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Original Article

Stress-strength reliability for general bivariate distributions

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Abstract An expression for the stress-strength reliability $R = P(X_1 < X_2)$ is obtained when the vector (X_1, X_2) follows a general bivariate distribution. Such distribution includes bivariate compound Weibull, bivariate compound Gompertz, bivariate compound Pareto, among others. In the parametric case, the maximum likelihood estimates of the parameters and reliability function R are obtained. In the non-parametric case, point and interval estimates of R are developed using Govindarajulu's asymptotic distribution-free method when X_1 and X_2 are dependent. An example is given when the population distribution is bivariate compound Weibull. Simulation is performed, based on different sample sizes to study the performance of estimates.

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1. Introduction

Research on stress-strength model and its generalizations has been collected in [1]. Several papers estimated the stress-strength reliability $R = P(X_1 < X_2)$ when the stress (or supply) X_1 and strength (or demand) X_2 are independent, in the frequentist and Bayes cases. See for example [2-13], among others. Estimation of R when (X_1, X_2) follows a bivariate exponential distribution is discussed in Chapter 3 in [1] and the references therein.

Estimation of R in the non-parametric set up was studied by several authors. See for example [14-19], among others. AL-Hussaini et al. [20] considered parametric estimation of R when X_1 and X_2 are independent and each of which is a finite mixture of lognormal components. Point and interval estimates were obtained and compared in the parametric versus non-parametric cases.

In this paper, R is estimated when the vector (X_1, X_2) follows a general bivariate distribution.

The rest of the paper is organized as follows: A univariate and bivariate distributions are given in Section 2. The model of stress-strength reliability is described in Section 3. Section 4

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deals with maximum likelihood and non-parametric estimations of R. Simulation study with illustrations followed by concluding remarks are given in Section 5.

2. Univariate and bivariate distributions

AL-Hussaini and Ateya [21], constructed multivariate distribution by compounding $L(\theta;\ \mathbf{x})$ with $\pi(\theta)$, where

$$L(\theta;\ \mathbf{x}) = \prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta), \tag{1}$$

$\mathbf{x} = (x_1, \ldots, x_n)$, $\theta \in \Omega$ is a one-dimensional parameter that belongs to a parameter space Ω,

$$f_{X_i|\Theta}(x_i|\theta) = \delta_i \theta\, z'_{\eta_i}(x_i)\ \exp[-\theta\,\delta_i\, z_{\eta_i}(x_i)], \quad 0 \le a < x_i < b \le \infty, \tag{2}$$

$z_{\eta_i}(x_i)$ is such that $f_{X_i|\Theta}(x_i|\theta)$ is a probability density function (PDF), θ, $\eta_i > 0$, a and b are positive real numbers such that a may assume the value 0 and b the value ∞.

The function $\pi(\theta)$ is given by

$$\pi(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta > 0,\ (\alpha,\ \beta > 0). \tag{3}$$

By compounding $L(\theta;\ \mathbf{x})$ with $\pi(\theta)$, given by (1) and (3), we obtain

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \int_0^{\infty} L(\theta;\ \mathbf{x})\,\pi(\theta)\,d\theta = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\left[\prod_{i=1}^{n}\gamma_i z'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{n}\gamma_i z_{\eta_i}(x_i)\right]^{-\alpha-n}, \tag{4}$$

where

$\gamma_i = \delta_i/\beta > 0$, α, $\eta_i > 0$, $0 \le a < x_i < b \le \infty$, $i = 1, \ldots, n$.

