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Oscillation of nonlinear neutral dynamic equations on time scales

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Abstract
The authors present necessary and sufficient conditions for the oscillation of a class of second order non-linear neutral dynamic equations with non-positive neutral coefficients by using Krasnosel'skii's fixed point theorem on time scales. The nonlinear function may be strongly sublinear or strongly superlinear.

Keywords: Oscillation, Nonoscillation, Neutral dynamic equation, Time scales, Fixed point theorem

Mathematics Subject Classification: 34C10, 34K11, 34N05, 39A10

Introduction

Neutral differential/difference equations find numerous applications in biology, engineering, economics, physics, neural networks, social sciences, etc (see, for example, [4, 12, 16]). In the last few decades, many authors have focused their interest on the study of the oscillation of solutions of neutral differential/difference equations with deviating arguments, and in this regard, we refer the reader to the monographs of Agarwal et al. [1, 2] and the papers [3, 7–11, 13–15, 22, 29].

Introduced by Stefan Hilger [17], the notion of time scales is not only to unify the theories of differential equations and difference equations, but also to extend some cases in between these classical ones. For details on the theory of dynamic equations on time scales and its applications as well as for basic concepts and notations, we refer the reader to the works of Bohner and Peterson [5, 6]. By employing a Riccati transformation technique and applying some inequalities, a large number of papers have been devoted to the oscillatory behavior of solutions to second order dynamic equations with nonnegative neutral coefficients; for example, see [3, 8, 9, 23–27] and the references cited therein. At the same time, there are comparatively few papers concerned with the oscillation of equations with nonpositive neutral coefficients; for example, see [7, 14, 18, 20, 28].

Bohner and Li [7] considered the second order dynamic equation

$$\left(r(\ell)|z^{\Delta}(\ell)|^{p-2}z^{\Delta}(\ell)\right)^{\Delta}+q(\ell)|x(\delta(\ell))|^{p-2}x(\delta(\ell))=0, \tag{1}$$

where $z(\ell) = x(\ell) - a(\ell)x(\tau(\ell))$, $p > 1$ is a constant, and $0 \leq a(\ell) \leq a_0 < 1$. They improved the papers [14, 18] by developing a new method for the analysis of the oscillation of (1) via a comparison principle.

Recently, Zhang et al. [28] discussed the neutral dynamic equation

$$\left(r(\ell)(z^{\Delta}(\ell))^{\alpha}\right)^{\Delta} + q(\ell)f(x(\delta(\ell))) = 0, \tag{2}$$

where $z(\ell) = x(\ell) - p(\ell)x(\tau(\ell))$, $\alpha \geq 1$ is a quotient of odd positive integers, $0 \leq p(\ell) \leq p_0 < 1$, and there is a positive constant k such that $\frac{f(x)}{x^{\alpha}} \geq k$ for all $x \neq 0$. They present some new oscillation criteria to ensure that a solution of (2) either oscillates or converges to zero.

Motivated by the results in [28] and the discussion above, in this work we wish to find conditions that are sufficient as well as necessary for the oscillation of second order nonlinear dynamic equations on time scales of the form

$$[a(\ell)\left(\nu^{\Delta}(\ell)\right)^{\alpha}]^{\Delta} + \Lambda(\ell)u^{\beta}(\tau(\ell)) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}, \tag{3}$$

where the time scale $\mathbb{T}$ satisfies $\sup \mathbb{T} = \infty$, $\ell \in [\ell_0, \infty)_{\mathbb{T}}$ with $\ell_0 \in \mathbb{T}$, and $\nu(\ell) = u(\ell) + q(\ell)u(m(\ell))$. A *solution* of (3) is a real function $u \in C^1_{rd}[\ell_0, \infty)_{\mathbb{T}}$ such that $a(\ell)(\nu^{\Delta}(\ell))^{\alpha} \in C^1_{rd}[\ell_0, \infty)_{\mathbb{T}}$ and which satisfies (3) on $[T_u, \infty)_{\mathbb{T}}$, where $T_u > \ell_0$ is chosen so that $\tau(\ell) > \ell_0$ for $\ell \geq T_u$, and $C_{rd}(\mathbb{T}, \mathbb{R})$ is the space of real valued right-dense continuous functions (see [5]). Throughout this paper, we restrict our attention to those solutions of (3) that exist on some half line $[\ell_u, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|u(\ell)| : \ell \geq T\} > 0$ for any $T > T_u$. Such a solution is said to be *oscillatory* if it is not eventually positive or eventually negative, and to be *nonoscillatory* otherwise.

Throughout, we assume that:

($\mathcal{H}_1$) α, β are quotient of odd positive integers, $\alpha > 1$, and $-1 < q_1 \leq q(\ell) \leq 0$;

($\mathcal{H}_2$) m, $\tau \in C_{rd}([l_0, \infty)_{\mathbb{T}}, \mathbb{T})$ with $m(\ell) \leq \ell$, $\tau(\ell) \leq \ell$, and $\lim_{\ell \to \infty} m(\ell) = \lim_{\ell \to \infty} \tau(\ell) = \infty$;

($\mathcal{H}_3$) $\Lambda, a \in C_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$ with $\Lambda(\ell) \not\equiv 0$ and

$$\int_{\ell_0}^{\infty} \frac{\Delta s}{a^{1/\alpha}(s)} = \infty.$$

Defining

$$\mathcal{A}(\ell) = \int_{\ell_0}^{\ell} \frac{\Delta s}{a^{1/\alpha}(s)},$$

we have $\lim_{\ell \to \infty} \mathcal{A}(\ell) = \infty$.

Methods

The approach used involves the construction of an appropriate Banach space and defining two mappings. The sum of these two mapping then yields an operator that is equivalent to an integral representation of the solution to the nonlinear dynamic equation (3) under investigation. By applying Krasnosel'skii's fixed point theorem on time scales, it is then possible to obtain a fixed point of the operator that in turn corresponds to a solution of Eq. (3). Once this is accomplished, various qualitative properties of solution can be obtained.

