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On Zero-Truncated Poisson- Muth Distribution

إعـــداد

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Abstract

This paper presents a novel distribution, referred to as the Zero-Truncated Poisson Muth (ZTPM) Distribution. an important statistical model for analyzing data that excludes zero values. This study examines the statistical properties of the Zero-Truncated Poisson Muth (ZTPM) distribution, covering its probability density function (PDF), cumulative distribution function (CDF), survival function, and hazard rate. Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method, applied across different parameter values and sample sizes. The study also derives moments, order statistics, and examines skewness and kurtosis of the distribution. Through practical examples, the ZTPM distribution demonstrates its flexibility in handling non-zero data, proving highly effective for parameter estimation and data analysis in fields such as reliability and survival analysis.

Keywords:

Zero-Truncated Poisson Muth- Maximum Likelihood Method- Skewness-Kurtosis- Mean Residual Life- Order Statistics.

1. Introduction:

Muth presented a probability distribution characterized by continuity, specifically applied in the realm of reliability theory by Jodrá Esteban, Jiménez Gamero, & Alba Fernández, [1]. If a random variable x possesses a Muth Distribution with parameter α.

$$F_1(t,\propto) = 1 - e^{\alpha t - \frac{1}{\propto}(e^{\alpha t}-1)}, 0 < \propto \leq 1 \quad (1)$$

The associated probability density function (pdf) is as follows

$$f_1(t,\propto) = (e^{\alpha t} - \propto)e^{\alpha t - \frac{1}{\propto}(e^{\alpha t}-1)}, t > 0 \quad (2)$$

Abouelmagd, Hamed, Handique, et al.,[2], A new family of distributions is studied, defined by the minimum of a Poisson-distributed number of independent and identically distributed random variables following the Topp Leone-G distribution. Several mathematical properties of this new family are derived, and the estimation of model parameters using Maximum Likelihood Estimation (MLE) is explored.

Additionally, two special cases within this family of distributions are discussed. a new class of lifetime distributions will be constructed using the ascendant order statistics. Orabi, Ahmed, & Ziedan,[3]. A new probability distribution, the Weibull-Generalized Truncated Poisson (WGTP) distribution, is introduced. The properties of the WGTP distribution are examined, and parameter estimation is carried out using both the Maximum Likelihood (ML)

method and the Expectation-Maximization (EM) algorithm. A comparison of this new distribution with other lifetime distributions is conducted using a real dataset provided by Abouelmagd, Hamed, Hamedani, et al., [4].

The aim of this study is to introduce a new family of continuous distributions with significant physical applications. Several statistical properties are derived, along with useful characterizations of the proposed distribution family. Five practical applications are provided to demonstrate its relevance. A modified goodness-of-fit test for the new family, in the case of complete data, is explored through two examples. As an initial step, the construction of the Nikulin-Rao-Robson statistic, based on chi-squared goodness-of-fit tests, is proposed for this family in complete data scenarios. The new test is based on the Nikulin-RaoRobson statistic separately proposed by NIKULIN & VOINOV; [5] As a second step, an application to real data has been proposed to show the applicability of the proposed test.

Badr, Hassan, El Din, & Ali, [6] introduces and discusses a new three-parameter lifespan distribution called Zero Truncated Poisson Pareto distribution ZTPP , building on compounding Pareto distribution as a continuous distribution and Zero-Truncated Poisson distribution as a discrete distribution.

2. The Zero Truncated Poisson Muth distribution:

A new distribution, known as the Zero-Truncated Poisson Muth (ZTPM) Distribution, is introduced. This distribution is formed by combining the continuous Muth Distribution, which has two parameters, with the discrete Zero-Truncated Poisson Distribution. The integration of these two distributions results in the Zero-Truncated Poisson Muth (ZTPM) Distribution.