- If, in (4), $n=1$, we obtain

$$f_{X_1}(x_1) = \alpha\,\gamma_i z'_{\eta_1}(x_1)[1+\gamma_1 z_{\eta_1}(x_1)]^{-\alpha-1}, \quad x_1 > 0. \tag{5}$$

- For ℓ_1=1, 2, $E(X_1^{\ell_1})$ is given by

$$E\left(X_1^{\ell_1}\right) = \alpha\int_0^1\left[z_{\eta_1}^{-1}\left(\frac{1-w_1}{\gamma_1 w_1}\right)\right]^{\ell_1} w_1^{\alpha-1}\,dw_1, \tag{6}$$

- where $z_{\eta_1}^{-1}(.)$ is the inverse function of $z_{\eta_1}(.)$.
- If, in (4), $n=2$, we obtain

$$f_{X_1,X_2}(x_1, x_2) = \alpha(\alpha+1) \times \left[\prod_{i=1}^{2}\gamma_i z'_{\eta_i}(x_i)\right]\left[1+\sum_{i=1}^{2}\gamma_i z_{\eta_i}(x_i)\right]^{-\alpha-2}. \tag{7}$$

- So that, for $\ell_1 = 1, 2, \ldots,$ and $\ell_2 = 1, 2, \ldots,$ we obtain

$$E\left(X_1^{\ell_1}X_2^{\ell_2}\right) = \alpha(\alpha+1) \times \int_0^1\int_0^1\left[z_{\eta_1}^{-1}\left(\frac{1-w_1}{\gamma_1 w_1}\right)\right]^{\ell_1}\left[z_{\eta_2}^{-1}\left(\frac{1-w_2}{\gamma_2 w_2}\right)\right]^{\ell_2} \times [w_1+w_2-w_1w_2]^{-\alpha-2}w_1^{\alpha}w_2^{\alpha}\,dw_1\,dw_2. \tag{8}$$

3. stress-strength reliability model

An expression for the stress-strength reliability R is given by the following theorem.

Theorem 3.1. *Suppose that a bivariate PDF of the vector* (X_1, X_2) *is given by* (7). *Then*

$$R = P(X_1 < X_2) = 1 - I, \tag{9}$$

where

$$I = \alpha\int_0^1 w^{\alpha-1}\left[1+\gamma_1 w\, z_{\eta_1}\left(z_{\eta_2}^{-1}\left(\frac{1-w}{\gamma_2 w}\right)\right)\right]^{-\alpha-1}dw. \tag{10}$$

Proof
Notice that

$$P(X_1 < X_2) = \int_0^{\infty}\int_0^{x_2} f_{X_1,X_2}(x_1, x_2)\,dx_1\,dx_2 = \alpha(\alpha+1)\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2) \times [1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-2} I(x_2)\,dx_2,$$

where

$$I(x_2) = \int_0^{x_2}\gamma_1 z'_{\eta_1}(x_1)[1+A(x_2)\,z_{\eta_1}(x_1)]^{-\alpha-2}dx_1,$$
$$A(x_2) = \gamma_1[1+\gamma_2 z_{\eta_2}(x_2)]^{-1}. \tag{11}$$

Let $v = [1+A(x_2)\,z_{\eta_1}(x_1)]^{-1}$. Then $z_{\eta_1}(x_1) = \frac{1}{A(x_2)}(\frac{1}{v} - 1)$. Therefore, $\frac{dv}{A(x_2)v^2} = -z'_{\eta_1}(x_1)\,dx_1$ and $(0, x_2) \to (1, v_0)$, $v_0 = [1+A(x_2)\,z_{\eta_1}(x_2)]^{-1}$. So that $I(x_2) = \frac{\gamma_1}{A(x_2)}\int_{v_0}^{1} v^{\alpha}\,dv = \frac{\gamma_1}{(\alpha+1)A(x_2)}\{1 - [1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1}\}$.
Then

$$P(X_1 < X_2) = \alpha\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2)[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-1} \times\left\{1-[1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1}\right\}dx_2.$$

Notice, from (11), that $\frac{\gamma_1}{A(x_2)} = 1+\gamma_2 z_{\eta_2}(x_2)$. Hence

$$P(X_1 < X_2) = \alpha\left\{\frac{[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha}}{-\alpha}\Bigg|_0^{\infty}\right\} - I = 1 - I,$$

where

$$I = \alpha\int_0^{\infty}\gamma_2 z'_{\eta_2}(x_2)[1+\gamma_2 z_{\eta_2}(x_2)]^{-\alpha-1} \times [1+A(x_2)\,z_{\eta_1}(x_2)]^{-\alpha-1}dx_2.$$

