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Original article

On the perturbation estimates of the maximal solution for the matrix equation $X + A^T\sqrt{X^{-1}}A = P$

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Abstract In this paper we investigate the nonlinear matrix equation $X + A^T\sqrt{X^{-1}}A = P$, for the existence of positive definite solutions. Bounds for $\|X_L^{-1}\|$ and $\|X^{-1}\|$ are derived where X_L is the maximal solution and X is any other positive definite solution of this matrix equation. A perturbation estimate for the maximal solution and an error bound for approximate solutions are derived. A numerical example is given to illustrate the reliability of the obtained results.

1. Introduction

Consider the nonlinear matrix equation

$$X + A^T\sqrt{X^{-1}}A = P, \tag{1.1}$$

where A and P are nxn nonsingular and positive definite matrices respectively. The existence and the uniqueness, the rate of convergence as well as the necessary and sufficient conditions for the existence of positive definite solutions of similar kinds of nonlinear matrix equations have been studied by several authors [1–15]. Perturbation analysis for the matrix equations $X \pm A^*X^{-n}A = Q$, $X + A^*F(X)A = Q$ and $X + A^*X^{-1}A = P$ are studied in [16–20] respectively. Throughout this paper we use X_L to denote the maximal solution of the matrix Eq. (1.1). The paper is organized as follows: First, in Section 2, we introduce some notations and a lemma that will be needed for developing the work. In Section 3, an iterative method for obtaining the maximal positive definite solution of the matrix Eq. (1.1) is proposed. We state and prove a theorem and a lemma for the existence of the maximal solution. We also, put conditions on the matrix A to derive bounds on the maximal solution X_L as well as any other positive definite solution X of the matrix Eq. (1.1) in terms of the matrix P. In Section 4, we investigate a perturbation estimate for the maximal solution of the matrix Eq. (1.1) and an error bound for approximation of the

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maximal solution is obtained. In Section 5, numerical example is given to illustrate the obtained results.

2. Preliminaries

The following properties and the lemma stated below will be needed for developing the work:

For square nonsingular matrices A, B, we have the following:

(i) If $A \geq B > 0$ then $\sqrt{A} \geq \sqrt{B} > 0$ and $A^{-1} \leq B^{-1}$.

(ii) The spectral norm is monotonic norm, that is if $0 < A_1 \leq A_2$, then $\|A_1\| \leq \|A_2\|$.

Lemma 2.1. *[21, Theorem 8.5.2, p.263] Let the matrices A, B and $C \in P(n)$[the set of all positive definite nxn matrices], be such that the integral $\int_0^\infty e^{At}Ce^{Bt}dt$ exists and $\lim_{t\to\infty} e^{At}Ce^{Bt} = 0$. Then the matrix $X = -\int_0^\infty e^{At}Ce^{Bt}dt$ is a solution of the matrix equation $AX + XB = C$.*

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Now, for solving our problem (1.1) we consider the following iterative process.

$$X_0 = P$$
$$X_{k+1} = P - A^T\sqrt{X_k^{-1}}A \quad k = 0, 1, 2, \ldots \tag{3.1}$$

Fact 3.1. If A is nonsingular matrix and the matrix Eq. (1.1) has a positive definite solution X, then the sequence $\{X_k\}$ derived from (3.1), is monotonic decreasing and bounded from below and hence converges to X_L (the maximal solution).

The statement of this fact and its proof are similar to Theorem 2.5 and Theorem 2.6 [22], where V. I. Hasanov considers the iteration $X_0 = \gamma Q$, $X_{s+1} = Q - A^*X_s^{-q}A$, $s = 0, 1, 2, \ldots$ to solve the matrix equation $X + A^*X^{-q}A = Q$.