The probability mass function for (ZTPM) is:

$$F(u,\propto,\lambda)=\frac{1-e^{-\lambda F_1(t,\propto)}}{1-e^{-\lambda}} \qquad (3)$$

And, probability density function for (ZTPM) is:

$$f(u,\propto,\lambda)=\frac{\lambda e^{-\lambda F_1(t,\propto)}.f_1(t,\propto)}{1-e^{-\lambda}} \qquad (4)$$

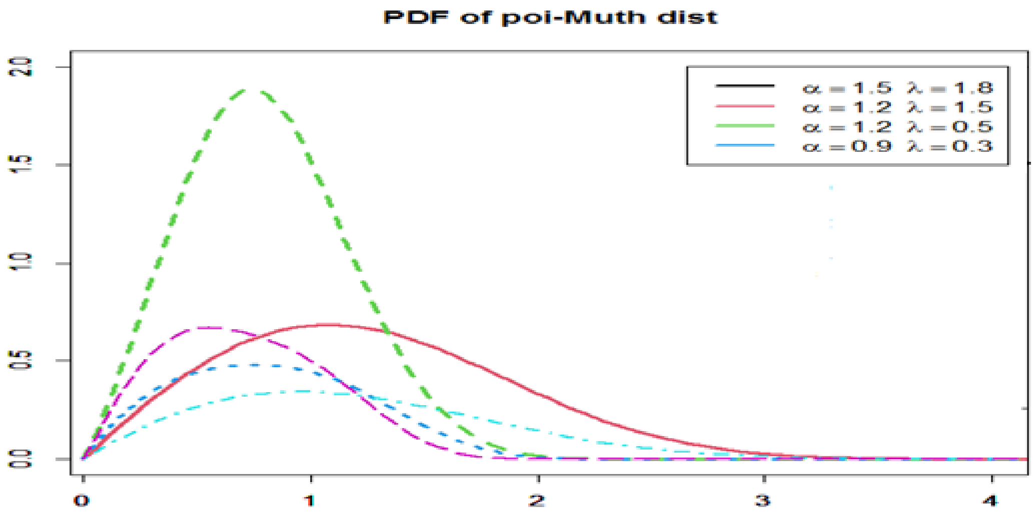
The probability mass function (cdf) for (ZTPM) is:

$$F(x,\propto,\lambda)=\frac{1-e^{-\lambda[1-e^{S_1}]}}{1-e^{-\lambda}} \qquad (5)$$

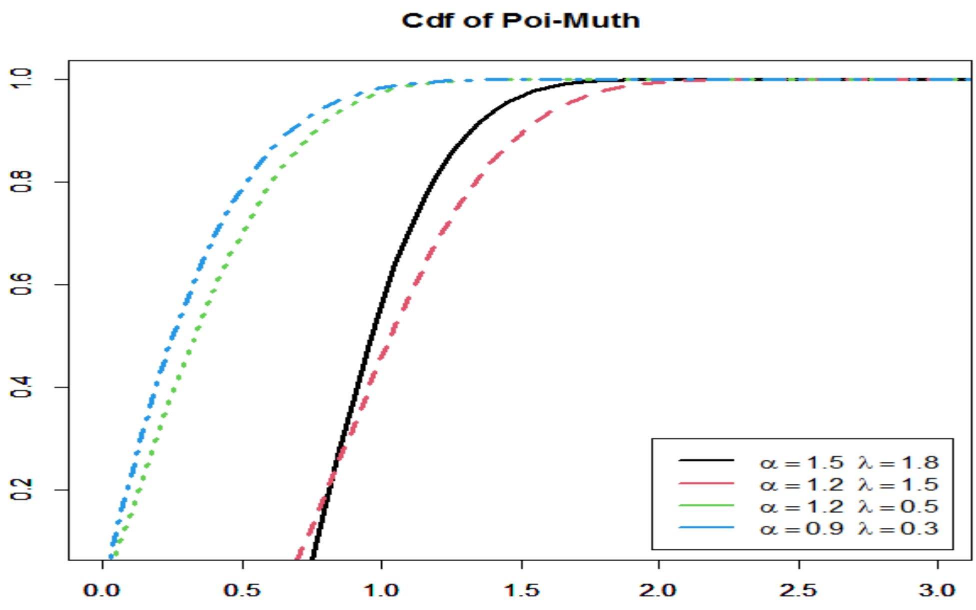
And, probability density function (pdf) for (ZTPM) is:

$$f(x,\propto,\lambda)=\frac{\lambda[e^{\alpha x}-\alpha].e^{\{-\lambda[1-e^{S_1}]+S_1\}}}{1-e^{-\lambda}} \qquad (6)$$

$$,\quad S_1=\alpha x-\frac{1}{\alpha}[e^{\alpha x}-1] \qquad (7)$$



Graph (1): the ZTPM probability density function (PDF) for various parameter values.



