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Short Communication

Comment on "T_i-spaces I, II"

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Abstract In this comment, we show that some assertions made in Bayoumi and Ibedou (2002) [1] and Bayoumi and Ibedou (2002) [2] are incorrect. Specifically, one implication from Theorem 3.1 made in Bayoumi and Ibedou (2002) [1] is erroneous. Consequently, Propositions 5.1 and 6.1 introduced in Bayoumi and Ibedou (2002) [2] are incorrect. In addition, one implication from Theorems 5.1 and 6.1, made in Bayoumi and Ibedou (2002) [2] are incorrect. We give some counterexamples to support our claim.

MATHEMATICS SUBJECT CLASSIFICATION: 54A05; 54A40; 54E55; 54C08

1. Introduction and preliminaries

In order for this comment to be clear, we need to review the terminology. Using Chang's [3] sense of fuzzy topological spaces, the concept of separation axioms is linked to fuzzy points and their stronger forms. In [1,2], the notions of separation axioms T_i, $i = 0, 1, 2, 3, 4$, in L-topological spaces depend on the notions of fuzzy neighborhood filters, ordinary points and crisp closed subsets of X.

In this comment L is a complete chain with differing least and last elements 0 and 1, respectively, $L_0 = L\backslash\{0\}$ and $L_1 = L\backslash\{1\}$. By a fuzzy set of a set X we mean a mapping $f : X \longrightarrow L$. L^X and $P(X)$ denote the sets of all fuzzy sets and of all ordinary subsets of X, respectively. For each $x \in X$ and $t \in L_0$, the fuzzy set x_t of X, whose value is t at x and 0 otherwise, is called a fuzzy point in X. For each $\alpha \in L$, the constant fuzzy set of X with value α will be denoted by $\bar{\alpha}$.

A fuzzy topology of a set X [3] is a subset τ of L^X which contains the constant fuzzy sets $\bar{0}$ and $\bar{1}$, and closed with respect to finite intersection and arbitrary union. The pair (X, τ) is called a fuzzy topological space and the elements of τ are called open

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fuzzy sets. The family of all closed fuzzy sets on X is denoted by τ'. The interior $int_\tau f$ (respectively, closure $cl_\tau f$) of a fuzzy set f is the greatest open fuzzy set less than or equal to f (respectively, is the smallest closed fuzzy set greater than or equal to f), that is, $int_\tau f = \bigvee_{g\in\tau, g\leq f} g$ (respectively, $cl_\tau f = \bigwedge_{g\in\tau', g\geq f} g$).

Definition 1.1 [4,5]**.** Let X be a non-empty set. A fuzzy filter on X is a mapping $\mathcal{M} : L^X \longrightarrow L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\bar{\alpha}) \leq \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\bar{1}) = 1$.
(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

A fuzzy filter $\mathcal{M}$ is called homogeneous if $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on X, then $\mathcal{M}$ is finer than $\mathcal{N}$, which is denoted by $\mathcal{M} \leq \mathcal{N}$, provided that $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \nleq \mathcal{N}$, we means that $\mathcal{M}$ is not finer than $\mathcal{N}$. Since L is a complete chain, then

$$\mathcal{M} \nleq \mathcal{N} \Longleftrightarrow \textit{there exists } f \in L^X \textit{such that } \mathcal{M}(f) < \mathcal{N}(f). \tag{1}$$

Proposition 1.1 [5]**.** *Let A be a set of fuzzy filters on X. Then, the following are equivalent:*

1. *The infimum $\bigwedge_{\mathcal{M}\in A} \mathcal{M}$ of A with respect to the finer relation of a fuzzy filter exists,*
2. *For each non-empty finite subset $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(f_1) \wedge \mathcal{M}_2(f_2) \wedge \ldots \wedge \mathcal{M}_n(f_n) \leq sup\ (f_1 \wedge f_2 \wedge \ldots \wedge f_n)$ for all $f_1, f_2, \ldots, f_n \in L^X$.*