Theorem 3.1. *If $\|A\| < \sqrt{\frac{2}{\sqrt{27}}}\ \|P^{-1}\|^{-\frac{3}{4}}$, then the maximal solution X_L of the matrix Eq. (1.1) satisfies $\|X_L^{-1}\| \leq (1+\eta)\ \|P^{-1}\|$. Moreover, for any other positive definite solution X we have $\|X^{-1}\| > (5-\eta)\ \|P\|^{-1}$, here $\eta = \|A\|^2\ \|P^{-1}\|^{\frac{3}{2}}(1+\eta)^{\frac{3}{2}} < 2$.*

Proof. It is clear that X is a solution of the matrix Eq. (1.1) if and only if $Y = X^{-1}$ satisfies

$$Y = P^{-1} + P^{-1}A^T\sqrt{Y}AY. \tag{3.2}$$

Now consider the sequence of matrices

$$\begin{aligned} &Y_0 = O \\ &Y_k = P^{-1} + P^{-1}A^T\sqrt{Y_{k-1}}AY_{k-1} \quad k = 1, 2, \ldots \end{aligned} \tag{3.3}$$

Using induction, it is easy to verify that:

$$\|Y_k\| \leq (1+\eta_k)\ \left\|P^{-1}\right\| \quad k = 1,\ 2, \cdots \tag{3.4}$$

where

$$\eta_1 = 0, \quad \eta_k = \|A\|^2\|P^{-1}\|^{\frac{3}{2}}\ (1+\eta_{k-1})^{\frac{3}{2}}. \tag{3.5}$$

Also, by induction we get $0 \leq \eta_k < \eta_{k+1} < 2, k = 1, 2, \ldots$ hence there exists a positive number η with $0 < \eta \leq 2$ such that $\lim_{k\to\infty} \eta_k = \eta$. Thus it follows from (3.5)

$$\eta = \|A\|^2\ \left\|P^{-1}\right\|^{\frac{3}{2}}(1+\eta)^{\frac{3}{2}} < 2. \tag{3.6}$$

Then from (3.4) and (3.6) we have

$$\|Y_k\| \leq (1+\eta_k)\|P^{-1}\| \leq (1+\eta)\|P^{-1}\|, \quad k = 1, 2, \ldots$$

which yields:

$$\begin{aligned} \|Y_{k+1} - Y_k\| &= \|P^{-1}(A^T\sqrt{Y_k}AY_k - A^T\sqrt{Y_{k-1}}AY_{k-1})\| \\ &\leq \|A\|\|P^{-1}\|\|\sqrt{Y_k}AY_k - \sqrt{Y_{k-1}}AY_{k-1}\| \\ &= \|A\|\|P^{-1}\|\|\sqrt{Y_k}A(Y_k - Y_{k-1}) \\ &\quad + (\sqrt{Y_k} - \sqrt{Y_{k-1}})AY_{k-1}\| \end{aligned} \tag{3.7}$$

It is clear that $Z = \sqrt{Y_k} - \sqrt{Y_{k-1}}$ is a positive definite solution of the matrix equation $\sqrt{Y_k}Z + Z\sqrt{Y_{k-1}} = Y_k - Y_{k-1}$.

According to Lemma 2.1, we have

$$Z = \int_0^\infty e^{-\sqrt{Y_k}t}(Y_k - Y_{k-1})e^{-\sqrt{Y_{k-1}}t}dt \tag{3.8}$$

From (3.7) and (3.8), then we get

$$\begin{aligned} &\|Y_{k+1} - Y_k\| \\ &\leq \|A\|\ \left\|P^{-1}\right\|\ \Big\|\sqrt{Y_k}\ A(Y_k - Y_{k-1}) \\ &\quad + \left(\int_0^\infty e^{-\sqrt{Y_k}t}(Y_k - Y_{k-1})e^{-\sqrt{Y_{k-1}}t}dt\right)\ A\ Y_{k-1}\Big\| \\ &\leq \|A\|^2\ \left\|P^{-1}\right\|\ \|Y_k - Y_{k-1}\| \\ &\quad \times \left\{\left\|\sqrt{Y_k}\right\| + \left(\int_0^\infty \left\|e^{-\sqrt{Y_k}t}\right\|\ \left\|e^{-\sqrt{Y_{k-1}}t}\right\|dt\right)\|Y_{k-1}\|\right\} \\ &\leq \|A\|^2\ \left\|P^{-1}\right\|\ \|Y_k - Y_{k-1}\| \\ &\quad \times \left\{\left\|\sqrt{Y_k}\right\| + \frac{1}{2\ \left\|P^{-1}\right\|^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}}\ \|Y_{k-1}\|\right\} \\ &= \left\{\|A\|^2\ \left\|P^{-1}\right\|^{\frac{3}{2}}\ \frac{3(1+\eta)^{\frac{1}{2}}}{2}\right\}\ \|Y_k - Y_{k-1}\|\ \leq \rho^k \end{aligned} \tag{3.9}$$