Let

$$w = [1+\gamma_2 z_{\eta_2}(x_2)]^{-1}. \tag{12}$$

Then, from (12)

$$\frac{1}{w} - 1 = \gamma_2 z_{\eta_2}(x_2) \text{ and } \frac{-dw}{w^2} = \gamma_2 z'_{\eta_2}(x_2)\,dx_2.$$

Hence, $x_2 = z_{\eta_2}^{-1}(\frac{1-w}{\gamma_2 w})$, $(0,\ \infty) \ \rightarrow \ (1,\ 0\,)$ and from (11) and (12), $A(x_2) = \gamma_1 w$.

So that

$$I = \alpha \int_0^{\infty} w^{\alpha-1}\left[1+\gamma_1\, w\, z_{\eta_1}\left(z_{\eta_2}^{-1}\left(\frac{1-w}{\gamma_2 w}\right)\right)\right]^{-\alpha-1} d\,w.$$

4. Maximum likelihood estimation of *R*

The likelihood function is given by

$$\begin{aligned} L(\theta,\ \mathbf{x}) &= \prod_{j=1}^{n} f_{X_1,X_2}\left(x_{1j},\ x_{2j}\right) \\ &= \prod_{j=1}^{n}\left\{\alpha\,(\alpha+1)\left[\prod_{i=1}^{2}\gamma_i\, z'_{\eta_i}(x_{ij})\right]\right. \\ &\quad \times\left.\left[1+\sum_{i=1}^{2}\gamma_i\, z_{\eta_i}(x_{ij})\right]^{-\alpha-2}\right\} \\ &= \alpha^n\,(\alpha+1)^n\,\gamma_1^n\,\gamma_2^n\left[\prod_{j=1}^{n} z'_{\eta_1}(x_{1j})z'_{\eta_2}(x_{2j})\right] \\ &\quad \times\prod_{j=1}^{n}\left[1+\gamma_1\, z_{\eta_1}(x_{1j})+\gamma_2\, z_{\eta_2}(x_{2j})\right]^{-\alpha-2}. \end{aligned}$$

The log-likelihood function is then given by

$$\begin{aligned} LL(\theta,\ \mathbf{x}) &= n\ln\alpha + n\ln(\alpha+1) + n\ln\gamma_1 + n\ln\gamma_2 \\ &\quad + \sum_{j=1}^{n}\ln z'_{\eta_1}(x_{1j}) + \sum_{j=1}^{n}\ln z'_{\eta_2}(x_{2j}) \\ &\quad - (\alpha+2)\sum_{j=1}^{n}\ln\left[1+\gamma_1\, z_{\eta_1}(x_{1j})\right. \\ &\quad \left.+\gamma_2\, z_{\eta_2}(x_{2j})\right]. \end{aligned} \tag{13}$$

Traditionally, the log-likelihood function $LL(\theta,\ \mathrm{x})$ is maximized by differentiating it with respect to the parameters, equating to zero and then solving the resulting likelihood equations. However, (13) is directly maximized by using it as the objective function (see Matlab R2013a Documentation, www.mathworks.com).

4.1. Non-parametric estimation of *R*

Asymptotic distribution-free two-sided 100 τ% confidence bounds of R when X_1 and X_2 are dependent were obtained by Govindarajulu [18] as

$$\hat{R}_{NP}\ \pm\left(\frac{\hat{R}_{NP}\,(1-\hat{R}_{NP})}{n}\right)^{1/2}\Phi^{-1}\left(\frac{1+\tau}{2}\right), \tag{14}$$

where

$\hat{R}_{NP} \equiv$ Non-parametric point estimate of $R = (\#z_j = x_{1j} - x_{2j} \le 0)/n$,

$\Phi(z)$ is the area, under the standard normal curve, up to z and the symbol # denotes the number of z_j that satisfies the inequality, $j = 1, \ldots, n$.

