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Original Article

On retracting properties and covering homotopy theorem for S-maps into S_χ-cofibrations and S_χ-fibrations

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Abstract In this paper we generalize the retracting property in homotopy theory for topological semigroups by introducing the notions of deformation S-retraction with its weaker forms and ES-homotopy extension property. Furthermore, the covering homotopy theorems for S-maps into S_χ-fibrations and S_χ-cofibrations are introduced and pullbacks for S_χ-fibrations behave properly.

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1. Introduction

The homotopy theory is an important part of mathematics which has many applications and numerous variants, generalizations, and adaptations. It has been improved to the shape theory in order to deal better with spaces with poor local properties. The concepts of Hurewicz fibrations [3] and retractions [1] have p layed very important roles for investigating the mutual relations among the topological spaces.

Under the notion of homotopy theory for topological spaces, Cerin in [2] introduced the definition of homotopy theory for topological semigroups. He extended some basic properties in homotopy theory to their analogous structures in homotopy theory for topological semigroups such as S-retraction, K-retraction, S-homotopically domination, S_χ-fibration and S_χ-cofibration.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we give the concepts of deformation S-retract, deformation K-retract, strong deformation S-retraction, and ES-homotopy extension property. The S_χ-fibrations and S_χ-cofibrations played very important roles for investigating the mutual relations of among these concepts. In

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Section 4 we introduce the covering homotopy theorems for S-maps into S_χ-fibrations and S_χ-cofibrations. We prove the pullbacks for S_χ-fibrations are S_χ-fibrations.

2. Preliminaries

In this section we provide some preliminary works that serve as background for the present study which were previously established by Cerin, in [2].

A *topological semigroup* or *S-space* is a pair (S, a) consisting a topological space S and a map (i.e., a continuous function) a: $S \times S \to S$ such that $a(x, a(y, z)) = a(a(x, y), z)$ for all $x, y, z \in S$. Let χ denotes the class of all S-spaces.

For every space S, the *natural S-space* is S-space (S, π_i), where π_i is a continuous associative multiplication on S given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in S$. We denote the class of all natural S-spaces (S, π) by $\mathcal{N}_\pi$, where $\pi = \pi_1, \pi_2$.

S-space (B, c) is called an *S-subspace* of (S, a) if B is a subspace of S and the map a takes the product $B \times B$ into B and $c(x, y) = a(x, y)$ for all $x, y \in B$. It is natural to denote the multiplication of S-subspace with the same symbol used for the multiplication on the S-space under consideration.

Let (S, a) and (O, e) be two S-spaces. The function f: $(S, a) \to (O, e)$ is called a *homomorphism* or *an S-map* if f is a map of a space S into O and $f(a(x, y)) = e(f(x), f(y))$ for all $x, y \in S$. Recall [2] that the usual composition and the usual product of two S-maps are S-maps.

For every a space S, by $P(S)$ we mean the space of all paths from the unit closed interval $I = [0, 1]$ into S with the compact-open topology. Recall [2] that for every S-space (S, a), $(P(S), \underline{a})$ is S-space where $\underline{a} : P(S) \times P(S) \to P(S)$ is a map defined by $\underline{a}(\alpha, \beta)(t) = a(\alpha(t), \beta(t))$ for all $\alpha, \beta \in P(S)$, $t \in I$. The shorter notion for this S-space will be $P(S, a)$.

Definition 2.1. The S-maps f, g: $(S, a) \to (O, e)$ are called S-homotopic and write $f \simeq {}_s g$ provided there is S-map H: $(S, a) \to P(O, e)$ called S-homotopy such that $H(s)(0) = f(s)$ and $H(s)(1) = g(s)$ for all $s \in S$.

Theorem 2.2. *The relation of S-homotopy $\simeq_s$ is an equivalence relation on the set of all S-maps of (S, a) into (O, e).*

Theorem 2.3. *If the S-maps f, g: $(S, a) \to (O, e)$ are S-homotopic then the relations $f \circ h \simeq {}_s g \circ h$ and $k \circ f \simeq {}_s k \circ g$ hold for all S-maps h into (S, a) and k from (O, e).*

Recall [2] that if the S-maps f, g: $(S, a) \to (O, e)$ are S-homotopic then the maps f, g: $S \to O$ are homotopic and the S-maps f, g: $(S, \pi) \to (O, \pi)$ are S-homotopic if and only if the maps f, g: $S \to O$ are homotopic.

Throughout this paper, for every S-homotopy H: $(S, a) \to P(O, e)$ and for every $t \in I$, by H_t (or $[H]_t$) we mean the S-map, [2], H_t: $(S, a) \to (O, e)$ which given by $H_t(s) = H(s)(t)$ for all $s \in S$. Also for every S-homotopy H: $(S, a) \to P[P(O), \underline{e}]$ and for every $r, t \in I$, by H_{rt} (or $[H]_{rt}$) we mean the S-map H_{rt}: $(S, a) \to (O, e)$ which given by $H_{rt}(s) = [H(s)(r)](t)$ for all $s \in S$.

Definition 2.4. S-map f: $(S, a) \to (O, e)$ is called S_χ-fibration if for every space $(X, c) \in \chi$, S-map g: $(X, c) \to (S, a)$, and S-homotopy G: $(X, c) \to P(O, e)$ with $G_0 = f \circ g$, there is S-homotopy H: $(X, c) \to P(S, a)$ such that $H_0 = g$ and $f \circ H_t = G_t$ for all $t \in I$.

Recall [2] that the map f: $S \to O$ is a Hurewicz fibration if and only if the S-map f: $(S, \pi) \to (O, \pi)$ is $S_{\mathcal{N}_\pi}$–fibration.

Definition 2.5. *S-map f: $(S, a) \to (O, e)$ is called* S_χ-cofibration*if for every space $(X, c) \in \chi$, S-map g: $(O, e) \to (X, c)$, and S-homotopy G: $(S, a) \to P(X, c)$ with $G_0 = g \circ f$, there is S-homotopy H: $(O, e) \to P(X, c)$ such that $H_0 = g$ and $H_t \circ f = G_t$ for all $t \in I$.*

Recall [2] that the map f: $S \to O$ is a cofibration if and only if the S-map f: $(S, \pi) \to (O, \pi)$ is $S_{\mathcal{N}_\pi}$–cofibration.