where

$$\rho = \|A\|^2\ \left\|P^{-1}\right\|^{\frac{3}{2}}\ \frac{3(1+\eta)^{\frac{1}{2}}}{2} < 1, \tag{3.10}$$

Thus, it follows that the matrix sequence $\{Y_k\}$ is convergent. Let $\hat{Y} = \lim_{k\to\infty} Y_k$, then $\hat{Y}$ is a solution of the matrix Eq. (3.2) and satisfies $\|\hat{Y}\| \leq\ (1+\eta)\ \|P^{-1}\|$.

It is easy to show $\hat{Y}$ is symmetric positive definite solution of the matrix Eq. (3.2). But since $\hat{Y}$ must be X_L^{-1} then the proof of the theorem is completed.

Lemma 3.2. *If $\|A\| < \sqrt{\frac{2}{\sqrt{27}}}\ \|P^{-1}\|^{-\frac{3}{4}}$, then the maximal solution X_L of the matrix Eq. (1.1) satisfies $\frac{1}{3}\ \|P\| \leq \|X_L\|\ \leq\ \|P\|$.*

The proof of this lemma is straightforward.

4. Perturbation estimate of the maximal solution for the equation $X + A^T\sqrt{X^{-1}}A = P$

In this section, we investigate a perturbation estimate for the maximal solution X_L of the matrix Eq. (1.1) using the lemma given in Section 3, also, we obtain an error bound for approximation of the maximal solution.

Consider the perturbed matrix equation

$$\tilde{X} + \tilde{A}^T\sqrt{\tilde{X}^{-1}}\tilde{A} = \tilde{P}, \tag{4.1}$$

where $\tilde{A}$ is nxn nonsingular matrix and $\tilde{P}$ is positive definite matrix. Denote $\Delta A = \tilde{A} - A$, $\Delta P = \tilde{P} - P$. We derive some perturbation estimates for ΔA and ΔP as follows:

Theorem 4.1. *Let*

(i) $\varepsilon = \sqrt{\frac{2}{\sqrt{27}}} - \|A\|\ \|P^{-1}\|^{\frac{3}{4}} > 0.$

(ii) $\|\Delta A\| < ((1 - \sqrt{\frac{2}{\sqrt{27}}})\ \varepsilon)\ \|P^{-1}\|^{-\frac{3}{4}}$

(iii) $\|\Delta P\| \le (1 - (1-\varepsilon)^{\frac{4}{3}})\ \|P^{-1}\|^{-1}$

then $\|\tilde{A}\|\ \|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ and $\|A\|\ \|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$.

Remark. The conditions (i), (ii) and (iii) of this theorem are similar to:

(1) the conditions of Theorem 3.1 [19, p.1415] for studying the matrix equations $X^s + A^*X^{-t}A = Q$ and $\tilde{X}^s + \tilde{A}^*\tilde{X}^{-t}\tilde{A} = \tilde{Q}$ by putting $s = 1,\ t = \frac{1}{2}$.
(2) the conditions of Theorem 10 [18] for studying the matrix equations $X - A^*X^{-n}A = Q$ and $\tilde{X} - \tilde{A}^*\tilde{X}^{-n}\tilde{A} = \tilde{Q}$ by putting $n = \frac{1}{2}$