Results

In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all ℓ large enough. Without loss of generality, in our proofs we only deal with positive solutions of (3).

The following two lemmas provide some inequalities that will be useful in our proofs.

Lemma 1 *Let $0 < \omega < 1$ be the ratio of odd positive integers and $A, B \geq 0$ with $A \geq B$. Then: $A^{\omega} - B^{\omega} \leq (A - B)^{\omega}$.*

Proof For $x \geq 1$ let $f(x) = (x-1)^{\omega} - x^{\omega} + 1$. Then,

$$f'(x) = \omega\Big[(x-1)^{\omega-1} - x^{\omega-1}\Big] = \left[\frac{x^{1-\omega} - (x-1)^{1-\omega}}{x^{1-\omega}(x-1)^{1-\omega}}\right] \geq 0$$

for $x > 1$. Therefore, $f(x) \geq f(1) = 0$ for $x \geq 1$. Letting $x = A/B$ proves the lemma. □

Lemma 2 [15] *Suppose that $\omega > 0$ and $|x|^{\Delta}$ is of one sign on $[t_0, \infty)$. Then*

$$\frac{|x|^{\Delta}}{(|x|^{\sigma})^{\omega}} \leq \frac{(|x|^{1-\omega})^{\Delta}}{1-\omega} \leq \frac{|x|^{\Delta}}{|x|^{\omega}} \quad \text{on } [t_0, \infty).$$

Lemma 3 below can be proved by following the lines of the proof of [20, Lemma 2.1].

Lemma 3 *Let u be an eventually positive solution of (3). Then v satisfies one of the following cases:*

(a) $v > 0, v^{\Delta} > 0,$ *and* $(a(v^{\Delta})^{\alpha})^{\Delta} \leq 0;$
(b) $v < 0, v^{\Delta} > 0,$ *and* $(a(v^{\Delta})^{\alpha})^{\Delta} \leq 0$

for $\ell \in \mathbb{T}$ sufficiently large.

Lemma 4 *Let u be an eventually positive solution of (3) such that v satisfies case (b) of Lemma 3. Then*

$$\lim_{\ell \to \infty} u(\ell) = 0.$$

Proof Let u be an eventually positive solution of (3) with $u(m(\ell)) > 0$ and $u(\tau(\ell)) > 0$ and such that Lemma 3(b) holds for $\ell \geq \ell_1$ for some $\ell_1 \geq \ell_0$. Then $v(\ell) < 0$ and $v^{\Delta}(\ell) > 0$ for $\ell \geq \ell_1$, so $v(\ell)$ is bounded.

We will consider two possibilities. First assume that $u(\ell)$ is bounded. Then,

$$\limsup_{\ell \to \infty} u(\ell) = L \quad \text{with} \quad 0 \leq L < \infty.$$

To show that $L = 0$, assume that $L > 0$. Then there is a sequence $\{\ell_k\} \to \infty$ such that $\{u(\ell_k)\} \to L$ as $\ell \to \infty$. Let $\epsilon = -L(1 + q_1)/2q_1 > 0$; then for large k, $u(m(\ell_k)) < L + \epsilon$, so

$$0 \geq \lim_{k\to\infty} v(\ell_k) = \lim_{k\to\infty} [u(\ell_k) + q(\ell_k)u(m(\ell_k)) > L + q_1(L + \epsilon) > L(1 + q_1)/2 > 0,$$

which is a contradiction.

Finally, to complete the proof, we need to show that $u(\ell)$ is not unbounded. If $u(\ell)$ is unbounded, then there is a sequence $\{\ell_j\} \to \infty$ such that $\{u(\ell_j)\} \to \infty$ as $j \to \infty$ and $u(\ell_j) = \max\{u(\ell) : \ell_0 \leq \ell \leq \ell_j\}$. Now $\{m(\ell_j)\} \to \infty$ and $m(\ell_j) \leq \ell_j$, so

$$u(m(\ell_j)) \leq \max\{u(\ell) : \ell_0 \leq \ell \leq \ell_j\} = u(\ell_j).$$

Hence, for large j,

$$v(\ell_j) = u(\ell_j) + q(\ell_j)u(m(\ell_j)) \geq u(\ell_j) + q_1 u(m(\ell_j)) \geq (1 + q_1)u(\ell_j) > 0,$$

which contradicts the fact that $v(\ell) < 0$. This completes the proof of the lemma. □

Our first result on the asymptotic behavior of solutions of Eq. (3) is as follows.

Theorem 5 *Let* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold and assume that* $\alpha \geq 1$ *and there is a constant* $\gamma \in \mathbb{R}_+$ *such that* $\beta < \gamma < \alpha$. *Then any solution of* (3) *either oscillates or satisfies* $\lim_{\ell\to\infty} u(\ell) = 0$ *if and only if*

$$(\mathcal{H}_4) \qquad \int_{\ell_0}^{\infty} \Lambda(s)\mathcal{A}^{\beta}(\tau(s))\Delta s = \infty.$$

Proof **Necessity:** To prove the necessity of the condition, assume that $(\mathcal{H}_4)$ does not hold. Then there exists $\ell_1 > \ell_0$ such that

$$\int_{\ell_1}^{\infty} \Lambda(s)\mathcal{A}^{\beta}(\tau(s)))\Delta s < \infty. \tag{4}$$

Let

$$\chi = \left\{ u : u \in C_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{R}) \;\middle|\; \sup_{\ell\in[\ell_0,\infty)_{\mathbb{T}}} \frac{u(\ell)}{\mathcal{A}(\ell)} < \infty \right\}.$$