Definition 2.6. *An S-subspace (B, a) of S-space (S, a) is called* S-retract*of (S, a) if there exists S-map R: $(S, a) \to (B, a)$ such that $R(s) = s$ for all $s \in B$. The S-map R is called* S-retraction*of (S, a) onto (B, a).*

Throughout this paper, j: $(B, a) \to (S, a)$ will denote to the inclusion S-map for every S-subspace (B, a) of S-space (S, a) and id the identity S-map.

Definition 2.7. An S-subspace (B, a) of S-space (S, a) is called K-retract of (S, a) if there exists S-map r: $(S, a) \to (B, a)$ such that $r \circ j \simeq {}_s id_B$. The S-map r is called K-retraction of (S, a) onto (B, a).

Notice that S-retract is an K-retract. The converse of the first claim is not true in general. In the following theorem, [2] proved a sufficient condition.

Theorem 2.8. *Let (B, a) be S-subspace of S-space (S, a) such that the inclusion S-map j: $(B, a) \to (S, a)$ is $S_{\{(B, a)\}}$-cofibration. Then (B, a) is S-retract of (S, a) if and only if (B, a) is K-retract of (S, a).*

3. Deformation S-retractions

Definition 3.1. An S-subspace (B, a) of S-space (S, a) is called a *deformation S-retract* of (S, a) if there exists S-retraction map R: $(S, a) \to (B, a)$ of (S, a) onto (B, a) such that $j \circ R \simeq {}_s id_S$. The S-homotopy between $j \circ R$ and id_S is called a deformation S-retraction.

Example 3.2. Let (S, a) be S-space and $s_o \in S$ be an idempotent element of (S, a) (i.e., $s_o a s_o = s_o$). Let

$$L(S, s_o) = \{\alpha \in P(S) : \alpha(0) = s_o\} \subset P(S)$$

and $\widetilde{s_o}$ be the constant path at s_o in $L(S, s_o)$. For every $\alpha, \beta \in L(S, s_o)$,

$$(\alpha \underline{a} \beta)(0) = \alpha(0) a \beta(0) = s_o a s_o = s_o.$$

That is, a pair $(L(S, s_o), \underline{a})$ is S-subspace of $P(S, a)$. Similarly, $(\{\widetilde{s_o}\}, \underline{a})$ is S-subspace of $(L(S, s_o), \underline{a})$. Define the S-retraction $R : (L(S, s_o), \underline{a}) \to (\{\widetilde{s_o}\}, \underline{a})$ by $F(\alpha) = \widetilde{s_o}$ for all $\alpha \in L(S, s_o)$. $(\{\widetilde{s_o}\}, \underline{a})$ is a deformation S-retract of $(L(S, s_o), \underline{a})$ such that $id_{L(S,s_o)} \simeq_s j \circ R$ by a deformation S-retraction F: $(L(S, s_o), \underline{a}) \to P(L(S, s_o), \underline{a})$ given by $F_{rt}(\alpha) = \alpha(r(1 - t))$ for all $r, t \in I$, $\alpha \in L(S, s_o)$, where $j : (\{\widetilde{s_o}\}, \underline{a}) \to (L(S, s_o), \underline{a})$ is the inclusion S-map.

The S-map f: $(S, a) \to (O, e)$ is called *S-homotopy equivalence* if there exists S-map g: $(O, e) \to (S, a)$ such that $f \circ g \simeq {}_s id_O$ and

$g \circ f \simeq {}_s id_S$. An S-subspace (B, a) of S-space (S, a) is called a *deformation K-retract* of (S, a) if the inclusion S-map j: $(B, a) \to (S, a)$ is S-homotopy equivalence.

Notice that a deformation S-retract is a deformation K-retract. Moreover, a deformation K-retract is a deformation H-retract (called a weak deformation retract in [[4], p. 30]). The converse of the last claim holds for multiplications π but fail in general, see Example (7) in ([4], P. 30), for the natural S-spaces (X, π) and (A, π). In the following theorem, we shall identify a sufficient condition when the converse of the first claim is true.

Theorem 3.3. *Let (B, a) be S-subspace of S-space (S, a) such that the inclusion S-map j: $(B, a) \to (S, a)$ is $S_{\{(B,a)\}}$-cofibration. Then (B, a) is a deformation S-retract of (S, a) if and only if (B, a) is a deformation K-retract of (S, a).*

Proof. We already noticed that the (only if) part is always true, it remains to show the (if) part. Since (B, a) is a deformation K-retract of (S, a), then there exists S-map r: $(S, a) \to (B, a)$ such that $r \circ j \simeq {}_s id_B$ and $j \circ r \simeq {}_s id_S$. For the first part, there exists S-homotopy F: $(B, a) \to P(B, a)$ such that $F_0 = r \circ j$ and $F_1 = id_B$. By hypothesis, there exists S-homotopy H: $(S, a) \to P(B, a)$ such that $H_0 = r$ and $H_t \circ j = F_t$ for all $t \in I$. Define the S-retraction R: $(S, a) \to (B, a)$ of (S, a) onto (B, a) by $R(s) = H(s)(1)$ for all $s \in S$. Note that for all $s \in B$, $R(s) = H(s)(1) = F(s)(1) = s$.

For the second part $j \circ r \simeq {}_s id_S$, there exists S-homotopy G: $(S, a) \to P(S, a)$ such that $G_1 = j \circ r$ and $G_0 = id_S$. Since $G_1 = j \circ r = j \circ H_0$, then we can define S-homotopy H': $(S, a) \to P(S, a)$ by

$$H'(s)(t) = \begin{cases} G(s)(2t) & \text{for all } t \in [0, 1/2], s \in S; \\ j[H(s)(2t-1)] & \text{for all } t \in [1/2, 1], s \in S. \end{cases}$$

Note that $H'_0 = G_0 = id_S$ and $H'_1 = j \circ H_1 = j \circ R$. That is, $j \circ R \simeq {}_s id_S$. Hence (B, a) is a deformation S-retract of (S, a). □

In Definition (3.1), the S-homotopy between $j \circ r$ and id_S, say F: $(S, a) \to P(S, a)$, is called a *strong deformation S-retraction* if $F(s)(t) = s$ for all $s \in B$, $t \in I$ and we say (B, a) is a *strong deformation S-retract* of (S, a).

In Example (3.2), F is a strong deformation S-retraction such that

$$F_{rt}(\widetilde{s_o}) = \widetilde{s_o}(r(1-t)) = s_o = \widetilde{s_o}(r)$$

for all $r, t \in I$.