5. Simulation study and illustrations

5.1. Generation of a random sample of size *n* from a bivariate distribution

It follows, from (5), that

$$F_{X_2}(x_2) = 1 - [1+\gamma_2\, z_{\eta_2}(x_2)]^{-\alpha}.$$

So that, if u_{2j} is uniform on (0, 1), then

$$x_{2j} = z_{\eta_2}^{-1}\left([\,(1-u_{2j})^{-1/\alpha} - 1\,]/\gamma_2\right). \tag{15}$$

Also, from (5) and (7),

$$\begin{aligned} f_{X_1|X_2}(x_1|x_2) &= \frac{f_{X_1,X_2}(x_1,\ x_2)}{f_{X_2}(x_2)} \\ &= (\alpha+1)\ \gamma_1^*\, z'_{\eta_1}(x_1)\left[1+\gamma_1^*\, z_{\eta_1}(x_1)\right]^{-\alpha-2}, \end{aligned}$$

where

$$\gamma_1^* = \frac{\gamma_1}{1+\gamma_2\, z_{\eta_2}(x_2)}. \tag{16}$$

So that

$$F_{X_1|X_2}(x_1|x_2) = 1 - \left[1+\gamma_1^*\, z_{\eta_1}(x_1)\right]^{-\alpha-1}.$$

Hence, if u_{1j} is uniform on (0, 1), then

$$x_{1j} = z_{\eta_1}^{-1}\left(\left[(1-u_{1j})^{-1/(\alpha+1)} - 1\right]/\gamma_1^*\right) \tag{17}$$

So, for given x_{2j} (obtained in (15)), γ_1^* is computed from (16) and hence x_{1j} is computed from (17).

5.2. Simulation study

The computations, based on one sample, are repeated 500 times by generating 500 samples, as in the one sample case, for $n = 50$, 100, 200. The average of the MLEs over the 500 repetitions, mean square errors (MSEs) and relative absolute biases (RABs) are displayed in Table 1.

In the case of non-parametric confidence interval (NPCI), $\hat{R}_{NP}$ is computed for each generated sample and hence the 95% confidence bounds. The lower and upper bounds are obtained by averaging over the 500 samples, the length of intervals and coverage probabilities (COVPs) are computed and reported in Table 2.

5.3. Illustrations

- Bivariate compound Weibull (bivariate Burr type XII) distribution.

If $z_{\eta_i}(x_i) = x_i^{\eta_i}$, then $z_{\eta_i}^{-1}(y_i) = y_i^{1/\eta_i}$.

Therefore, $z_{\eta_1}\left(z_{\eta_2}^{-1}\left(\frac{1-w}{\gamma_2 w}\right)\right) = z_{\eta_1}\left(\frac{1-w}{\gamma_2 w}\right)^{1/\eta_2} = \left(\frac{1-w}{\gamma_2 w}\right)^{\eta_1/\eta_2}$.

Table 1 MLEs, MSEs and RABs based on 500 repetitions.

n	$\hat{\alpha}$ MSE RAB	$\hat{\gamma}_1$ MSE RAB	$\hat{\gamma}_1$ MSE RAB	$\hat{\eta}_1$ MSE RAB	$\hat{\eta}_2$ MSE RAB	$\hat{R}_{ML}$ MSE RAB
50	2.26429	2.16075	1.70928	1.84898	2.44697	0.64214
	0.97014	1.37425	0.95476	0.05159	0,09174	0.00256
	0.40216	0.50436	0.50813	0.10321	0.10548	0.06337
100	2.21282	1.99828	1.56468	1.81725	2.43426	0.64312
	0.57312	0.81018	0.46581	0.02277	0.05497	0.00138
	0.26845	0.35859	0.34575	0.06519	0.07904	0.04688
200	2.08572	1.98732	1.56385	1.80998	2.41675	0.64277
	0.23361	0.42255	0.25620	0.01310	0.02265	0.00070
	0.17940	0.26252	0.25585	0.05117	0.04946	0.03285

Population parameters: $\alpha = 2$, $\gamma_1 = 1.9$, $\gamma_2 = 1.5$, $\eta_1 = 1.8$, $\eta_2 = 2.4$, $R{=}0.6421$.