Definition 1.2 [6]**.** For each fuzzy topological space (X, τ) and each $x \in X$, a fuzzy neighborhood filter of the space (X, τ) at x is a mapping $\mathcal{N}(x) : L^X \longrightarrow L$ defined by

$$\mathcal{N}(x)(f) = (\text{int}_\tau f)(x), \tag{2}$$

for all $f \in L^X$ which is a fuzzy filter on X. The fuzzy neighborhood filter $\mathcal{N}(F)$ at a set $F \subseteq X$ is defined by means of $\mathcal{N}(x)$ and $x \in F$ as

$$\mathcal{N}(F) = \bigvee_{x\in F} \mathcal{N}(x). \tag{3}$$

For each $x \in X$, the mapping $\dot{x} : L^X \longrightarrow L$ defined by

$$\dot{x}(f) = f(x), \tag{4}$$

for all $f \in L^X$, is a homogeneous fuzzy filter on X.

Definition 1.3 [7]**.** For each fuzzy topological space (X, τ) the closure operator of τ is the mapping cl that is assigned to each fuzzy filter $cl\mathcal{M}$ such that

$$cl\mathcal{M}(f) = \bigvee_{cl_\tau \rho\leq f} \mathcal{M}(\rho). \tag{5}$$

$cl\mathcal{M}$ is called the closure of $\mathcal{M}$.

Definition 1.4. A fuzzy topological space (X, τ) is called

1. T_0-space if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ or $\dot{y} \nleq \mathcal{N}(x)$ [1].
2. T_1-space if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ and $\dot{y} \nleq \mathcal{N}(x)$ [1].
3. T_2-space (or Hausdorff space) if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist [1].
4. Regular space if $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist for all $x \in X$, $F \subseteq X$ with $x \notin F$ and $cl_\tau F = F$.
   T_3-space if it is regular space and T_1-space [2].
5. Normal space if for all $F_1, F_2 \subseteq X$, such that $cl_\tau F_1 = F_1$, $cl_\tau F_2 = F_2$ and $F_1 \cap F_2 = \emptyset$, we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist.
   T_4-space if it is normal and T_1-space [2].

2. Counterexamples

In this section we point out where the errors occur in [1] and [2]. We then give counterexamples to confirm our claim.

(a) In [1, Theorem 3.1, p. 190], the authors introduced a characterization of T_1-spaces. The implication (1)$\Longrightarrow$(3) (i.e., “If (X, τ) is a T_1-space, then $cl\dot{x} = \dot{x}$ for each $x \in X$”) is not necessarily true, as we show in the following example.

Example 1. Let $L = [0, 1]$, $X = \{x, y\}$, $\tau = \{\bar{0}, \bar{1}, x_1, y_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}\}$ and $\tau' = \{\bar{0}, \bar{1}, y_1, x_1 \vee y_{\frac{1}{2}}, y_{\frac{1}{2}}\}$. Then (X, τ) is T_1. However, $cl\dot{y} \neq \dot{y}$. Indeed, one can find $f = x_1 \vee y_{\frac{3}{4}} \in L^X$ such that $cl\dot{y}(f) = \bigvee_{cl_\tau \rho\leq x_1\vee y_{\frac{3}{4}}} \dot{y}(\rho) = \dot{y}(x_1 \vee y_{\frac{1}{2}}) = \frac{1}{2} \neq \frac{3}{4} = \dot{y}(x_1 \vee y_{\frac{3}{4}})$.

(b) In [2, p. 203], Lemma 5.1 states that “for every fuzzy topological space (X, τ) and each $x \in X$ we have $cl\dot{x} = \dot{x}$ implies $cl_\tau\{x\} = \{x\}$”. This statement has been used as a sufficient condition to prove that: (*i*) “every T_3-space is a T_2-space” (see [2, Proposition 5.1, p. 203]), and (*ii*) “every T_4-space is a T_3-space” (see [2, Proposition 6.1, p. 209]). In fact, the condition $cl_\tau\{x\} = \{x\}$ for all $x \in X$ is not equivalent to T_1-spaces.
In Example 1, (X, τ) is a T_1-space but there exists $x \in X$ such that $cl_\tau x_1 = x_1 \vee y_{\frac{1}{2}} \neq x_1$. Hence, (*i*) and (*ii*) are not necessarily true.