Proof. Since $\Delta P = \tilde{P} - P$ therefore, $\tilde{P}^{-1} = P^{-1} - P^{-1}(\Delta P)\tilde{P}^{-1}$

Taking norm and using condition (iii), we get

$$\|\tilde{P}^{-1}\| \le \|P^{-1}\| + \|P^{-1}\|\|\Delta P\|\|\tilde{P}^{-1}\|$$
$$\le \|P^{-1}\| + (1 - (1-\varepsilon)^{\frac{4}{3}})\|\tilde{P}^{-1}\|$$

and so we have,

$$\|\tilde{P}^{-1}\| \le \frac{\|P^{-1}\|}{(1-\varepsilon)^{\frac{4}{3}}}$$

Therefore,

$$\|\tilde{P}^{-1}\|^{\frac{3}{4}} \le \frac{\|P^{-1}\|^{\frac{3}{4}}}{1-\varepsilon} \tag{4.2}$$

Combining (ii) and (4.2) we obtain

$$\|\tilde{A}\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} \le (\|A\| + \|\tilde{A} - A\|)\|\tilde{P}^{-1}\|^{\frac{3}{4}}$$
$$\le \frac{(\|A\| + \|\Delta A\|)\|P^{-1}\|^{\frac{3}{4}}}{1-\varepsilon}$$
$$< \frac{(\|A\| + ((1 - \sqrt{\frac{2}{\sqrt{27}}})\varepsilon)\|P^{-1}\|^{-\frac{3}{4}})\|P^{-1}\|^{\frac{3}{4}}}{1-\varepsilon}$$
$$= \frac{(\|A\|\ \|P^{-1}\|^{\frac{3}{4}} + ((1 - \sqrt{\frac{2}{\sqrt{27}}})\varepsilon))}{1-\varepsilon}$$
$$= \sqrt{\frac{2}{\sqrt{27}}}. \tag{4.3}$$

From condition (i) and inequality (4.2) it is easy to verify:

$$\|A\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} \le \frac{\|A\|\|P^{-1}\|^{\frac{3}{4}}}{(1 - \sqrt{\frac{2}{\sqrt{27}}} + \|A\|\|P^{-1}\|^{\frac{3}{4}})} < \sqrt{\frac{2}{\sqrt{27}}}$$

which completes the proof of the theorem.

Theorem 4.2. *Consider the two matrix equations* $X + A^T\sqrt{X^{-1}}A = P$, $\tilde{X} + \tilde{A}^T\sqrt{\tilde{X}^{-1}}\tilde{A} = \tilde{P}$ *with*

$$\|\tilde{A}\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}} \quad \text{and} \quad \|A\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}} \tag{4.4}$$

then the maximal solutions X_L and $\tilde{X}_L$ of these two equations exist and satisfy

$$\frac{\|\Delta X_L\|}{\|X_L\|} \le \frac{1}{\delta}\left(\frac{3\ \|\Delta P\|}{\|P\|} + \frac{4\ \|\Delta A\|}{\|A\|}\right) \tag{4.5}$$

where $\delta = 1 - \sqrt{\frac{\sqrt{27}}{2}}\|A\|\ \|P^{-1}\|^{\frac{3}{4}} > 0.$

Proof. Using the obtained results of Theorem 4.1 where $\|A\|\|P^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ that is: $\|\tilde{A}\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ and $\|A\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ hold, then "by setting $0 < \eta \le 2$ in Theorem 3.1" and from Lemma 3.2 we get that the maximal solutions X_L and $\tilde{X}_L$ of the two Eqs. (1.1) and (4.1) exist and satisfy that

$$\|X_L^{-1}\| < 3\|P^{-1}\|, \quad \|\tilde{X}_L^{-1}\| < 3\|\tilde{P}^{-1}\| \tag{4.6}$$

and

$$\frac{1}{3}\|P\| \le \|X_L\| \le \|P\|, \quad \frac{1}{3}\|\tilde{P}\| \le \|\tilde{X}_L\| \le \|\tilde{P}\|. \tag{4.7}$$