Clearly, χ is a Banach space with the norm $\|u\| = \sup_{\ell\in[\ell_0,\infty)_{\mathbb{T}}} \frac{u(\ell)}{\mathcal{A}(\ell)}$. For any $\varsigma_1 > 0$, $\varsigma_2 > 0$, and $\ell^* \in [\ell_0, \infty)_{\mathbb{T}}$ with $\varsigma_1 < (1 + q_1)\varsigma_2$, let $\Omega_{\varsigma_1,\varsigma_2} \subset \chi$ be given by

$$\Omega_{\varsigma_1,\varsigma_2} = \{u \in \chi : \varsigma_1[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)] \leq u(\ell) \leq \varsigma_2[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)],\ \ell \in [\ell_0, \infty)_{\mathbb{T}}\}.$$

By (4), we can find $\ell^* > \ell_1$, ς_1, ς_2, and ς_3 such that $(\varsigma_1)^{\alpha} < \varsigma_3 < ((1 + q_1)\varsigma_2)^{\alpha}$ and

$$\int_{\ell^*}^{\infty} \Lambda(s)\mathcal{A}^{\beta}(\tau(s)))\Delta s \leq \frac{((1 + q_1)\varsigma_2)^{\alpha} - \varsigma_3}{\varsigma_2^{\beta}}. \tag{5}$$

Define two maps Γ_1 and Γ_2 on $\Omega_{\varsigma_1,\varsigma_2}$ by

$$(\Gamma_1 u)(\ell) = \begin{cases} (\Gamma_1 u)(\ell^*), & \ell \in [\ell_0, \ell^*)_{\mathbb{T}}, \\ -q(\ell)u(m(\ell)), & \ell \in [\ell^*, \infty)_{\mathbb{T}} \end{cases}$$

and

$$(\Gamma_2 u)(\ell) = \begin{cases} (\Gamma_2 u)(\ell^*), & \ell \in [\ell_0, \ell^*)_{\mathbb{T}}, \\ \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)}\left[\varsigma_3 + \int_s^\infty \Lambda(\theta) u^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha} \Delta s, & \ell \in [\ell^*, \infty)_{\mathbb{T}}. \end{cases}$$

First, we show that for any $u_1, u_2 \in \Omega_{\varsigma_1,\varsigma_2}$, we have $\Gamma_1 u_1 + \Gamma_2 u_2 \in \Omega_{\varsigma_1,\varsigma_2}$. To do this, let $u_1, u_2 \in \Omega_{\varsigma_1,\varsigma_2}$. Note that $u(\ell) \le \varsigma_2 \mathcal{A}(\ell)$, so $u^\beta(\tau(\ell)) \le \varsigma_2^\beta \mathcal{A}^\beta(\tau(\ell))$. This, together with (5) implies that for $\ell \ge \ell^*$,

$$\begin{aligned} (\Gamma_1 u_1)(\ell) + (\Gamma_2 u_2)(\ell) &= -q(\ell)u_1(m(\ell)) + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)}\left[\varsigma_3 + \int_s^\infty \Lambda(\theta) u_2^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha} \Delta s \\ &\le -q(\ell)u_1(m(\ell)) + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)}\left[\varsigma_3 + \int_s^\infty \varsigma_2^\beta \Lambda(\theta) \mathcal{A}^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha} \Delta s \\ &\le -q_1\varsigma_2[\mathcal{A}(m(\ell)) - \mathcal{A}(\ell^*)] + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)}\left(\varsigma_3 + ((1+q_1)\varsigma_2)^\alpha - \varsigma_3\right)\right]^{1/\alpha} \Delta s \\ &\le -q_1\varsigma_2[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)] + (1+q_1)\varsigma_2[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)] \\ &\le \varsigma_2[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)] \end{aligned}$$

and

$$\begin{aligned} (\Gamma_1 u_1)(\ell) + (\Gamma_2 u_2)(\ell) &= -q(\ell)u_1(m(\ell)) + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)}\left[\varsigma_3 + \int_s^\infty \Lambda(\theta) u_2^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha} \Delta s \\ &\ge \left[\int_{\ell^*}^{\ell} \varsigma_3 \frac{1}{a(s)}\right]^{1/\alpha} \Delta s \\ &= \varsigma_3^{1/\alpha}[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)] \\ &\ge \varsigma_1[\mathcal{A}(\ell) - \mathcal{A}(\ell^*)]. \end{aligned}$$

Therefore, $\Gamma_1 u_1 + \Gamma_2 u_2 \in \Omega_{\varsigma_1,\varsigma_2}$.

Next, we show that Γ_1 is a contraction mapping on $\Omega_{\varsigma_1,\varsigma_2}$. Now for $u_1, u_2 \in \Omega_{\varsigma_1,\varsigma_2}$ and $\ell \ge \ell^*$, we have

$$|(\Gamma_1 u_1)(\ell) - (\Gamma_1 u_2)(\ell)| \le |q(\ell)||u_1(m(\ell)) - u_2(m(\ell))| \le -q_1|u_1(m(\ell)) - u_2(m(\ell))|,$$

that is,

$$\|\Gamma_1 u_1 - \Gamma_1 u_2\| \le -q_1\|u_1 - u_2\|.$$

Since $0 \le -q_1 < 1$, Γ_1 is a contraction.