Graph (2): the ZTPM cumulative distribution function (CDF) for various parameter values.

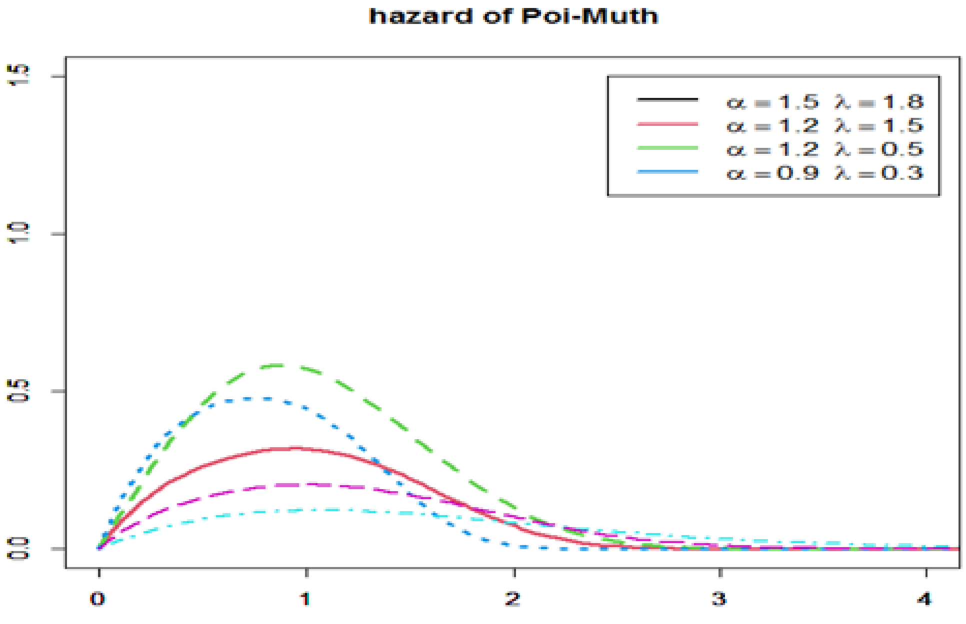
3. Some statistical properties:

The survival function and hazard rate function for the ZTPM distribution are defined for a random variable x, with the probability density function $f(x)$, cumulative distribution function $F(x)$, and survival function $S(x)$, respectively, given by:

$$S(x) = \frac{e^{-\lambda}\left[e^{\left[1-e^{S_1}\right]} - 1\right]}{1 - e^{-\lambda}} \tag{8}$$

Hazard Rate Functions:

$$h(x) = \frac{\lambda[e^{\alpha x} - \alpha].\, e^{\{-\lambda\left[1-e^{S_1}\right]+S_1\}}}{e^{-\lambda}\left[e^{\left[1-e^{S_1}\right]} - 1\right]} \tag{9}$$



Graph(3): the ZTPM hazard function (HF) for various parameter values.

3.1. Quantile function:

The quantile function of the KMD distribution can be derived by inverting equation (5) as follows:

$$q = \frac{1 - e^{-\lambda\left[1-e^{S_1}\right]}}{1 - e^{-\lambda}}$$

$$q\left[1 - e^{-\lambda}\right] = \left[1 - e^{-\lambda\left[1-e^{S_1}\right]}\right]$$

$$e^{-\lambda\left[1-e^{S_1}\right]} = 1 - q\left[1 - e^{-\lambda}\right]$$

$$-\lambda\left[1 - e^{S_1}\right] = ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]$$

$$\left[1 - e^{S_1}\right] = \frac{-ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}$$

$$e^{S_1} = 1 + \frac{ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}$$

$$S_1 = ln\left\{1 + \frac{ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}\right\}$$

$$\alpha^2 x - \left(\frac{1}{\alpha}\right)\left[e^{\alpha x} - 1\right] = ln\left\{1 + \frac{ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}\right\} \qquad (10)$$

$$\alpha^2 x - \left[e^{\alpha x} - 1\right] = \alpha ln\left\{1 + \frac{ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}\right\} \qquad (11)$$

$$\alpha^2 x - \left[e^{\alpha x} - 1\right] = C$$

By using the LEMMA;[7].