One can easily check that a strong deformation S-retract is a deformation S-retract. The converse of this claim fail in general, see Example (8) in ([4], P. 30), for the natural S-spaces (X, π) and (A, π). In Theorem (3.7), we shall identify a sufficient condition when the converse of the first claim is true.

Definition 3.4. Let (B, a) be an S-subspace of S-space (S, a) and (O, e) be any S-space. An S-homotopy G: $(B, a) \to P[P(O), \underline{e}]$ is called a S_{01}-extended map to (S, a) provided for every $t \in I$, the two S-maps G_{0t}, G_{1t}: $(B, a) \to (O, e)$ have extension S-maps to S, denoted by EG_{0t}, EG_{1t}: $(S, a) \to (O, e)$, respectively.

For every a closed subspace B of a space S, S_B^{01} will be denote to the closed subspace $(S \times \{0\}) \cup (B \times I) \cup (S \times \{1\})$ of $S \times I$. In the above definition, for every $(s, r) \in S_B^{01}$, *E_{sr}-path in O induced by G* (denoted E_{sr}^G) is a path in O given by

$$E_{sr}^G(t) = \begin{cases} EG_{0t}(s) & s \in S, r = 0; \\ G_{rt}(s) & s \in B, r \in I; \\ EG_{1t}(s) & s \in S, r = 1 \end{cases}$$

for all $t \in I$. Note that E_{sr}^G is a continuous, since B is a closed subspace of S.

Definition 3.5. A closed S-subspace (B, a) of S-space (S, a) is said to have ES-homotopy extension property in (S, a) with respect to (O, e) if, given S-homotopy g: $(S, a) \to P(O, e)$ and S_{01}-extended map G: $(B, a) \to P[P(O), \underline{e}]$ to S with $E_{sr}^G(0) = g(s)(r)$ for all $(s, r) \in S_B^{01}$, there exists S-homotopy H: $(S, a) \to P[P(O), \underline{e}]$ such that $H_{r0} = g_r$ for all $r \in I$ and $H_{rt}(s) = E_{sr}^G(t)$ for all $(s, r) \in S_B^{01}$, $t \in I$.

Example 3.6. Let (S, a) be any S-space and (B, a) be any closed S-subspace of (S, a). Let $s_o \in S$ be an idempotent element of (S, a). Then (B, a) has ES-homotopy extension property in (S, a) with respect to $(\{s_o\}, a)$. Note that we have only one S-homotopy g: $(S, a) \to P(\{s_o\}, a)$ given by $g(s) = \widetilde{s_o}$ for all $s \in S$ and one S_{01}-extended map G: $(B, a) \to P[P(\{s_o\}), \underline{a}]$ to S given by $G(s)(r) = \widetilde{s_o}$ for all $s \in B$ with extensions $EG_{0t}(s) = s_o$ and $EG_{1t}(s) = s_o$ for all $s \in S$, $t \in I$. For every $(s, r) \in S_B^{01}$, $E_{sr}^G = \widetilde{s_o}$ and we observe that

$$E_{sr}^G(0) = \widetilde{s_o}(0) = s_o = \widetilde{s_o}(r) = g(s)(r).$$

Define S-homotopy H: $(S, a) \to P[P(\{s_o\}), \underline{a}]$ by $H_{rt}(s) = s_o$ for all $s \in S$ and $r, t \in I$. Note that $H_{r0} = g_r$ for all $r \in I$ and $H_{rt}(s) = E_{sr}^G(t)$ for all $(s, r) \in S_B^{01}$, $t \in I$.

Theorem 3.7. *Let (B, a) be a closed S-subspace of S-space (S, a) such that (B, a) has ES-homotopy extension property in (S, a) with respect to (S, a). Then (B, a) is a strong deformation S-retract of (S, a) if and only if (B, a) is a deformation S-retract of (S, a).*

Proof. We already noticed above that the (only if) part is always true, it remains to show the (if) part. Since (B, a) is a deformation S-retract of (S, a), there exist S-retraction map R: $(S, a) \to (B, a)$ and S-homotopy F: $(S, a) \to P(S, a)$ such that $F_0 = id_S$ and $F_1 = j \circ R$. Define S-homotopy G: $(B, a) \to P[P(S), \underline{a}]$ by

$$G_{rt}(s) = F(s)(r(1-t))$$

for all $r, t \in I$, $s \in B$. For every $t \in I$, define S-maps EG_{0t}, EG_{1t}: $(S, a) \to (S, a)$ by

$$EG_{0t}(s) = s, \ \ EG_{1t}(s) = F(R(s))(1-t)$$

for all $s \in S$, respectively. Note that for every $t \in I$,

$$G_{0t}(s) = F(s)(0) = s = EG_{0t}(s)$$

and since R is S-retraction of (S, a) onto (B, a), then

$$G_{1t}(s) = F(s)(1-t) = F(R(s))(1-t) = EG_{1t}(s)$$

for all $s \in B$. Then EG_{0t} and EG_{1t} are extension S-maps of G_{0t} and G_{1t} to S, respectively. That is, S-homotopy G is S_{01}-extended map to S. For every $(s, r) \in S_B^{01}$, the E_{sr}-path in O induced by G is given by

$$E_{sr}^G(t) = \begin{cases} s & s \in S, r = 0; \\ F(s)(r(1-t)) & s \in B, r \in I; \\ F(R(s))(1-t) & s \in S, r = 1 \end{cases}$$

for all $t \in I$.

Note that $E_{sr}^G(0) = F(s)(r)$ for all $s \in B, r \in I$, $E_{s0}^G(0) = s = F(s)(0)$, and

$$E_{s1}^G(0) = F(R(s))(1) = (j \circ R)(R(s)) = j[R(R(s))] \\ = j[R(s)] = F(s)(1)$$

for all $s \in S$. That is, $E_{sr}^G(0) = F(s)(r)$ for all $(s, r) \in S_B^{01}$. Since (B, a) has ES-homotopy extension property in (S, a) w.r.t (S, a), then there exists S-homotopy H: $(S, a) \to P[P(S), \underline{a}\,]$ such that $H_{r0} = F_r$ for all $r \in I$ and $H_{rt}(s) = E_{sr}^G(t)$ for all $(s, r) \in S_B^{01}$, $t \in I$.