To show that Γ_2 is completely continuous, we will first show that Γ_2 is continuous. So fix $\ell \ge \ell^*$ and let $u_k \in \Omega_{\varsigma_1,\varsigma_2}$ be such that $u_k(\ell) \to u(\ell)$ as $k \to \infty$. By taking a subsequence if necessary and again calling it $u_k(\ell)$, we can assume that $u_k(\ell) - u(\ell)$ is of fixed sign, say $u_k(\ell) \ge u(\ell)$ for $k = 1, 2, \ldots$. Since $\Omega_{\varsigma_1,\varsigma_2}$ is closed, $u(\ell) \in \Omega_{\varsigma_1,\varsigma_2}$. By Lemma 1 with $\omega = 1/\alpha \le 1$, we obtain

$$\begin{aligned}|(\Gamma_2 u_k)(\ell)-(\Gamma_2 u)(\ell)| &= \Bigg|\int_{\ell^*}^{\ell}\left[\frac{1}{a(s)}\left[\varsigma_3+\int_s^\infty \Lambda(\theta)u_k^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha}\Delta s\\ &\quad -\int_{\ell^*}^{\ell}\left[\frac{1}{a(s)}\left[\varsigma_3+\int_s^\infty \Lambda(\theta)u^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha}\Delta s\Bigg|\\ &\le \int_{\ell^*}^{\ell}\left[\frac{1}{a(s)}\int_s^\infty \Lambda(\theta)\left|u_k^\beta(\tau(\theta))-u^\beta(\tau(\theta))\right|\Delta\theta\right]^{1/\alpha}\Delta s.\end{aligned}$$

Since $|u_k^\beta(\tau(\theta))-u^\beta(\tau(\theta))|\to 0$ as $k\to\infty$, an application of Lebesgue's dominated convergence theorem shows that $\lim_{k\to\infty}|(\Gamma_2 u_k)(\ell)-(\Gamma_2 u)(\ell)|\to 0$, so $\Gamma_2 u$ is continuous.

To show that Γ_2 is relatively compact, it suffices to show that the family of functions $\{\Gamma_2 u : u\in\Omega_{\varsigma_1,\varsigma_2}\}$ is uniformly bounded and equicontinuous on $[\ell^*,\infty)_{\mathbb{T}}$. Clearly, $\Gamma_2 u$ is uniformly bounded. To see that Γ_2 is equicontinuous, let $\epsilon>0$ be given and choose $\delta>0$ such that $\ell_3>\ell_2\ge\ell^*$ and $|\ell_2-\ell_1|<\delta$ implies $|\mathcal{A}(\ell_3)-\mathcal{A}(\ell_2)|<\epsilon\left\{\frac{1}{[(1+q_1)\varsigma_2]^\alpha-\varsigma_3}\right\}^{1/\alpha}$. Then,

$$\begin{aligned}&|(\Gamma_2 u)(\ell_3)-(\Gamma_2 u)(\ell_2)|\\ &=\left|\int_{\ell^*}^{\ell_3}\left[\frac{1}{a(s)}\int_s^\infty \Lambda(\theta)u^\beta(\tau(\theta))\Delta\theta\right]^{1/\alpha}\Delta s-\int_{\ell^*}^{\ell_2}\left[\frac{1}{a(s)}\int_s^\infty \Lambda(\theta)u^\beta(\tau(\theta))\Delta\theta\right]^{1/\alpha}\Delta s\right|\\ &=\left|\int_{\ell_2}^{\ell_3}\left[\frac{1}{a(s)}\int_s^\infty \Lambda(\theta)u^\beta(\tau(\theta))\Delta\theta\right]^{1/\alpha}\Delta s\right|\\ &\le\left|\int_{\ell_2}^{\ell_3}\left[\frac{1}{a(s)}\int_s^\infty \Lambda(\theta)\varsigma_2^\beta\mathcal{A}^\beta(\tau(\theta))\Delta\theta\right]^{1/\alpha}\Delta s\right|\\ &\le[\mathcal{A}(\ell_3)-\mathcal{A}(\ell_2)]\left[\int_s^\infty \Lambda(\theta)\varsigma_2^\beta\mathcal{A}^\beta(\tau(\theta))\Delta\theta\right]^{1/\alpha}\Delta s<\epsilon.\end{aligned}$$

Thus, $\{\Gamma_2 u : u\in\Omega_{\varsigma_1,\varsigma_2}\}$ is uniformly bounded and equicontinuous on $[\ell^*,\infty)_{\mathbb{T}}$, and so $\Gamma_2 u$ is relatively compact. By Krasnosel'skii's fixed point theorem [29, Lemma 5], $\Gamma_1+\Gamma_2$ has a unique fixed point $u\in\Omega_{\varsigma_1,\varsigma_2}$, i.e., $\Gamma_1 u+\Gamma_2 u=u$. That is,

$$u(\ell)=-q(\ell)u(m(\ell))+\int_{\ell^*}^{\ell}\left[\frac{1}{a(s)}\left[\varsigma_3+\int_s^\infty \Lambda(\theta)u^\beta(\tau(\theta))\Delta\theta\right]\right]^{1/\alpha}\Delta s,\quad \ell\in[\ell^*,\infty)_{\mathbb{T}}.$$

is a nonoscillatory solution of (3).

Sufficiency: Now assume that v is a nonoscillatory solution of (3). Then Lemma 3 holds for $\ell\in[\ell_1,\infty)_{\mathbb{T}}$ for some $\ell_1\ge\ell_0$, and there are two possible cases.

Case a Since $a(v^\Delta)^\alpha$ is nonincreasing and positive for $\ell\in[\ell_1,\infty)_{\mathbb{T}}$, we can find $\mathcal{C}>0$ and $\ell_2>\ell_0$ such that

$$a(\ell)(v^\Delta(\ell))^\alpha\le\mathcal{C}\ \text{ for }\ \ell\in[\ell_2,\infty)_{\mathbb{T}}.$$

Integrating from ℓ_2 to ℓ gives

$$\nu(\ell) \leq \nu(\ell_2) + \mathcal{C}^{1/\alpha} \int_{\ell_2}^{\ell} \frac{\Delta s}{a^{1/\alpha}(s)} = \nu(\ell_2) + \mathcal{C}^{1/\alpha}(\mathcal{A}(\ell) - \mathcal{A}(\ell_2)).$$