It is clear that X_L and $\tilde{X}_L$ satisfy the two matrix equations

$$X_L + A^T\sqrt{X_L^{-1}}A = P \quad \text{and} \quad \tilde{X}_L + \tilde{A}^T\sqrt{\tilde{X}_L^{-1}}\tilde{A} = \tilde{P}$$

Set $\Delta A = \tilde{A} - A$ and $\Delta X_L = \tilde{X}_L - X_L$. If we consider the identity

$$\Delta P = \tilde{P} - P$$
$$= \tilde{X}_L - X_L + \tilde{A}^T\sqrt{\tilde{X}_L^{-1}}\tilde{A} - A^T\sqrt{X_L^{-1}}A$$
$$= \Delta X_L + (\Delta A^T + A^T)\sqrt{\tilde{X}_L^{-1}}(\Delta A + A) - A^T\sqrt{X_L^{-1}}A$$
$$= \Delta X_L - A^T(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}})$$
$$A + \Delta A^T\sqrt{\tilde{X}_L^{-1}}\tilde{A} + A^T\sqrt{\tilde{X}_L^{-1}}\Delta A$$

$$\|\Delta X_L - A^T(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}})A\|$$
$$\ge \|\Delta X_L\| - \|A^T(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}})A\|$$

$$\geq \|\Delta X_L\| - \|A\|^2\|\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}}\|$$
$$\geq \|\Delta X_L\| - \|A\|^2\|\sqrt{\tilde{X}_L^{-1}}(\sqrt{\tilde{X}_L} - \sqrt{X_L})\sqrt{X_L^{-1}}\|$$
$$\geq \|\Delta X_L\| - \|A\|^2\|\sqrt{\tilde{X}_L^{-1}}\|\|\sqrt{X_L^{-1}}\|\|\sqrt{\tilde{X}_L} - \sqrt{X_L}\| \quad (4.8)$$

It is clear that $Z = \sqrt{\tilde{X}_L} - \sqrt{X_L}$ is a positive definite solution of the matrix equation $\sqrt{\tilde{X}_L}Z + Z\sqrt{X_L} = \tilde{X}_L - X_L$.

According to Lemma 2.1, we have

$$Z = \int_0^\infty e^{-\sqrt{\tilde{X}_L}t}(\tilde{X}_L - X_L)e^{-\sqrt{X_L}t}dt \quad (4.9)$$

Note that, $\sqrt{\tilde{X}_L}$ and $\sqrt{X_L}$ are positive definite and hence $\int_0^\infty e^{-\sqrt{\tilde{X}_L}t}(\tilde{X}_L - X_L)e^{-\sqrt{X_L}t}dt$ exists and $e^{-\sqrt{\tilde{X}_L}t}(\tilde{X}_L - X_L)e^{-\sqrt{X_L}t} \to 0$ as $t \to \infty$.

From (4.8) and (4.9) we get

$$\|\Delta X_L - A^T(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}})A\|$$
$$\geq \|\Delta X_L\| - \|A\|^2\|\sqrt{\tilde{X}_L^{-1}}\|\|\sqrt{X_L^{-1}}\| \times \left\|\int_0^\infty e^{-\sqrt{\tilde{X}_L}t}(\tilde{X}_L - X_L)e^{-\sqrt{X_L}t}dt\right\|$$
$$\geq \|\Delta X_L\| - \|A\|^2\|\sqrt{\tilde{X}_L^{-1}}\|\|\sqrt{X_L^{-1}}\|\|\tilde{X}_L - X_L\| \times \left(\int_0^\infty \left\|e^{-\sqrt{\tilde{X}_L}t}\right\|\left\|e^{-\sqrt{X_L}t}\right\|dt\right)$$
$$\geq \|\Delta X_L\|\left\{1 - \frac{\sqrt{3}}{2}\|A\|^2\left\|\sqrt{\tilde{X}_L^{-1}}\right\| \left\|\sqrt{X_L^{-1}}\right\| \|P^{-1}\|^{\frac{1}{2}}\right\}$$