(c) In [2, Theorem 5.1, p. 203], a characterization of regular spaces has been introduced (see (1)$\Longrightarrow$(3), i.e., “if (X, τ) is a regular space, then $cl\mathcal{N}(x) = \mathcal{N}(x)$ for each $x \in X$”. In fact this result is not correct as we show in the following example.

Example 2. Let $L = [0, 1]$, $X = \{x, y, z\}$, $\tau = \{\bar{0}, \bar{1}, z_1, x_1 \vee y_1 \vee z_{\frac{1}{2}}, z_{\frac{1}{2}}\}$ and $\tau' = \{\bar{0}, \bar{1}, x_1 \vee y_1, z_{\frac{1}{2}}, x_1 \vee y_1 \vee z_{\frac{1}{2}}\}$. We next show that (X, τ) is a regular space. Observe that the only closed fuzzy set in X is $F = \{x, y\}$ with $z \notin F$. Note also that $\mathcal{N}(z) \wedge \mathcal{N}(F)$ does not exist. Indeed, if $f = z_1$ and $g = x_1 \vee y_1 \vee z_{\frac{1}{2}}$ then $\mathcal{N}(z)(z_1) \wedge \mathcal{N}(F)(x_1 \vee y_1 \vee z_{\frac{1}{2}}) = 1 > \frac{1}{2} = sup(f \wedge g)$. However, $cl\mathcal{N}(x) \neq \mathcal{N}(x)$ for some $x \in X$. For instance, take $z \in X$ with $f = z_1$, then $cl\mathcal{N}(z)(f) = \frac{1}{2} \neq 1 = \mathcal{N}(z)(f)$.

(d) The last claim, a characterization of normal spaces is given in [2, Theorem 6.1, p. 209]. More specifically, the implication of (1)$\Longrightarrow$(3) (i.e., “if (X, τ) is a normal space, then $cl\mathcal{N}(F) = \mathcal{N}(F)$ for all $F \in P(X)$ with $F = clF$ ”). This result is incorrect as the next example shows.

Example 3. Let $L = [0, 1]$, $X = \{x, y\}$, $\tau = \{\bar{0}, \bar{1}, x_1, y_1, x_{\frac{1}{3}}, y_{\frac{1}{3}}, x_{\frac{1}{3}} \vee y_{\frac{1}{3}}, x_1 \vee y_{\frac{1}{3}}, x_{\frac{1}{3}} \vee y_1\}$ and $\tau' = \{\bar{0}, \bar{1}, y_1, x_1,$

$x_{\frac{2}{3}} \vee y_1, x_1 \vee y_{\frac{2}{3}}, x_{\frac{2}{3}} \vee y_{\frac{2}{3}}, x_{\frac{2}{3}}, y_{\frac{2}{3}}\}$. Then, (X, τ) is a normal space because the only closed fuzzy sets in X are $F_1 = \{x\}$ and $F_2 = \{y\}$ such that $F_1 \cap F_2 = \varnothing$ and $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist. For example, if we take $f = x_1$ and $g = y_1$, then $\mathcal{N}(F_1)(x_1) \wedge \mathcal{N}(F_2)(y_1) = 1 > 0 = sup(f \wedge g)$. However, $cl\mathcal{N}(F) \neq \mathcal{N}(F)$ for some $F \in P(X)$. For instance, if we take $F = \{x\}$ and $f = x_{\frac{1}{3}} \vee y_{\frac{1}{3}}$ this implies that $cl\mathcal{N}(F)(f) = 0 \neq \frac{1}{3} = \mathcal{N}(F)(f)$.

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