Since $\lim_{\ell\to\infty} \mathcal{A}(\ell) = \infty$,

$$\nu(\ell) \leq \mathcal{C}^{1/\alpha}\mathcal{A}(\ell) \tag{6}$$

for ℓ sufficiently large, say $\ell \geq \ell_3$. Now $\beta < \gamma$ and (6) imply

$$\nu^{\beta}(\tau(\ell)) = \nu^{\beta-\gamma}(\tau(\ell))\nu^{\gamma}(\tau(\ell)) \geq [\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\ell))]^{\beta-\gamma}\nu^{\gamma}(\tau(\ell)).$$

Therefore, (3) becomes

$$[a(t)(\nu^{\Delta}(\ell))^{\alpha}]^{\Delta} + \Lambda(\ell)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\ell))]^{\beta-\gamma}\nu^{\gamma}(\tau(\ell)) \leq 0.$$

Integrating the last inequality from $\ell \geq \ell_3$ to ∞ gives

$$\lim_{t\to\infty} a(t)(\nu^{\Delta}(t))^{\alpha} - a(\ell)(\nu^{\Delta}(\ell))^{\alpha} + \int_{\ell}^{\infty} \Lambda(s)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(s))]^{\beta-\gamma}\nu^{\gamma}(\tau(s))\Delta s \leq 0,$$

which implies

$$a(\ell)(\nu^{\Delta}(\ell))^{\alpha} \geq \int_{\ell}^{\infty} \Lambda(s)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(s))]^{\beta-\gamma}\nu^{\gamma}(\tau(s))\Delta s.$$

As a result,

$$\nu^{\Delta}(\ell) \geq \left[\frac{1}{a(\ell)} \int_{\ell}^{\infty} \Lambda(s)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(s))]^{\beta-\gamma}\nu^{\gamma}(\tau(s))\Delta s\right]^{1/\alpha}. \tag{7}$$

Integrating this from ℓ_3 to ℓ, we have

$$\nu(\ell) \geq \int_{\ell_3}^{\ell} \left[\frac{1}{a(s)} \int_{s}^{\infty} \Lambda(\theta)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\theta))]^{\beta-\gamma}\nu^{\gamma}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s.$$

Consequently,

$$\nu(\ell) \geq [\mathcal{A}(\ell) - \mathcal{A}(\ell_1)]\left[\int_{s}^{\infty} \Lambda(\theta)[\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\theta))]^{\beta-\gamma}\nu^{\gamma}(\tau(\theta))\Delta\theta\right]^{1/\alpha}. \tag{8}$$

Clearly, $\int_{\ell_3}^{\ell} \frac{1}{a(s)}\Delta s = \mathcal{A}(\ell) - \mathcal{A}(\ell_3) = \pi(\ell)\mathcal{A}(\ell)$, where $\pi(\ell) = \frac{\mathcal{A}(\ell)-\mathcal{A}(\ell_3)}{\mathcal{A}(\ell)}$. In view of $(\mathcal{H}_3)$, we have $\lim_{\ell\to\infty} \pi(\ell) = 1$, so there exists $\ell_4 \geq \ell_3$ and $\pi^* \in (0,1)$ such that $\pi(\ell) \geq \pi^*$, that is,

$$\mathcal{A}(\ell) - \mathcal{A}(\ell_3) \geq \pi^*\mathcal{A}(\ell) \text{ for } \ell \in [\ell_4, \infty)_{\mathbb{T}}. \tag{9}$$

Setting

$$\Psi(\ell) = \int_{\ell}^{\infty} \Lambda(s)(\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(s))^{\beta-\gamma}\nu^{\gamma}(\tau(s))\Delta s, \tag{10}$$

in (8), we have

$$\nu(\ell) \geq [\mathcal{A}(\ell) - \mathcal{A}(\ell_1)]\Psi^{1/\alpha}(\ell).$$

and in view of (9),

$$\nu(\ell) \geq \pi^* \mathcal{A}(\ell)\Psi^{1/\alpha}(\ell)$$

for $\ell \in [\ell_4, \infty)_{\mathbb{T}}$. From the preceding inequality, it is easy to verify that

$$\frac{\nu^\gamma(\ell)}{\mathcal{C}^{\gamma/\alpha}\mathcal{A}^\gamma(\ell)} \geq \left(\frac{\pi^*}{\mathcal{C}^{1/\alpha}}\right)^\gamma \Psi^{\gamma/\alpha}(\ell)$$

which implies that

$$\frac{\nu^\gamma(\tau(\ell))}{\mathcal{C}^{\gamma/\alpha}\mathcal{A}^\gamma(\tau(\ell))} \geq \left(\frac{\pi^*}{\mathcal{C}^{1/\alpha}}\right)^\gamma \Psi^{\gamma/\alpha}(\tau(\ell))$$

for $\ell \in [\ell_5, \infty)_{\mathbb{T}} \subset [\ell_4, \infty)_{\mathbb{T}}$. From (10), we have

$$\begin{aligned}
\Psi^\Delta(\ell) &= \left(\int_\ell^\infty \Lambda(s)(\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(s))^{\beta-\gamma}\nu^\gamma(\tau(s))\Delta s\right)^\Delta \\
&= -\Lambda(\ell)(\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\ell))^{\beta-\gamma}\nu^\gamma(\tau(\ell)) \\
&= -\Lambda(\ell)(\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\ell))^{\beta}\left(\frac{\nu(\tau(\ell))}{\mathcal{C}^{1/\alpha}\mathcal{A}(\tau(\ell))}\right)^\gamma \\
&\leq -(\pi^*)^\gamma \mathcal{C}^{(\beta-\gamma)/\alpha}\Lambda(\ell)\mathcal{A}^\beta(\tau(\ell))\Psi^{\gamma/\alpha}(\tau(\ell)).
\end{aligned}$$