From (4.6) we get

$$\left\|\Delta X_L - A^T\left(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}}\right)A\right\|$$
$$\geq \|\Delta X_L\| \left\{1 - \sqrt{\frac{\sqrt{27}}{2}\|A\|\| P^{-1}\|^{\frac{3}{4}}}\right\} = \delta\,\|\Delta X_L\|$$

where $\delta = 1 - \sqrt{\frac{\sqrt{27}}{2}\| A \| \|P^{-1}\|^{\frac{3}{4}}} > 0$

So we have

$$\delta\|\Delta X_L\| \leq \|\Delta X_L - A^T(\sqrt{X_L^{-1}} - \sqrt{\tilde{X}_L^{-1}})A\|$$
$$= \|\Delta P - \Delta A^T\sqrt{\tilde{X}_L^{-1}}\tilde{A} - A^T\sqrt{\tilde{X}_L^{-1}}\Delta A\|$$
$$\leq \|\Delta P + \Delta A^T\sqrt{\tilde{X}_L^{-1}}\tilde{A} + A^T\sqrt{\tilde{X}_L^{-1}}\Delta A\|$$
$$\leq \|\Delta P\| + \| A\|\|\sqrt{\tilde{X}_L^{-1}}\|\|\Delta A\| + \|\Delta A\|\|\sqrt{\tilde{X}_L^{-1}}\|\|\tilde{A}\| \quad (4.10)$$

From (4.6) we get

$$\delta\|\Delta X_L\| \leq \|\Delta P\| + \sqrt{3}\|A\|\|\tilde{P}^{-1}\|^{\frac{1}{2}}\|\Delta A\| + \sqrt{3}\|\Delta A\|\|\tilde{A}\|\|\tilde{P}^{-1}\|^{\frac{1}{2}}$$
$$\leq \|\Delta P\| + \sqrt{3}\|\Delta A\| \times (\|A\|\|\tilde{P}^{-1}\|^{\frac{3}{4}} + \|\tilde{A}\|\|\tilde{P}^{-1}\|^{\frac{3}{4}})$$

From (4.4) we get

$$\delta\,\|\Delta X_L\| \leq \|\Delta P\| + 2\sqrt{3}\sqrt{\frac{2}{\sqrt{27}}}\,\|\Delta A\| \quad (4.11)$$

Therefore, $\|\Delta X_L\| \leq \frac{1}{\delta}\,(\|\Delta P\| + 2\sqrt{\frac{2}{\sqrt{3}}}\,\|\Delta A\|)$

$$\frac{\|\Delta X_L\|}{\|X_L\|} \leq \frac{1}{\delta}\left(\frac{\|\Delta P\|}{\|X_L\|} + 2\sqrt{\frac{2}{\sqrt{3}}}\frac{\|\Delta A\|}{\|X_L\|}\right)$$
$$= \frac{1}{\delta}\left(\frac{\|\Delta P\|}{\|P\|}\frac{\|P\|}{\|X_L\|} + 2\sqrt{\frac{2}{\sqrt{3}}}\frac{\|\Delta A\|}{\|A\|}\frac{\|A\|}{\|X_L\|}\right) \quad (4.12)$$

But since $\|X_L\| \geq \frac{1}{3}\| P \| \geq \frac{1}{3}\|P^{-1}\|^{\frac{-3}{4}} > \sqrt{\frac{\sqrt{3}}{6}}\,\|A\|$.

Then we have $\frac{\|P\|}{\|X_L\|} \leq 3$ and $\frac{\|A\|}{\|X_L\|} < \sqrt{\frac{6}{\sqrt{3}}}$.

Substituting with this result in (4.12) yields that

$$\frac{\|\Delta X_L\|}{\|X_L\|} \leq \frac{1}{\delta}\left(\frac{3\,\|\Delta P\|}{\|P\|} + \frac{4\,\|\Delta A\|}{\|A\|}\right)$$

which ends the proof of the theorem.