Define S-homotopy F': $(S, a) \to P(S, a)$ by $F'(s)(r) = H_{r1}(s)$ for all $r \in I$, $s \in S$. Note that

$$F'(s)(0) = H_{01}(s) = E_{s0}^G(1) = s$$

and

$$F'(s)(1) = H_{11}(s) = E_{s1}^G(1) = F(R(s))(0) = R(s) = (j \circ R)(s)$$

for all $s \in S$. That is, F' is S-homotopy between id_S and $j \circ R$. Since R is S-retraction, then F' is a deformation S-retraction. For a strong property, we note that for every $s \in B$, $r \in I$,

$$F'(s)(r) = H_{r1}(s) = E_{sr}^G(1) = F(s)(0) = s.$$

Hence (B, a) is a strong deformation S-retract of (S, a). □

In the following theorem, recall [2] that the function f: $S \to O$ of a natural S-space (S, π) into (O, π) is S-map if and only if it is continuous.

Theorem 3.8. *Let (B, π) be a closed S-subspace of S-space (S, π). Then (B, π) has ES-homotopy extension property in (S, π) w.r.t any S-space $(O, \pi) \in \mathcal{N}_\pi$ if and only if the inclusion S-map $j : (S_B^{01}, \pi) \to (S \times I, \pi)$ is $S_{\mathcal{N}_\pi}$-cofibration.*

Proof. Suppose (B, π) has ES-homotopy extension property in (S, π) with respect to S-space (O, π). Let g': $(S \times I, \pi) \to (O, \pi)$ be S-map and $G' : (S_B^{01}, \pi) \to P(O, \pi)$ be S-homotopy with $G'_0 = g' \circ j$. Define S-homotopy G: $(B, \pi) \to P[P(O), \underline{\pi}\,]$ by $G_{rt}(s) = G'(s, r)(t)$ for all $r, t \in I$, $s \in B$. For every $t \in I$, define S-maps EG_{0t}, EG_{1t}: $(S, \pi) \to (O, \pi)$ by

$$EG_{0t}(s) = G'((s, 0), t), \;\; EG_{1t}(s) = G'((s, 1), t)$$

for all $s \in S$, respectively. Note that for every $t \in I$, EG_{0t} and EG_{1t} are extension S-maps of G_{0t} and G_{1t} to S, respectively. That is, S-homotopy G is S_{01}-extended map to S. For every $(s, r) \in S_B^{01}$, the E_{sr}-path in O induced by G is given by $E_{sr}^G(t) = G'(s, r)(t)$ for all $t \in I$.

Define S-map g: $(S, \pi) \to P(O, \pi)$ by $g(s)(r) = g'(s, r)$ for all $s \in S$, $r \in I$. Note that

$$E_{sr}^G(0) = G'((s, r), 0) = g'(s, r) = g(s)(r)$$

for all $(s, r) \in S_B^{01}$. Then there exists S-homotopy H: $(S, \pi) \to P[P(O), \underline{\pi}\,]$ such that $H_{r0}(s) = g_r(s) = g'(s, r)$ for all $r \in I$, $s \in S$ and $H_{rt}(s) = E_{sr}^G(t) = G'(s, r)(t)$ for all $(s, r) \in S_B^{01}$, $t \in I$. Hence j is $S_{\mathcal{N}_\pi}$-cofibration.

Conversely, suppose $j : (S_B^{01}, \pi) \to (S \times I, \pi)$ is an $S_{\mathcal{N}_\pi}$-cofibration. Let g: $(S, \pi) \to P(O, \pi)$ be S-homotopy and G: $(B, \pi) \to P[P(O), \underline{\pi}\,]$ be S_{01}-extended map to (S, π) with $E_{sr}^G(0) = g(s)(r)$ for all $(s, r) \in S_B^{01}$. Define S-map g': $(S \times I, \pi) \to (O, \pi)$ by $g'(s, r) = g(s)(r)$ for all $r \in I$, $s \in S$ and define S-homotopy $G' : (S_B^{01}, \pi) \to P(O, \pi)$ by $G'(s, r)(t) = E_{sr}^G(t)$ for all $(s, r) \in S_B^{01}$, $t \in I$. Note that

$$G'(s, r)(0) = E_{sr}^G(0) = g(s)(r) = g'(s, r)$$

for all $(s, r) \in S_B^{01}$. That is, $G'_0 = g' \circ j$. Since j is $S_{\mathcal{N}_\pi}$-cofibration, then there exists S-homotopy H': $(S \times I, \pi) \to P(O, \pi)$ such that $H'_0 = g'$ and $H' \circ j = G'$. Then the desired S-homotopy H: $(S, \pi) \to P[P(O), \underline{\pi}\,]$ is defined by $H_{rt}(s) = H'(s, r)(t)$ for all $r, t \in I$, $s \in S$. □

In the following theorem, we show the role of S_χ-fibrations in finding the extensions S-maps with a deformation S-retract property.

Theorem 3.9. *Let f: $(S, a) \to (O, e)$ be S_χ-fibration. Let (B, c) be S-subspace of S-space (X, c) such that (B, c) is a deformation S-retract of (X, c). If h: $(B, c) \to (S, a)$ and k: $(X, c) \to (O, e)$ are S-maps such that $f \circ h = kB$, then there exists S-map h': $(X, c) \to (S, a)$ such that $f \circ h' = k$ and $h' \mid B \simeq {}_s h$.*

Proof. Since (B, c) is a deformation S-retract of (X, c), then there exist S-retraction map R: $(X, c) \to (B, c)$ and S-homotopy F: $(X, c) \to P(X, c)$ such that $F_0 = j \circ R$ and $F_1 = id_X$. Define S-map g: $(X, c) \to (S, a)$ and S-homotopy G: $(X, c) \to P(O, e)$ by $g = h \circ R$ and $G_t = k \circ F_t$ for all $t \in I$, respectively. Note that

$$G_0 = k \circ F_0 = k \circ (j \circ R) = (k \circ j) \circ R = k|B \circ R \\ = (f \circ h) \circ R = f \circ g.$$

Since f: $(S, a) \to (O, e)$ be S_χ-fibration, then there exists S-homotopy H: $(X, c) \to P(S, a)$ such that $H_0 = g$ and $f \circ H_t = G_t$ for all $t \in I$. Define h': $(X, c) \to (S, a)$ by $h' = H_1$. Note that

$$f \circ h' = f \circ H_1 = G_1 = k \circ F_1 = k$$

and for all $x \in B$,

$$H_0(x) = g(x) = (h \circ R)(x) = h(R(x)) = h(x).$$

Since $h' = H_1$, then $h \simeq {}_s h' \mid B$ by S-homotopy $H|B$: $(B, c) \to P(S, a)$. □

4. Covering homotopy theorem

The main results of this section are covering homotopy theorems for S-maps into S_χ-fibrations and into S_χ-cofibrations.