From Lemma 2 with $\omega = \gamma/\alpha$ and $x = \Psi(\ell)$ and the fact that $\gamma < \alpha$, it follows that

$$\begin{aligned}
-[\Psi^{1-\gamma/\alpha}(\ell)]^\Delta &\geq -(1-\gamma/\alpha)\Psi^{-\gamma/\alpha}(\ell)\Psi^\Delta(\ell) \\
&\geq (\pi^*)^\gamma \mathcal{C}^{(\beta-\gamma)/\alpha}(1-\gamma/\alpha)\Psi^{-\gamma/\alpha}(\ell)\Lambda(\ell)\mathcal{A}^\beta(\tau(\ell))\Psi^{\gamma/\alpha}(\tau(\ell)) \\
&= (\pi^*)^\gamma \mathcal{C}^{(\beta-\gamma)/\alpha}(1-\gamma/\alpha)\Lambda(\ell)\mathcal{A}^\beta(\tau(\ell))
\end{aligned} \tag{11}$$

for $\ell \in [\ell_5, \infty)_{\mathbb{T}}$. Integrating (11) from ℓ_5 to ℓ,

$$-\Psi^{1-\gamma/\alpha}(\ell) + \Psi^{1-\gamma/\alpha}(\ell_5) \geq (\pi^*)^\gamma \mathcal{C}^{(\beta-\gamma)/\alpha}(1-\gamma/\alpha)\int_{\ell_5}^\ell \Lambda(s)\mathcal{A}^\beta(\tau(s))\Delta s$$

so

$$\int_{\ell_5}^\ell \Lambda(s)\mathcal{A}^\beta(\tau(s))\Delta s \leq \frac{\mathcal{C}^{(\gamma-\beta)/\alpha}}{(\pi^*)^\gamma(1-\gamma/\alpha)}\Psi^{1-\gamma/\alpha}(\ell_5)$$

contradicting $(\mathcal{H}_4)$.

Case b Now suppose $\nu < 0$ for $\ell \in [\ell_0, \infty)_{\mathbb{T}}$. Then $u(\ell) \to 0$ as $\ell \to \infty$ by Lemma 4. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 6 *Under the assumption of Theorem 5, every unbounded solution of (3) oscillates if and only if $(\mathcal{H}_4)$ holds.*

Theorem 7 *Let $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ hold, $\sigma(\tau(\ell)) = \tau(\sigma(\ell))$, $a^{\Delta}(\ell) \geq 0$, and there is a constant $\gamma \in \mathbb{R}_+$ such that $\alpha < \gamma < \beta$. Then any solution $u(\ell)$ of (3) is either oscillatory or satisfies $\lim_{\ell\to\infty} u(\ell) = 0$ if and only if*

$$(\mathcal{H}_5) \qquad \lim_{\ell\to\infty} \int_{\ell_0}^{\ell} \int_s^{\infty} \left(\frac{\Lambda(\theta)}{a(s)}\right)^{1/\alpha} \Delta\theta \Delta s = \infty .$$

Proof **Necessity:** Assume that $(\mathcal{H}_5)$ does not hold so that there exists $\ell_1 > \ell_0$ such that

$$\int_{\ell_1}^{\infty} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)\Delta\theta\right]^{1/\alpha} \Delta s < \infty. \tag{12}$$

Letting

$$\chi = \left\{ u : u \in C_{rd}([\ell_0,\infty)_{\mathbb{T}}, \mathbb{R}) \middle| \sup_{\ell\in[\ell_0,\infty)_{\mathbb{T}}} u(\ell) < \infty \right\},$$

we see that χ is a Banach space with the norm $\|u\| = \sup_{\ell\in[\ell_0,\infty)_{\mathbb{T}}} u(\ell)$. Choose $\varsigma_1 > 0$ and $\varsigma_2 > 0$ so that $\varsigma_1 - q_1\varsigma_2 < \varsigma_2$ and consider $\Omega_{\varsigma_1,\varsigma_2} \subset \chi$ to be

$$\Omega_{\varsigma_1,\varsigma_2} = \{u \in \chi : \varsigma_1 \leq u(\ell) \leq \varsigma_2, \ \ \ell \in [\ell_0,\infty)_{\mathbb{T}}\}.$$

By (12), we can find $\ell^* > \ell_1$ and $\varsigma_3 > 0$ such that $\varsigma_1 < \varsigma_3 < (1+q_1)\varsigma_2$ and

$$\int_{\ell^*}^{\infty} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)\Delta\theta\right]^{1/\alpha} \Delta s \leq \frac{(1+q_1)\varsigma_2 - \varsigma_3}{\varsigma_2^{\beta/\alpha}}. \tag{13}$$

Define two maps Γ_1 and Γ_2 on Ω by

$$(\Gamma_1 u)(\ell) = \begin{cases} (\Gamma_1 u)(\ell^*), & \ell \in [\ell_0,\ell^*)_{\mathbb{T}}, \\ \varsigma_3 - q(\ell)u(m(\ell)), & \ell \in [\ell^*,\infty)_{\mathbb{T}} \end{cases}$$

and

$$(\Gamma_2 u)(\ell) = \begin{cases} (\Gamma_2 u)(\ell^*), & \ell \in [\ell_0,\ell^*)_{\mathbb{T}}, \\ \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta) u^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s, & \ell \in [\ell^*,\infty)_{\mathbb{T}}. \end{cases}$$

To show that $\Gamma_1 + \Gamma_2 : \Omega \to \Omega$, let $u_1, u_2 \in \Omega$. Then from (13),

$$\begin{aligned} (\Gamma_1 u_1)(\ell) + (\Gamma_2 u_2)(\ell) &= \varsigma_3 - q(\ell)u_1(m(\ell)) + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta) u_2^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s \\ &\leq \varsigma_3 - q_1\varsigma_2 + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta) u_2^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s \\ &\leq \varsigma_3 - q_1\varsigma_2 + \varsigma_2^{\beta/\alpha} \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)\Delta\theta\right]^{1/\alpha} \Delta s \\ &\leq \varsigma_2 \end{aligned}$$

and

$$(\Gamma_1 u_1)(\ell) + (\Gamma_2 u_2)(\ell) \geq \varsigma_3 - q(\ell)u_1(m(\ell)) \geq \varsigma_3 \geq \varsigma_1$$

for $\ell \geq \ell^*$. Hence, $\Gamma_1 u_1 + \Gamma_2 u_2 \in \Omega_{\varsigma_1,\varsigma_2}$.