Remark. From the above theorem we can see that the condition number of the matrix Eq. (1.1) at its maximal solution X_L is equal to $\frac{1}{\delta} = \frac{1}{(1-\sqrt{\frac{\sqrt{27}}{2}\| A \| \| P^{-1}\|^{\frac{3}{4}}})}$ and is denoted by $k\,(A, P)$ which is not too large, especially for the case $0 < \|A\|\,\| P^{-1}\|^{\frac{3}{4}} < \frac{1}{3}\sqrt{\frac{2}{\sqrt{27}}}$ and the condition number in this case is such that $1 < k\,(A, P) < \frac{3}{2}$. Putting $P = I$ in the matrix Eq. (1.1) we get the matrix equation:

$$X + A^T\sqrt{X^{-1}}A = I \quad (4.13)$$

and applying Theorems 4.1 and 4.2 once again to the matrix Eq. (4.13) yields that

$$\frac{\|\Delta X_L\|}{\|X_L\|} \leq \frac{4}{(1 - \sqrt{\frac{\sqrt{27}}{2}}\| A \|)}\frac{\|\Delta A\|}{\|A\|}.$$

To derive an error bound of $\tilde{X}$ we consider the following theorem.

Theorem 4.3. *Let $\tilde{X}$ approximate the maximal solution X_L of the matrix Eq. (1.1). If the following conditions: (1) $\|A\|\,\|P^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ and (2) $\|\tilde{X}^{-1}\| \leq 3\|\,(\tilde{X} + A^T\sqrt{\tilde{X}^{-1}}A)^{-1}\|$ hold and if the residual $R(\tilde{X}) \equiv \tilde{X} + A^T\sqrt{\tilde{X}^{-1}}A - P$ satisfies that $\|R(\tilde{X})\| \leq (1 - (1 - \sqrt{\frac{2}{\sqrt{27}}} + \|A\|\,\|P^{-1}\|^{\frac{3}{4}})^{\frac{4}{3}})\,\|P^{-1}\|^{-1}$ then*

$$\frac{\|\tilde{X} - X_L\|}{\|X_L\|} \leq \frac{3}{\delta}\frac{\|R(\tilde{X})\|}{\|P\|} \quad (4.14)$$

Proof. It is clear that $\tilde{X}$ is a solution of the matrix equation $X + A^T\sqrt{X^{-1}}A = \tilde{P}$, where $\tilde{P} = P + R(\tilde{X})$ and satisfies that $\|\tilde{X}^{-1}\| \leq 3\,\|\,(\tilde{X} + A^T\sqrt{\tilde{X}^{-1}}A)^{-1}\| = 3\,\|\,\tilde{P}^{-1}\|$.

Applying once again the same proof of (4.5), inequality (4.14) can be verified.

Table 1 The numerical results for q_1, q_2, $\|\tilde{A}-A\|$ and $\|\tilde{P}-P\|$.

α	q_1	q_2	$\|\tilde{A}-A\|$	$\|\tilde{P}-P\|$
1.0000 e-08	2.1762 e-09	9.2675 e-07	1.0000 e-08	6.6000 e-09
3.0000 e-08	6.5286 e-09	2.7803 e-06	3.0000 e-08	1.9800 e-08
5.0000 e-08	1.0881 e-08	4.6338 e-06	5.0000 e-08	3.3000 e-08
7.0000 e-08	1.5233 e-08	6.4873 e-06	7.0000 e-08	4.6200 e-08
9.0000 e-08	1.9586 e-08	8.3408 e-06	9.0000 e-08	5.9400 e-08

5. Numerical example

In this section, we report a numerical example for different values of α to illustrate the results derived in Section 4, namely Theorem 4.2 for the matrix Eq. (1.1). We implemented the process (3.1) in MATLAB (writing our own program) and run the program on a PC Pentium IV. For the stopping condition we take $\varepsilon < 1.0e-5$.