Recall [2] that for every S-map f: $(S, a) \to (O, e)$, $\widehat{f}$: $P(S, a) \to P(O, e)$ is S-map given by $\widehat{f}(\alpha) = f \circ \alpha$ for all $\alpha \in P(S, a)$. By $\mathcal{P}_1$ and $\mathcal{P}_2$ we mean the usual first and the second projection maps (or S-maps), respectively.

Theorem 4.1. *Let f: $(S, a) \to (O, e)$ be S_χ-fibration and let h, h': $(X, c) \to P(S, a)$ be two S-maps. Let $h_0 \simeq_s h'_0$ and $\widehat{f} \circ h \simeq_s \widehat{f} \circ h'$*

by S-homotopies K*:* $(X, c) \to P(S, a)$ *and* G*:* $(X, c) \to P[P(O), \underline{e}\,]$*, respectively. If* $G_{0t} = f \circ K_t$ *for all* $t \in I$*, then there exists S-homotopy* H*:* $(X, c) \to P[P(S), \underline{a}\,]$ *between* h *and* h' *such that* $H_{0t} = K_t$ *and* $f \circ H_{rt} = G_{rt}$ *for all* $r, t \in I$*.*

Proof. Let $M = (I \times \{0\}) \cup (\{0\} \times I) \cup (I \times \{1\}) \subset I \times I$. For every $(r, t) \in M$, define S-map $\Gamma^{(r,t)}$: $(X, c) \to (S, a)$ by

$$\Gamma^{(r,t)}(x) = \begin{cases} h(x)(r) & t = 0; \\ K(x)(t) & r = 0; \\ h'(x)(r) & t = 1 \end{cases}$$

for all $x \in X$. Recall ([4], P. 100) that there is a homeomorphism λ: $I \times I \to I \times I$ taking M onto $I \times \{0\}$. By hypothesis, note that for every $(r, t) \in M$,

$$(f \circ \Gamma^{(r,t)})(x) = G_{rt}(x) = (G(x)(r))(t)$$

for all $x \in X$. For every $r \in I$, define an S-map g^r: $(X, c) \to (S, a)$ and S-homotopy G^r: $(X, c) \to P(O, e)$ by $g^r(x) = \Gamma^{\lambda^{-1}(r,0)}(x)$ and

$$G^r(x)(t) = [G(x)(\mathcal{P}_1[\lambda^{-1}(r, t)])](\mathcal{P}_2[\lambda^{-1}(r, t)])$$

for all $x \in X$, $t \in I$, respectively. Note that for every $r \in I$,

$$\begin{aligned} G^r(x)(0) &= (G(x)(\mathcal{P}_1[\lambda^{-1}(r, 0)]))(\mathcal{P}_2[\lambda^{-1}(r, 0)]) \\ &= (f \circ \Gamma^{(\mathcal{P}_1[\lambda^{-1}(r,0)], \mathcal{P}_2[\lambda^{-1}(r,0)])})(x) \\ &= (f \circ \Gamma^{\lambda^{-1}(r,0)})(x) = (f \circ g^r)(x) \end{aligned}$$

for all $x \in X$. That is, $G^r_0 = f \circ g^r$. Then for every $r \in I$, since f is S_χ-fibration, there exists S-homotopy H^r: $(X, c) \to P(S, a)$ such that $H^r_0 = g^r$ and $f \circ H^r_t = G^r_t$ for all $t \in I$. Define S-homotopy H: $(X, c)) \to P[P(S), \underline{a}\,]$ by

$$(H(x)(r))(t) = H^{\mathcal{P}_1[\lambda(r,t)]}(x)(\mathcal{P}_2[\lambda(r, t)])$$

for all $x \in X$, $r, t \in I$. Note that

$$\begin{aligned} (H(x)(r))(0) &= H^{\mathcal{P}_1[\lambda(r,0)]}(x)(\mathcal{P}_2[\lambda(r, 0)]) = H^{\mathcal{P}_1[\lambda(r,0)]}(x)(0) \\ &= g^{\mathcal{P}_1[\lambda(r,0)]}(x) \\ &= \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(r,0)],0)}(x) = \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(r,0)], \mathcal{P}_2[\lambda(r,0)])}(x) \\ &= \Gamma^{\lambda^{-1}(\lambda(r,0))}(x) = \Gamma^{(r,0)}(x) = h(x)(r) \end{aligned}$$

and

$$\begin{aligned} (H(x)(r))(1) &= H^{\mathcal{P}_1[\lambda(r,1)]}(x)(\mathcal{P}_2[\lambda(r, 1)]) = H^{\mathcal{P}_1[\lambda(r,1)]}(x)(0) \\ &= g^{\mathcal{P}_1[\lambda(r,1)]}(x) \\ &= \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(r,1)],0)}(x) = \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(r,1)], \mathcal{P}_2[\lambda(r,1)])}(x) \\ &= \Gamma^{\lambda^{-1}(\lambda(r,1))}(x) = \Gamma^{(r,1)}(x) = h'(x)(r) \end{aligned}$$

for all $x \in X$, $r \in I$. Then H is S-homotopy between h and h'. Also note that

$$\begin{aligned} H_{ot}(x) &= (H(x)(0))(t) = H^{\mathcal{P}_1[\lambda(0,t)]}(x)(\mathcal{P}_2[\lambda(0, t)]) \\ &= H^{\mathcal{P}_1[\lambda(r,0)]}(x)(0) \\ &= g^{\mathcal{P}_1[\lambda(0,t)]}(x) = \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(0,t)],0)}(x) \\ &= \Gamma^{\lambda^{-1}(\mathcal{P}_1[\lambda(0,t)], \mathcal{P}_2[\lambda(0,t)])}(x) \\ &= \Gamma^{\lambda^{-1}(\lambda(0,t))}(x) = \Gamma^{(0,t)}(x) = K_t(x) \end{aligned}$$

