: R. Askey (Ed.), Theory and Application of Special Functions, Academic Press, New York, 1975, pp. 5–495.

[20] C.F. Dunkl, Y. Xu, Orthogonal polynomials of several variables, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001.