Table 2 95% non-parametric confidence intervals for R, their lengths and COVPs (in %), based on 500 repetitions. Population parameters: $\alpha = 2$, $\gamma_1 = 1.9$, $\gamma_2 = 1.5$, $\eta_1 = 1.8$, $\eta_2 = 2.4$, $R{=}0.6421$.

n	NPCI	Length	COVP
50	(0.509095, 0.772425)	0.263330	96.0
100	(0.550057, 0.736783)	0.186725	93.2
200	(0.574802, 0.707398)	0.132597	96.8

Now,

$$I = \alpha \int_0^1 w^{\alpha-1}\left[1+\gamma_1 w\left(\frac{1-w}{\gamma_2 w}\right)^{\eta_1/\eta_2}\right]^{-\alpha-1} d\,w$$

$$= \alpha \sum_{j=0}^{\infty} k_j \gamma_1^j \int_0^1 w^{\alpha+j-1}\left(\frac{1-w}{\gamma_2 w}\right)^{j\eta_1/\eta_2} d\,w,$$

$$k_j = (-1)^j \binom{\alpha+j}{j}, \quad 0 < \gamma_1 < \gamma_2^{\eta_1/\eta_2}$$

$$= \alpha \sum_{j=0}^{\infty} k_j^* \int_0^1 w^{\alpha+j(1-\eta_1/\eta_2)-1}(1-w)^{j\eta_1/\eta_2}\, d\,w\,,$$

$$k_j^* = k_j\left(\frac{\gamma_1}{\gamma_2^{\eta_1/\eta_2}}\right)^j$$

$$= \alpha \sum_{j=0}^{\infty} k_j^* B[\,\alpha + j\{1-(\eta_1/\eta_2)\},\ 1 + j\eta_1/\eta_2\,].$$

Then, from (10),

$$R = 1 - \alpha \sum_{j=0}^{\infty} k_j^* B[\,\alpha + j\{1-(\eta_1/\eta_2)\},$$
$$1 + j\eta_1/\eta_2\,]\,,\ \eta_2 > \eta_1,$$

$$k_j^* = (-1)^j \binom{\alpha+j}{j}\left(\frac{\gamma_1}{\gamma_2^{\eta_1/\eta_2}}\right)^j.$$

- Bivariate compound exponential distribution.

If $\eta_1 = \eta_2 = 1$, the bivariate compound Weibull distribution reduces to the bivariate compound exponential distribution. It can be shown, in this case, that

$$I = \left(1 + \frac{\gamma_1}{\gamma_2}\right)^{-1} \text{ and hence } \quad R = \frac{\gamma_1}{\gamma_1+\gamma_2}\,.$$

In this case, $R \to 1$ as $(\gamma_2/\gamma_1) \to 0$.

5.4. One sample illustration

For a given vector of parameters $\boldsymbol{\theta} = (\alpha, \gamma_1, \gamma_2, \eta_1, \eta_2)$, a random sample of size $n{=}50$ is generated according to (15) and (17), in which $z_{\eta_i}(x) = x^{\eta_i}$. This leads to $z_{\eta_i}^{-1}(y) = y^{1/\eta_i}$. So that, from (15),

$$x_{2j} = \left(\left[(1-u_{2j})^{-1/\alpha} - 1\right]/\gamma_2\right)^{1/\eta_2},$$

and, from (17),

$$x_{1j} = \left(\left[(1-u_{1j})^{-1/(\alpha+1)} - 1\right]/\gamma_1^*\right)^{1/\eta_1},$$

for $j = 1, \ldots, n$, where γ_1^* is computed from (16). The sample is presented in Table 3. Based on the generated sample in Table 3, the MLE $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\eta}_1, \hat{\eta}_2)$ of the vector of parameters $\boldsymbol{\theta}$ is given by

$$\hat{\alpha} = 1.97538,\ \hat{\gamma}_1 = 2.01733,\ \hat{\gamma}_2 = 1.53845,\ \hat{\eta}_1 = 1.59707,$$
$$\hat{\eta}_2 = 2.4271,\ \hat{R}_{ML} = 0.68634\,.$$

The estimate $\hat{R}_{ML}$of R is obtained by replacing the parameters in R by their MLEs.