$$x = -\frac{1}{\alpha} W\left(\frac{C - 1}{\alpha}\right) \qquad (12)$$

$$x = -\frac{1}{\alpha} W\left(\frac{\alpha ln\left\{1 + \frac{ln\left[1 - q\left(1 - e^{-\lambda}\right)\right]}{\lambda}\right\} - 1}{\alpha}\right) \qquad (13)$$

For any $\alpha \in [0,1]$, $q \in [0,1]$ $and\ x > 0$

which the Lambert function W^{-1} in equation (13) is given by [8],[9].

The quantile function $Q(u;\lambda,\alpha) is$:

$$Q(u;\lambda,\alpha) = -\frac{1}{\alpha}W\left(\frac{\alpha ln\left\{1+\frac{ln\left[1-q\left(1-e^{-\lambda}\right)\right]}{\lambda}\right\}-1}{\alpha}\right) \quad (11)$$

The median $M(u)$ of the KM distribution can be determined by setting $q = 0.5$, as shown below:

$$M(u) = -\frac{1}{\alpha}W\left(\frac{\alpha ln\left\{1+\frac{ln\left[1-0.5\left(1-e^{-\lambda}\right)\right]}{\lambda}\right\}-1}{\alpha}\right) \quad (12)$$

Using Wolfram Alpha (https://www.wolframalpha.com/), we can calculate the median. Similarly, by setting $q = 0.25$ and $q = 0.75$ we can obtain the quartiles.

3.2. Skewness and kurtosis:

The quantile function is essential in estimation and simulation, particularly for heavy-tailed distributions. In such cases, skewness and kurtosis measures based on quantiles are more reliable than those based on moments, as higher moments

may not exist; Kenney & Keeping, [10]and [11] are defined as follows:

$$Bsk = \frac{q_{0.75} - 2q_{0.5} + q_{0.25}}{q_{0.75} - q_{0.25}}$$

$$Mkur = \frac{q_{0.875} - q_{0.625} - q_{0.375} + q_{0.125}}{q_{0.75} - q_{0.25}}$$

The above measures are less sensitive to outliers.

Table:(1) below presents the values of Bowley skewness and Moor kurtosis for different values of λ and α.

table: (1)

Q.00	parm 1	parm 2	parm 3	parm 4	parm 5	parm 6	parm 7
Q.25	1.765	1.882	1.945	1.987	2.015	2.034	2.046
Q.5	2.030	2.145	2.231	2.267	2.304	2.328	2.340
Q.75	2.289	2.391	2.454	2.499	2.528	2.545	2.554
Q.12	1.921	2.011	2.064	2.105	2.131	2.149	2.160
Q.375	2.105	2.201	2.263	2.311	2.342	2.361	2.372
Q.625	2.209	2.304	2.366	2.413	2.444	2.463	2.474
Q.875	2.459	2.549	2.606	2.649	2.675	2.692	2.702
Bskewness	-0.490	-0.622	-0.710	-0.744	-0.812	-0.839	-0.856
Mkurtosis	1.980	3.224	4.785	7.125	10.438	15.294	22.637

parm $(i) = (\lambda\ and\ \alpha), i = 1, 2, \dots, 7$ **for** $\lambda = 1.5$**and** $\alpha = 2$.