To see that Γ_1 is a contraction, let $u_1, u_2 \in \Omega_{\varsigma_1,\varsigma_2}$ and $\ell \geq \ell^*$. Then,

$$|(\Gamma_1 u_1)(\ell) - (\Gamma_1 u_2)(\ell)| \leq |q(\ell)||u_1(m(\ell)) - u_2(m(\ell))| \leq -q_1|u_1(m(\ell)) - u_2(m(\ell))|,$$

so

$$\|\Gamma_1 u_1 - \Gamma_1 u_2\| \leq -q_1\|u_1 - u_2\|,$$

i.e., Γ_1 is a contraction mapping.

To show that Γ_2 is completely continuous, we begin by letting $u_k \in \Omega_{\varsigma_1,\varsigma_2}$ be such that $u_k(\ell) \to u(\ell)$ as $k \to \infty$. Since $\Omega_{\varsigma_1,\varsigma_2}$ is closed, $u(\ell) \in \Omega_{\varsigma_1,\varsigma_2}$. Now

$$|(\Gamma_2 u_k)(\ell) - (\Gamma_2 u)(\ell)| \leq \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)|u_k^{\beta}(\tau(\theta)) - u^{\beta}(\tau(\theta))|\Delta\theta\right]^{1/\alpha} \Delta s.$$

Since $|u_k^{\beta}(\tau(\theta)) - u^{\beta}(\tau(\theta))| \to 0$ as $k \to \infty$, an application of Lebesgue's dominated convergence theorem implies $\lim_{k\to\infty} |(\Gamma_2 u_k)(\ell) - (\Gamma_2 u)(\ell)| \to 0$. Hence, $\Gamma_2 u$ is continuous. To show that $\Gamma_2 u$ is relatively compact, it suffices to show that the family of functions $\{\Gamma_2 u : u \in \Omega_{\varsigma_1,\varsigma_2}\}$ is uniformly bounded and equicontinuous on $[\ell^*, \infty)_{\mathbb{T}}$. The uniform boundedness is clear.

To show $\Gamma_2 u$ is equicontinuous, let $\epsilon > 0$ be given and choose $\delta > 0$ such that $\ell_3 > \ell_2 \geq \ell^*$ and $|\ell_2 - \ell_1| < \delta$ implies

$$\int_{\ell_2}^{\ell_3} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)\Delta\theta\right]^{1/\alpha} \Delta s < \frac{\epsilon}{\varsigma_2^{\beta/\alpha}}.$$

Then,

$$\begin{aligned}
&|(\Gamma_2 u)(\ell_3) - (\Gamma_2 u)(\ell_2)| \\
&\quad = \left|\int_{\ell^*}^{\ell_3} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)u^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s - \int_{\ell^*}^{\ell_2} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)u^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s\right| \\
&\quad = \left|\int_{\ell_2}^{\ell_3} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)u^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s\right| \\
&\quad \leq \varsigma_2^{\beta/\alpha} \int_{\ell_2}^{\ell_3} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)\Delta\theta\right]^{1/\alpha} \Delta s < \epsilon.
\end{aligned}$$

Therefore, $\Gamma_2 u$ is relatively compact, and by Krasnosel'skii's fixed point theorem [, Lemma 5], 29$\Gamma_1 + \Gamma_2$ has a unique fixed point $u \in \Omega_{\varsigma_1,\varsigma_2}$. It follows that

$$u(\ell) = \varsigma_3 - q(\ell)u(m(\ell)) + \int_{\ell^*}^{\ell} \left[\frac{1}{a(s)} \int_s^{\infty} \Lambda(\theta)u^{\beta}(\tau(\theta))\Delta\theta\right]^{1/\alpha} \Delta s,\ \ell \in [\ell^*, \infty)_{\mathbb{T}}$$

is a nonoscillatory solution of (3).

Sufficiency: Let u be a nonoscillatory solution of (3) with Lemma 3 holding for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. We again consider two cases.

Case a Let $v > 0$; then $u(\ell) \geq v(\ell)$ for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. From the fact that $v^{\Delta}(\ell) > 0$ for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$, it follows that $v(\tau(\ell)) \geq v(\tau(\ell_1)) = \mathcal{C}$ for $\ell \in [\ell_2, \infty)_{\mathbb{T}}$ for some $\ell_2 \geq \ell_1$. Since $\gamma < \beta$,

$$v^{\beta}(\tau(\ell)) = v^{\beta-\gamma}(\tau(\ell))v^{\gamma}(\tau(\ell)) \geq \mathcal{C}^{\beta-\gamma}v^{\gamma}(\tau(\ell)). \tag{14}$$

Using (14) in (3), we obtain

$$[a(\ell)(v^{\Delta}(\ell))^{\alpha}]^{\Delta} + \mathcal{C}^{\beta-\gamma}\Lambda(\ell)v^{\gamma}(\tau(\ell)) \leq 0,$$

and an integration from ℓ to ∞ gives

$$\lim_{t\to\infty} a(t)(v^{\Delta}(t))^{\alpha} - a(\ell)(v^{\Delta}(\ell))^{\alpha} + \mathcal{C}^{\beta-\gamma}\int_{\ell}^{\infty} \Lambda(s)v^{\gamma}(\tau(s))\Delta s \leq 0,$$

that is,

$$\mathcal{C}^{\beta-\gamma}\int_{\ell}^{\infty} \Lambda(s)v^{\gamma}(\tau(s))\Delta s \leq a(\ell)(v^{\Delta}(\ell))^{\alpha} \leq a(\tau(\ell))(v^{\Delta}(\tau(\ell)))^{\alpha}.$$