and

$$\begin{aligned} (f \circ H_{rt})(x) &= (f \circ H_r(x))(t) = (f \circ H^{\mathcal{P}_1[\lambda(r,t)]}(x))(\mathcal{P}_2[\lambda(r, t)]) \\ &= G^{\mathcal{P}_1[\lambda(r,t)]}(x)(\mathcal{P}_2[\lambda(r, t)]) \\ &= \{G(x)(\mathcal{P}_1[\lambda^{-1}\{\mathcal{P}_1[\lambda(r, t)], \mathcal{P}_2[\lambda(r, t)]\}])\} \\ &\quad (\mathcal{P}_2[\lambda^{-1}\{\mathcal{P}_1[\lambda(r, t)], \mathcal{P}_2[\lambda(r, t)]\}]) \\ &= \{G(x)(\mathcal{P}_1[\lambda^{-1}\{\lambda(r, t)\}])\}(\mathcal{P}_2[\lambda^{-1}\{\lambda(r, t)\}]) \\ &= \{G(x)(\mathcal{P}_1[r, t])\}(\mathcal{P}_2[r, t]) \\ &= (G(x)(r))(t) = G_{rt}(x) \end{aligned}$$

for all $r, t \in I$, $x \in X$. That is, $H_{0t} = K_t$ and $f \circ H_{rt} = G_{rt}$ for all $r, t \in I$. $\square$

Corollary 4.2. *Let* f*:* $(S, a) \to (O, e)$ *be* S_χ*-fibration. Let* h, h'*:* $(X, c) \to P(S, a)$ *be S-maps such that* $h_0 = h'_0$ *and* $\widehat{f} \circ h = \widehat{f} \circ h'$*. Then there exists S-homotopy* H*:* $(X, c) \to P[P(S), \underline{a}\,]$ *between* h *and* h' *such that* $H_{0t} = h_0 = h'_0$ *and* $f \circ H_{rt} = f \circ h_r$ *for all* $r, t \in I$*.*

Proof. Define S-homotopy K: $(X, c) \to P(S, a)$ by $K(x)(t) = h_0(x)$ and S-homotopy G: $(X, c) \to P[P(O), \underline{e}\,]$ by $(G(x)(r))(t) = (f \circ h_r)(x)$ for all $r, t \in I$, $x \in X$. Then by using the above theorem, one can get the desired S-homotopy. $\square$

In the following example, we give some applications for Theorem (4.1).

Example 4.3. The two pairs $(\mathbb{R}, \pi)$ and $(\mathbb{R}^2, \pi)$ are S-spaces with the usual real space $\mathbb{R}$ and the usual product space $\mathbb{R}^2$, respectively. Let $b, b' : (X, \pi) \to (\mathbb{R}, \pi)$ be any two S-maps from any S-space (X, π) into $(\mathbb{R}, \pi)$. Define $S_{\mathcal{N}_\pi}$-fibration $f : (\mathbb{R}^2, \pi) \to (\mathbb{R}, \pi)$ by $f(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Define two S-maps $h, h' : (X, \pi) \to P(\mathbb{R}^2, \pi)$ by

$$h(x)(r) = (b(x), r) \text{ and } h'(x)(r) = (b'(x), 1 - r)$$

for all $x \in X$, $r \in I$. Define S-homotopies $K : (X, \pi) \to P(\mathbb{R}^2, \pi)$ and $G : (X, \pi) \to P[P(\mathbb{R}), \underline{\pi}]$ by

$$\begin{aligned} &K(x)(t) = (tb'(x) + (1 - t)b(x), t) \text{ and} \\ &\quad G_{rt}(x) = tb'(x) + (1 - t)b(x) \end{aligned}$$

for all $x \in X$, $r, t \in I$. Note that $K(x)(0) = (b(x), 0) = h(x)(0)$,

$$\begin{aligned} &K(x)(1) = (b'(x), 1) = h'(x)(0), \\ &\quad G_{r0}(x) = b(x) = [(\widehat{f} \circ h)(x)](r) \end{aligned}$$

and $G_{r1}(x) = b'(x) = [(\widehat{f} \circ h')(x)](r)$ for all $x \in X$, $r, t \in I$. That is, $h_0 \simeq_s h'_0$ and $\widehat{f} \circ h \simeq_s \widehat{f} \circ h'$ by S-homotopies K and G, respectively. Since $G_{0t} = f \circ K_t$ for all $t \in I$, then the desired S-homotopy $H : (X, \pi) \to P[P(\mathbb{R}^2), \underline{\pi}]$ is given by

$$H_{rt}(x) = [tb'(x) + (1 - t)b(x), r + t - 2rt]$$

for all $x \in X$, $r, t \in I$.

Let f: $(S, a) \to (O, e)$ and k: $(O', e') \to (O, e)$ be S-maps. The S-space $(S_k, e' \times a)$ is called a *pullback S-space* of (S, a) induced from f by k where $S_k = \{(x, s) \in O' \times S | k(x) = f(s)\}$. The S-map f^k: $(S_k, e' \times a) \to (O', e')$ which is given by $f^k(x, s) = x$ for all $(x, s) \in S_k$ is called a *pullback S-map* of f induced by k.

One notable exception is that the pullbacks of some fibration types need not be an fibrations such as approximate fibrations.

In the following theorem, we show that the pullbacks of S_χ-fibration maps are S_χ-fibrations.

Theorem 4.4. *Let f: $(S, a) \to (O, e)$ be S_χ-fibration and k: $(O', e') \to (O, e)$ be S-map. Then the pullback f^k of f induced by k is S_χ-fibration.*

Proof. Let $(X, c) \in \chi$, g': $(X, c) \to (S_k, e' \times a)$ be S-map, and G': $(X, c) \to P(O', e')$ be S-homotopy with $G'_0 = f^k \circ g'$. Define S-map g: $(X, c) \to (S, a)$ by $g(x) = \mathcal{P}_2(g'(x))$ and S-homotopy G: $(X, c) \to P(O, e)$ by $G(x) = k \circ G'(x)$ for all $x \in X$. Note that

$$G(x)(0) = (k \circ G'(x))(0) = k(G'(x)(0)) = k[f^k(g'(x))]$$
$$= k(\mathcal{P}_1(g'(x))) = f(\mathcal{P}_2(g'(x))) = f(g(x))$$

for all $x \in X$. That is, $G_0 = f \circ g$. Since f is S_χ-fibration, then there is S-homotopy H: $(X, c) \to P(S, a)$ such that $H_0 = g$ and $f \circ H_t = G_t$ for all $t \in I$.