3.3. Ordinary Moments:

The r th ordinary moments It was formulated by Mariano,[12] ,The r th ordinary moments of the Zero-Truncated Poisson- Muth Distribution is given by:

$$\mu_r' = E(x^r) = \int_{\alpha}^{\infty} x^r \ f(x, \propto, \lambda) dx$$

$$\mu_r' = \int_{\alpha}^{\infty} x^r \ \frac{\lambda[e^{\alpha x} - \alpha].\, e^{\{-\lambda[1-e^{S_1}]+S_1\}}}{1 - e^{-\lambda}} dx$$

Table: (2) below presents the values of the first four moments: mean (μ), m2, m3 , m4, the variance (μ_2), the coefficient of variation (CV), the skewness (sk), and the kurtosis (kur) of distribution for λ and α .

table:(2)

Moments	α=0.5, λ=1.5	$\alpha = 0.5, \lambda = 0.7$	$\alpha = 0.9, \lambda = 0.2$	$\alpha = 0.7, \lambda = 0.4$
μ	0.523934	0.426718	0.480866	0.392354
μ_2'	0.708002	0.529801	0.429769	0.364174
μ_3'	1.108628	0.745829	0.440081	0.382698
μ_4'	1.92596	1.148053	0.494527	0.437419
Sk	1.256652	1.381879	0.926608	1.168615
Kur	0.993067	1.087622	0.480208	0.77642

3.4. Order Statistics:

Let $x_1, x_2, \dots, x_n$ be random sample having order statistics The pdf of x_n of order statistics is given by:

$$f_{L.n}(x) = \frac{n!}{(n-L)!\,(r-L)!}[f(x)][F(x)]^{L-1}[1-F(x)]^{n-L}$$

we get the pdf for Zero-Truncated Poisson- Muth Distribution of the L_{th} order Statistics as follow:-

$$f_{L.n}(x) = \frac{n!}{(n-L)!\,(r-L)!}\left[\frac{\lambda[e^{\alpha x}-\alpha].e^{\{-\lambda[1-e^{S_1}]+S_1\}}}{1-e^{-\lambda}}\right]\left[\frac{1-e^{-\lambda[1-e^{S_1}]}}{1-e^{-\lambda}}\right]^{L-1}\left[1-\frac{1-e^{-\lambda[1-e^{S_1}]}}{1-e^{-\lambda}}\right]^{n-L}$$

when L=1 and when L=n the pdf of order statistic

$$f_{1.n}(x) = n\left[\frac{\lambda[e^{\alpha x}-\alpha].e^{\{-\lambda[1-e^{S_1}]+S_1\}}}{1-e^{-\lambda}}\right]\left[1-\frac{1-e^{-\lambda[1-e^{S_1}]}}{1-e^{-\lambda}}\right]^{n-1}$$

$$f_{n.n}(x) = n\left[\frac{\lambda[e^{\alpha x}-\alpha].e^{\{-\lambda[1-e^{S_1}]+S_1\}}}{1-e^{-\lambda}}\right]\left[1-\frac{1-e^{-\lambda[1-e^{S_1}]}}{1-e^{-\lambda}}\right]^{n-1}$$

3.5. Mean Residual life (MRL):

The mean residual life (MRL), also known as life expectancy at age t represents the expected additional lifespan for an individual who has already reached age t The concept of mean residual life has been explored by Siddiqui & Çağlar, [13] .

The MRL has many important applications in fuzzy set engineering , modeling ,insurance assessment of human life expectancy,

demography, and economic etc .The MRL is the conditional expectation $E(x - t|x > t)$ where $t > 0$. The MRL function cab be simply represented with the survival function $S(x)$. For a random lifetime X, the MRL is :

$$\mathrm{MRL} = \frac{1}{S(x)} \int_{t}^{\infty} S(x)dx \quad , \qquad S(x) > 0$$

when $S(0) = 1$ and $= 0$, the MRL equal the average lifetime. when the MRL is represented with $S(x)$. We refer to the theoretical mean residual life (TMRL) as the MRL derived from a ZTPM distribution. when we calculate the MRL from a random sample $x_1, x_2, ..., x_n$ of size n. the result is known as the empirical mean residual life (EMRL). This can be computed using the following expression:

$$\mathrm{EMRL} = \frac{1}{(n-k)} \sum_{k=1}^{n-1} (x_{k+1} - x_k)$$

where $x_{(k)}$ is the k^{th} orders statistic of the sample . Table(3) given below present the MRL of ZTPM.

table:(3)

	time	MRL
α=0.2, λ=1.5	5	1.301069
	10	1.152569
	15	1.103045
	20	1

4. The Renyi Entropy:

Entropy is a thermodynamic quantity used as a measure of uncertainty variation of systems studied by Rényi,[14] .