Using the fact that $a^{\Delta}(\ell) \geq 0$, we see that

$$(v^{\Delta}(\tau(\ell)))^{\alpha} \geq \frac{\mathcal{C}^{\beta-\gamma}}{a(\ell)}\int_{\ell}^{\infty} \Lambda(s)v^{\gamma}(\tau(s))\Delta s,$$

which implies

$$\begin{aligned} v^{\Delta}(\tau(\ell)) &\geq \frac{\mathcal{C}^{(\beta-\gamma)/\alpha}}{a^{1/\alpha}(\ell)}\left[\int_{\ell}^{\infty} \Lambda(s)v^{\gamma}(\tau(s))\Delta s\right]^{1/\alpha} \\ &\geq \frac{\mathcal{C}^{(\beta-\gamma)/\alpha}}{a^{1/\alpha}(\ell)}\left[\int_{\sigma(\ell)}^{\infty} \Lambda(s)v^{\gamma}(\tau(s))\Delta s\right]^{1/\alpha} \\ &\geq \frac{\mathcal{C}^{(\beta-\gamma)/\alpha}}{a^{1/\alpha}(\ell)}\left[\int_{\sigma(\ell)}^{\infty} \Lambda(s)\Delta s\right]^{1/\alpha}(v^{\sigma}(\tau(\ell)))^{\gamma/\alpha}, \end{aligned}$$

that is

$$v^{\Delta}(\tau(\ell))(v^{\sigma}(\tau(\ell)))^{-\gamma/\alpha} \geq \frac{\mathcal{C}^{(\beta-\gamma)/\alpha}}{a^{1/\alpha}(\ell)}\left[\int_{\sigma(\ell)}^{\infty} \Lambda(s)\Delta s\right]^{1/\alpha}.$$

Since $\alpha < \gamma$, by Lemma 2

$$\frac{\mathcal{C}^{(\beta-\gamma)/\alpha}}{a^{1/\alpha}(\ell)}\left[\int_{\sigma(\ell)}^{\infty} \Lambda(s)\Delta s\right]^{1/\alpha} \leq v^{\Delta}(\tau(\ell))(v^{\sigma}(\tau(\ell)))^{-\gamma/\alpha} \leq \frac{[v^{1-\gamma/\alpha}(\tau(\ell))]^{\Delta}}{1-\gamma/\alpha}.$$

Integrating the preceding inequality from ℓ_2 to ℓ gives

$$\mathcal{C}^{(\beta-\gamma)/\alpha}\int_{\ell_2}^{\ell}\left[\frac{1}{a(s)}\int_s^{\infty}\Lambda(\theta)\Delta\theta\right]^{1/\alpha}\Delta s \le \frac{1}{1-\gamma/\alpha}\int_{\ell_2}^{\ell}\left[\nu^{1-\gamma/\alpha}(\tau(s))\right]^{\Delta}\Delta s$$
$$= \frac{1}{\gamma/\alpha-1}\left[\nu^{1-\gamma/\alpha}(\tau(\ell_2))-\nu^{1-\gamma/\alpha}(\tau(\ell))\right]$$
$$\le \frac{1}{\gamma/\alpha-1}\nu^{1-\gamma/\alpha}(\tau(\ell_2)),$$

contradicting $(\mathcal{H}_5)$.

Case b If $\nu < 0$, then $u(\ell) \to 0$ by Lemma 4. This proves the theorem. □

The following corollary is analogous to Corollary 6.

Corollary 8 *Under the assumption of Theorem 5, every unbounded solution of (3) oscillates if and only if $(\mathcal{H}_5)$ holds.*

Discussion

First, we constructed an appropriate Banach space as the setting on which to defining two mappings Γ_1 and Γ_2. The sum of these two mappings is an operator that is equivalent to an integral representation of the solution to the nonlinear dynamic equation (3) under investigation. By applying Krasnosel'skii's fixed point theorem on time scales, it was then possible to obtain a fixed point of the operator that in turn corresponds to a solution of Eq. (3). Once this was accomplished, various results on the qualitative properties of solution were obtained. For example, we found sufficient conditions for positive solutions to converge to zero (Lemma 4). In addition, we were able to prove necessary and sufficient conditions for a solution to either oscillate or converge to zero (Theorems 5 and 7) , and necessary and sufficient conditions for unbounded solutions to oscillate (Corollaries 6 and 8).

Conclusion

In this work, we discuss two classes of oscillation criteria for (3). Note that Theorem 5 and Theorem 7 guarantee that a solution of (3) either oscillates or converges to zero. In Corollaries 6 and 8, we restrict the solutions to make (3) oscillatory. Here, we formulate some interesting problem for future research:

1. Is it possible to find necessary and sufficient conditions for the oscillation of

$$[a(\ell)((u(\ell)+q(\ell)u(m(\ell)))^{\Delta})^{\alpha}]^{\Delta}+\Lambda(\ell)u^{\beta}([\tau(\ell)])=0$$

   under the assumption $\beta < \gamma < \alpha$ or $\alpha < \gamma < \beta$?
2. Following the work in [19, 21], is it possible to find necessary and sufficient conditions for the oscillation of the forced equation

$$[a(\ell)((u(\ell)+q(\ell)u(m(\ell)))^{\Delta})^{\alpha}]^{\Delta}+\Lambda(\ell)u^{\beta}(\tau(\ell))=f(\ell),$$

   with either $\beta < \gamma < \alpha$ or $\alpha < \gamma < \beta$?

Authors' contributions

Each of the authors contributed equally to the research, writing, and preparation of this manuscript. In particular, GC, SG, and JG participated in the writing and revising of the manuscript. The formulation of the results was a joint effort by GC, SG, and JG. Although JG is serving as the corresponding author, GC and SG approved the final version of the paper before it was submitted. All authors read and approved the final manuscript.

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Declarations

Competing interests

The authors declare that they have no competing interests.

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