Define S-homotopy H': $(X, c) \to P(S_k, e' \times a)$ by $H'(x)(t) = [G'(x)(t), H(x)(t)]$ for all $x \in X$, $t \in I$. Note that $f \circ H' = G'$ and

$$H'(x)(0) = [G'(x)(0), H(x)(0)] = [f^k(g'(x)), g(x)]$$
$$= [\mathcal{P}_1(g'(x)), \mathcal{P}_2(g'(x))] = g'(x)$$

for all $x \in X$. That is, $H'_0 = g'$. Hence f^k is S_χ-fibration. □

In the following theorem, we use Corollary (4.2) to show that the pullback S_χ-fibrations, which induced by S-homotopic S-maps, have S-homotopy equivalent total S-spaces.

Theorem 4.5. *Let f: $(S, a) \to (O, e)$ be S_χ-fibration and k, k': $(O', e') \to (O, e)$ be two S-maps. If k and k' are S-homotopic, then the total S-spaces S_k and $S_{k'}$ of pullback S_χ-fibrations f^k: $(S_k, e' \times a) \to (O', e')$ and $f^{k'}$: $(S_{k'}, e' \times a) \to (O', e')$ are S-homotopy equivalent.*

Proof. Define two S-maps d: $(S_k, e' \times a) \to (S, a)$ and d' : $(S_{k'}, e' \times a) \to (S, a)$ by $d(x, s) = s$ and $d'(x', s') = s'$ for all $(x, s) \in S_k$, $(x', s') \in S_{k'}$, where

$$S_k = \{(x, s) \in O' \times S | k(x) = f(s)\},$$
$$S_{k'} = \{(x, s) \in O' \times S | k'(x) = f(s)\},$$

respectively. Note that $f \circ d = k \circ f^k$ and $f \circ d' = k' \circ f^{k'}$. Since k and k' are S-homotopic, then there exists S-homotopy F: $(O', e') \to P(O, e)$ such that $F_0 = k$ and $F_1 = k'$.

Consider S-homotopy $F \circ f^k$: $(S_k, e' \times a) \to P(O, e)$ with S-map d and S-homotopy $F \circ f^{k'}$: $(S_{k'}, e' \times a) \to P(O, e)$ with S-map d'. Since

$$[F \circ f^k]_0 = f \circ d, \ [F \circ f^{k'}]_1 = f \circ d',$$

and f is S_χ-fibration, then there exist two S-homotopies H: $(S_k, e' \times a) \to P(S, a)$ and H' : $(S_{k'}, e' \times a) \to P(S, a)$ such that

$$H_0 = d, \ \widehat{f} \circ H = F \circ f^k, \ H'_1 = d',$$

and $\widehat{f} \circ H' = F \circ f^{k'}$.

Let $\mu : (S_k, e' \times a) \to (S_{k'}, e' \times a)$ and $\mu' : (S_{k'}, e' \times a) \to (S_k, e' \times a)$ be two S-maps defined by the properties $H_1 = d' \circ \mu$ and $H'_0 = d \circ \mu'$, respectively. In Corollary (4.2), take $h = H \circ \mu'$ and $h' = H'$. Note that

$$h_0 = H_0 \circ \mu' = d \circ \mu' = H'_0 = h'_0$$

and

$$f \circ h_t = f \circ H_t \circ \mu' = F_t \circ f^k \circ \mu' = F_t \circ f^{k'} = f \circ H'_t = f \circ h'_t$$

for all $t \in I$. That is, $h_0 = h'_0$ and $\widehat{f} \circ h = \widehat{f} \circ h'$. Then $H \circ \mu' \simeq {}_s H'$. Hence $\mu \circ \mu' \simeq_s id_{S_{k'}}$. Again in Corollary (4.2), take $h = H' \circ \mu$ and $h' = H$. Similarly, we get that $\mu' \circ \mu \simeq_s id_{S_k}$. Hence the total S-spaces S_k and $S_{k'}$ are S-homotopy equivalent. □

The following theorem is the analogous result of Theorem (4.1) in the S_χ-cofibration theory which its proof is similar as the proof of Theorem (4.1).

Theorem 4.6. *Let f: $(S, a) \to (O, e)$ be S_χ-cofibration and let h, h': $(O, e) \to P(X, c)$ be two S-maps. Let $h_0 \simeq_s h'_0$ and $h \circ f \simeq {}_s h' \circ f$ by S-homotopies K: $(O, e) \to P(X, c)$ and G: $(S, a) \to P[P(X), \underline{c}\,]$, respectively. If $G_{0t} = K_t \circ f$ for all $t \in I$, then there exists S-homotopy H: $(O, e) \to P[P(X), \underline{c}\,]$ between h and h' such that $H_{0t} = K_t$ and $H_{rt} \circ f = G_{rt}$ for all $r, t \in I$.*

The proof of following corollary is also similar as the proof of Corollary (4.2).

Corollary 4.7. *Let f: $(S, a) \to (O, e)$ be S_χ-cofibration. Let h, h': $(O, e) \to P(X, c)$ be two S-maps such that $h_0 = h'_0$ and $h \circ f = h' \circ f$. Then there exists S-homotopy H: $(O, e) \to P[P(X), \underline{c}\,]$ between h and h' such that $H_{0t} = h_0 = h'_0$ and $H_{rt} \circ f = h \circ f$ for all $r, t \in I$.*

In the following example, we give some applications for Theorem (4.6) which are the analogous applications of Theorem (4.1) in Example (4.3).