In the non-parametric case, based on the same sample of Table 3, $\hat{R}_{NP} = 33/50{=}0.66$, $\Phi^{-1}(0.975) = 1.96$. So that a 95% confidence interval is computed, from (14), as

$0.5287 < R < 0.7913$.

The estimates $\hat{R}_{ML} = 0.68634$ and $\hat{R}_{NP} = 0.66$, may be compared with the population reliability$R = 0.6421$.

With the choice of parameters, given by $\alpha = 2$, $\gamma_1 = 1.9$, $\gamma_2 = 1.5$, $\eta_1 = 1.8$, $\eta_2 = 2.4$ and the use of (6) and (8), computations show that

$$E(X_1) = 0.5514,\ E(X_2) = 0.6676,\ E(X_1^2) = 0.5558,$$
$$E(X_2^2) = 0.6225,\ E(X_1X_2) = 0.4589.$$

So that the correlation coefficient between them is given by$\rho = 0.4307$.

Table 3 Generated set of $n = 50$ observations. Population parameters: $\alpha = 2,\ \gamma_1 = 1.9,\ \gamma_2 = 1.5,\ \eta_1 = 1.8,\ \eta_2 = 2.4,\ R = 0.6421$.

j	x_{ij}		j	x_{ij}	
	$i=1$	$i=2$		$i=1$	$i=2$
1	1.16646	2.11035	26	0.42309	1.04666
2	1.43574	1.31085	27	0.84110	0.41853
3	0.50119	0.51287	28	0.04213	0.87408
4	0.10190	0.46011	29	0.22835	0.83931
5	0.37354	0.78868	30	0.44766	0.24363
6	0.09173	1.15789	31	0.22159	0.58736
7	0.43846	0.89242	32	0.10774	0.40205
8	0.64801	0.77310	33	0.06643	0.56653
9	0.47627	0.94641	34	0.24915	0.29079
10	0.10260	0.25517	35	0.04460	0.70876
11	0.41067	0.27463	36	0.50962	0.45172
12	0.96175	0.67096	37	0.17354	0.23069
13	0.22385	0.26464	38	0.71964	1.25463
14	0.29477	0.17776	39	0.28707	0.63871
15	0.09554	0.60930	40	0.41848	0.57844
16	0.14133	0.59567	41	0.67778	0.20745
17	1.09799	0.67704	42	0.28943	0.30234
18	0.55671	0.36253	43	0.43629	0.27983
19	0.54836	0.35771	44	0.81176	0.65261
20	0.26641	0.70760	45	0.72305	0.53620
21	0.05998	0.88468	46	0.42336	0.25148
22	0.04728	0.29756	47	2.56028	1.20121
23	1.28437	1.53950	48	1.30957	0.80369
24	0.25652	0.40859	49	0.70702	1.61168
25	0.43140	0.57118	50	0.78045	0.99724

5.5. Concluding remarks

An expression for point estimate of stress-strength reliability function $R = P(X_1 < X_2)$ is obtained, when (X_1, X_2) follows a general bivariate distribution, where X_1, X_2 are dependent. MLEs of the parameters and reliability function R are obtained and computed when the distribution is bivariate compound Weibull (bivariate Burr type XII). Simulation is conducted when the sample size is $n = 50$, 100, 200 and MSEs and RABs are computed over 500 samples. Table 1 shows that the MSEs and RABs decrease as the sample size increases, which give credibility to estimation computations.

In the non-parametric set up, point estimation of R and the two-sided confidence intervals are computed, based on Govindarajulu's suggestion, assuming dependence of X_1 and X_2. The intervals improve (shorter length) as the sample size increases, as shown in Table 2. The coverage probabilities (COVPs) are close to the nominal value of 95%. It may be observed that while the parametric interval estimation of R is computationally involved, Govindarajulu's bounds are quite easy to compute. For large values of n, a non-parametric interval tends to be close to the parametric interval.

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