Consider the matrix Eq. (1.1) with the matrices A and P as follows:

$$A = \begin{pmatrix} 0.0050 & -0.0025 & 0.0075 & 0.0100 \\ 0.0175 & 0.0150 & -0.0125 & 0.0225 \\ 0.0100 & 0.0200 & 0.0250 & 0.0150 \\ -0.0075 & 0.0125 & 0.0050 & 0.0200 \end{pmatrix},$$

$$P = \begin{pmatrix} 1.0 & 0.3 & 0.0 & 0.0 \\ 0.3 & 3.0 & 0.1 & 0.0 \\ 0.0 & 0.1 & 2.0 & 0.2 \\ 0.0 & 0.0 & 0.2 & 1.0 \end{pmatrix}.$$

The matrices A and P satisfy the condition $\|A\|\,\|P^{-1}\|^{\frac{3}{4}} < \sqrt{\frac{2}{\sqrt{27}}}$ where $\|A\|\,\|P^{-1}\|^{\frac{3}{4}} = 0.0489 < \sqrt{\frac{2}{\sqrt{27}}}$ and the condition number $1 < k\,(A,P) = 1.0855 < \frac{3}{2}$ which is not too large. The maximal solution obtained by the algorithm for the matrix Eq. (1.1) is found to be $X_L = \begin{pmatrix} 0.9997 & 0.2998 & -0.0001 & -0.0002 \\ 0.2998 & 2.9995 & 0.0997 & -0.0006 \\ -0.0001 & 0.0997 & 1.9994 & 0.1998 \\ -0.0002 & -0.0006 & 0.1998 & 0.9991 \end{pmatrix}$. Applying the iterative process (3.1) for Eq. (4.1) where $\tilde{A} = (\alpha\, I) + A$ and $\tilde{P} = (0.66\,\alpha)I + P$.

The obtained results are summarized in the table below:

$$q_1 = \frac{\|\Delta X_L\|}{\|X_L\|} \quad \text{and}$$

$$q_2 = \frac{1}{(1-\sqrt{\frac{\sqrt{27}}{2}}\,\|A\|\,\|P^{-1}\|^{\frac{3}{4}})}\left(\frac{3\,\|\Delta P\|}{\|P\|} + \frac{4\,\|\Delta A\|}{\|A\|}\right)$$

Remarks.

(1) The first column of Table 1 shows different values of the real number α.

q_1 in the second column refers to $\frac{\|\Delta X_L\|}{\|X_L\|}$ while q_2 in the third column refers to $\frac{1}{(1-\sqrt{\frac{\sqrt{27}}{2}}\,\|A\|\,\|p^{-1}\|^{\frac{3}{4}})}\left(\frac{3\,\|\Delta P\|}{\|P\|} + \frac{4\,\|\Delta A\|}{\|A\|}\right)$. For different values of the real number α and applying the iterative process (3.1) for Eq. (4.1), where the perturbed matrix $\tilde{A} = (\alpha\, I) + A$ and the corresponding perturbed matrix $\tilde{P} = (0.66\,\alpha)I + P$, it is clear that the second and third columns show that inequality (4.5) in Theorem 4.2 is satisfied. From Table 1 we see that , the values of q_1 and q_2 increase as the value of α increases.

(2) The fourth and the fifth columns of Table 1 show the values of $\|\tilde{A}-A\|$ and $\|\tilde{P}-P\|$ respectively. It is noted that $\|\tilde{A}-A\| \leq \theta$ and $\|\tilde{P}-P\| \leq \theta$, where $\theta \leq 10^{-7}$.

6. Conclusion

In this paper we are concerned with the nonlinear matrix equation $X + A^T\sqrt{X^{-1}}A = P$. An elegant property of the maximal solution of this matrix equation is presented. Also, a perturbation estimate for the maximal solution X_L of this matrix equation and an error bound for approximate solutions are given. Numerical example is given to illustrate the results, where the obtained numerical results show that the method is reliable.

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