Example 4.8. It's clear that for $n = 1, 2, 3, \ldots$, the S-space $(\mathbb{S}^n, \pi)$ is a closed S-subspace of S-space $(\mathbb{D}^{n+1}, \pi)$ and both of them are closed S-subspace of S-space $(\mathbb{R}^{n+1}, \pi)$, where $\mathbb{S}^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$ is the unit sphere of dimension n, $\mathbb{D}^{n+1} = \{x \in \mathbf{R}^{n+1} : |x| \le 1\}$ is the unit disk of dimension $n + 1$ and $\mathbb{R}^{n+1}$ is the Euclidean space of dimension $n + 1$. Let d, d' : $(\mathbb{D}^{n+1}, \pi) \to (\mathbb{R}, \pi)$ be any two S-maps from S-space $(\mathbb{D}^{n+1}, \pi)$ into the usual real S-space $(\mathbb{R}, \pi)$. It's clear that the inclusion S-map $j : (\mathbb{S}^n, \pi) \to (\mathbb{D}^{n+1}, \pi)$ is $S_{\mathcal{N}_\pi}$-cofibration. Define two S-maps h, h' : $(\mathbb{D}^{n+1}, \pi) \to P(\mathbb{R}, \pi)$ by

$$h(x)(r) = r + d(x) \text{ and } h'(x)(r) = 1 - r + d'(x)$$

for all $x \in \mathbb{D}^{n+1}$, $r \in I$. Define S-homotopies

$$K : (\mathbb{D}^{n+1}, \pi) \to P(\mathbb{R}, \pi) \text{ and } G : (\mathbb{S}^n, \pi) \to P[P(\mathbb{R}), \underline{\pi}]$$

by

$$K(x)(t) = t(1 + d'(x)) + (1 - t)d(x) \text{ and } G_{rt}(y)$$
$$= r + t - 2rt + d(y) - td(y) + td'(y)$$

for all $x \in \mathbb{D}^{n+1}$, $y \in \mathbb{S}^n$, r, $t \in I$. Note that $K(x)(0) = d(x) = h(x)(0)$,

$$K(x)(1) = 1 + d'(x) = h'(x)(0),$$
$$G_{r0}(y) = r + d(y) = [(h \circ j)(y)](r)$$

and $G_{r1}(y) = 1 - r + d'(y) = [(h' \circ j)(y)](r)$ for all $x \in \mathbb{D}^{n+1}$, $y \in \mathbb{S}^n$, $r, t \in I$. That is, $h_0 \simeq_s h'_0$ and $h \circ j \simeq {}_s h' \circ j$ by S-homotopies

K and G, respectively. Since $G_{0t} = K_t \circ j$ for all $t \in I$, then the desired S-homotopy $H : (\mathbb{D}^{n+1}, \pi) \to P[P(\mathbb{R}), \underline{\pi}]$ is given by

$$H_{rt}(x) = r + t - 2rt + d(x) - td(x) + td'(x)$$

for all $x \in \mathbb{D}^{n+1}$, $r, t \in I$.

Theorem 4.9. *Let (B, π) be closed S-subspace of S-space (S, π). The inclusion S-map j: $(B, \pi) \to (S, \pi)$ is $S_{\mathcal{N}_\pi}$-cofibration if and only if (S_B^0, π) is S-retract of $(S \times I, \pi)$, where $S_B^0 = (S \times \{0\}) \cup (B \times I) \subset S \times I$.*

Proof. Let j: $(B, \pi) \to (S, \pi)$ be $S_{\mathcal{N}_\pi}$-cofibration. Define S-map $g : (S, \pi) \to (S_B^0, \pi)$ by $g(s) = (s, 0)$ for all $s \in I$ and define S-homotopy $G : (B, \pi) \to P(S_B^0, \pi)$ by $G(s)(t) = (s, t)$ for all $s \in B$, $t \in I$. Note that $G_0 = g \circ j$, then there is S-homotopy $H : (S, \pi) \to P(S_B^0, \pi)$ such that $H_0 = g$ and $H_t \circ j = G_t$ for all $t \in I$. Then define the S-retraction $R : (S \times I, \pi) \to (S_B^0, \pi)$ by $R(s, t) = H(s)(t)$ for all $(s, t) \in S \times I$. That is, (S_B^0, π) is S-retract of $(S \times I, \pi)$.

Conversely, suppose $R : (S \times I, \pi) \to (S_B^0, \pi)$ is S-retraction. Define S-map $R' : (S, \pi) \to P(S_B^0, \pi)$ by $R'(s)(t) = R(s, t)$ for all $s \in S$, $t \in I$. Then for every an-space $(X, \pi) \in \mathcal{N}_\pi$, S-map g: $(S, \pi) \to (X, \pi)$, and S-homotopy G: $(B, \pi) \to P(X, \pi)$ with $G_0 = g \circ j$, define S-homotopy H: $(S, \pi) \to P(X, \pi)$ by

$$H(s)(t) \begin{cases} (g \circ \mathcal{P}_1)(R(s, t)) & (s, t) \in R^{-1}(S \times \{0\}); \\ (G \circ R')(s)(t) & (s, t) \in R^{-1}(B \times I) \end{cases}$$

for all $s \in S$, $t \in I$. H is continuous, since $S \times \{0\}$ and $B \times I$ are closed subspace of $S \times I$. Then

$$H(s)(0) = (g \circ \mathcal{P}_1)(R(s, 0)) = (g \circ \mathcal{P}_1)(s, 0) = g(s)$$

for all $s \in I$ and

$$(H_t \circ j)(s) = (G_t \circ R'_t)(s) = G_t(s)$$

for all $s \in B$, $t \in I$. Hence j: $(B, \pi) \to (S, \pi)$ is $S_{\mathcal{N}_\pi}$-cofibration. □

Corollary 4.10. *Let (B, π) be closed S-subspace of S-space (S, π). j: $(B, \pi) \to (S, \pi)$ be an inclusion $S_{\mathcal{N}_\pi}$-cofibration. Then its S-retraction $R : (S \times I, \pi) \to (S_B^0, \pi)$ is unique up to S-homotopy.*

Proof. Let $R, R' : (S \times I, \pi) \to (S_B^0, \pi)$ be two S-retractions. Let $X = S_B^0$ and h, h': $(S, \pi) \to P(X, \pi)$ be two S-maps given by $h'(s)(r) = R'(s, r)$ and $h(s)(r) = R(s, r)$ for all $s \in S$, $t \in I$, respectively. Since R and R' are S-retractions of $S \times I$ onto S_B^0, then

$$h(s)(0) = R(s, 0) = (s, 0) = R'(s, 0) = h'(s)(0)$$

for all $s \in S$ and

$$(h \circ j)(s, r) = R(s, r) = (s, r) = R'(s, r) = (h' \circ j)(s, r)$$

for all $s \in B$, $r \in I$. Then by Corollary (4.7), there exists S-homotopy H': $(S, \pi) \to P[P(X), \underline{\pi}\ ]$ between h and h' such that $H'_{0t} = h_0 = h'_0$ and $H'_{rt} \circ j = h \circ j$ for all $r, t \in I$. Define the desired S-homotopy H: $(S \times I, \pi) \to P(X, \pi)$ by $H(s, r)(t) = H'_{rt}(s)$ for all $s \in S$ and $r, t \in I$. □

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