If x is a random variable has the Zero-Truncated Poisson Muth Distribution then the Renyi entropy of x is defined as:

$$R(p) = \frac{1}{1-p} \ln\left[\int_0^\infty f^p(x)dx \quad\right]$$

Where $f(x)$ is given by (6).The value of $R(p)$can be evaluated using the integrate function of the R software.

The values of $R(p)$ for different values of the parameters λ and α are presented in Table (4) given below:

Table (4)

Theoretical MRL		
α, λ	$\nu = 0.5$	$\nu = 0.7$
1.5,1.2	0.952809	1.528623
1.2,1.5	1.036202	1.232592
0.9,1.2	1.227322	1.397229
1.7,0.7	1.356533	4.044979
0.5,0.7	1.53646	1.73585
1.5,0.5	1.635067	5.544234
0.9,0.2	2.545582	10.05108

5. The Maximum Likelihood Estimation:

The maximum likelihood function is a commonly used method for estimating the parameters of various distributions. It was formulated by Mood,[15] .

We will use it in this study and study its results. The Maximum Likelihood method is a traditional and widely used approach for estimating the parameters of the ZTPM model. It simplifies the estimation process by taking the logarithm of the likelihood function, making it easier to apply both analytically and numerically, especially for large sample sizes. The likelihood function is expressed as follows:

$$\ell(\theta) = \prod_{i=1}^{n} g(x_i;\theta)$$

$$\begin{aligned} \ln l(\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left\{\lambda[e^{\alpha x} - \alpha].\, e^{\{-\lambda[1-e^{S_1}]+S_1\}}\left[1 - e^{-\lambda}\right]^{-1}\right\} \\ - (2\alpha + 1)\sum_{i=1}^{n} \ln\left\{\left(1 - e^{-\lambda[1-e^{S_1}]}\right)\left(1 - e^{-\lambda}\right)^{-1}\right\} \\ + \sum_{i=1}^{n} \ln\left\{1 - \alpha\left[\left(1 - e^{-\lambda[1-e^{S_1}]}\right)\left(1 - e^{-\lambda}\right)^{-1}\right]^{\alpha}\right\} \\ - \frac{1}{\alpha}\sum_{i=1}^{n}\left\{\left[\left(1 - e^{-\lambda[1-e^{S_1}]}\right)\left(1 - e^{-\lambda}\right)^{-1}\right]^{-\alpha}\right\} \end{aligned}$$

The elements of the score function: -

$$\frac{\partial\, ln\, l(\theta)}{\partial\alpha} = \frac{n}{\alpha^2} - 2\sum_{i=1}^{n} ln\left[\left(1-e^{-\lambda[1-e^{S_1}]}\right)\left(1-e^{-\lambda}\right)^{-1}\right] - \sum_{i=1}^{n}\frac{\left[\left(1-e^{-\lambda[1-e^{S_1}]}\right)\left(1-e^{-\lambda}\right)^{\alpha}\right]\left[1+ln\left(1-e^{-\lambda[1-e^{S_1}]}\right)\left(1-e^{-\lambda}\right)\right]}{1-\alpha\left[\left(1-e^{-\lambda[1-e^{S_1}]}\right)(1-e^{-\lambda})^{-1}\right]^{\alpha}} + \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[\left(1-e^{-\lambda[1-e^{S_1}]}\right)\left(1-e^{-\lambda}\right)^{-1}\right]^{-\alpha}(1+\alpha)ln\left[\left(1-e^{-\lambda[1-e^{S_1}]}\right)\left(1-e^{-\lambda}\right)\right]$$

$$\frac{\partial\, ln\, l(\theta)}{\partial\lambda} = \sum_{i=1}^{n}\frac{\frac{\partial f(x)}{\partial\lambda}}{f(x)} - (2\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial f(x)}{\partial\lambda}}{F(x)} - \alpha^2\frac{F(x)^{\alpha-1}\frac{\partial F(x)}{\partial\lambda}}{1-\alpha F(x)^{\alpha}} + \sum_{i=1}^{n}\left[F(x)^{\alpha-1}\frac{\partial F(x)}{\partial\lambda}\right]$$

we use R software to obtain the optimal estimates for the parameters λ and α which maximize the likelihood function.

5.1. Parameters of simulation :

1. **Number of replications = 5000**
2. **Sample sizes are: $n = 10, 20$.**
3. **Maximum Likelihood Method**
4. **Parameters of the Zero-Truncated Poisson Muth distribution are**
5. **Computed measure: Average (Avg.) and root mean square error (RMSE)**

Table (5): Average and RMSE for different estimation methods of ZTPM distribution at different sample sizes.

sample size	parameter	AVG	RMSE
10	α=0.1	0.2751384	0.2118674
	λ=0.1	2	1.9
10	α=0.2	0.259705	0.1531813
	λ=0.2	2	1.8
10	α=0.3	0.2504074	0.1499231
	λ=0.3	2	1.7
10	α=0.4	0.2551199	0.2019407
	λ=0.4	1.99	1.599247
10	α=0.5	0.2545282	0.2831923
	λ=0.5	1.98	1.495168
10	α=0.2	0.2561508	0.2814336
	λ=0.2	2	1.8
10	α=0.3	0.257866	0.1528999
	λ=0.3	1.999	1.699835
10	α=0.5	0.2585935	0.1483501
	λ=1.5	2	1.9
20	α=0.1	0.2558036	0.2111585
	λ=0.1	2	1.9
20	α=0.2	0.2546193	0.1534458
	λ=0.2	2	1.8
20	α=0.3	0.2528781	0.1494979
	λ=0.3	2	1.7

5.2. Analysis of parameter estimates using the maximum likelihood method:

- **Different sample sizes:**

The estimates were made at different sample sizes (10, 20), reflecting how sample size affects parameter estimates. Larger sample sizes typically increase the accuracy of the estimates, resulting in lower RMSE values.

- **Parameter estimates:**

The parameter estimation results (means λ and α show good convergence to the true values used to generate the data. This demonstrates the ability of the maximum likelihood method to estimate the parameters accurately.

- **RMSE value:**

RMSE values reflect the level of accuracy of the estimates. The lower the RMSE value, the higher the accuracy. The RMSE is expected to decrease with increasing sample size, which can be seen in the results.

- **Variation in estimates:**

The table shows the variability in estimates across different parameter values. This suggests that the distribution model reacts differently to changes in the parameters, emphasizing the importance of choosing different parameter values in practical applications.

- **Final results:**

It can be concluded that the resulting estimates were consistent with the true values, providing confidence in the use of the Zero-Truncated Poisson Muth model in future research. Further analysis of larger data sets or application of different estimation methods may be useful to confirm these results.

6. Conclusion:

In conclusion, this study introduced the Zero-Truncated Poisson Muth (ZTPM) Distribution, a novel statistical model tailored for data that exclude zero values.

The research thoroughly examined the key statistical properties of this distribution, including its probability density function, cumulative distribution function, survival function, hazard rate, and moments.

The Maximum Likelihood Estimation (MLE) method was successfully employed to estimate the parameters across

various sample sizes, demonstrating its robustness and reliability.

The ZTPM distribution has proven to be a flexible and efficient model for analyzing non-zero data, especially in fields such as reliability, survival analysis, and other applications where zero values are not permitted. Additionally, the study has highlighted the distribution's ability to model skewness and kurtosis, making it a versatile tool for statistical analysis.

Future research could further expand on this work by applying the ZTPM distribution to more complex real-world datasets, investigating alternative parameter estimation methods, and exploring its potential in broader applications.

Overall, the ZTPM distribution is a valuable contribution to statistical modeling, offering a powerful framework for analyzing non-zero datasets.

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