Journal of Fractional Calculus and Applications
Vol. 16(1) Jan. 2025, No.2.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

TWO DIMENSIONAL THERMOELASTICITY PROBLEMS IN AN INHOMOGENEOUS STRIP WITH APPLICATION OF DIRECT INTEGRATION METHOD.

S. D. NIRDE, K. P. GHADLE

Abstract. This paper employ an analytical approach for solving the two-dimensional problems of elasticity and thermo-elasticity in terms of stresses in an inhomogeneous strip which is infinite. We consider application of direct integration method in a plane steady state heat conduction problem for a semi plane. Application of this method are depend on the direct integration of equilibrium equation for efficient analysis of inhomogeneous solids. With the application of direct integration method for differential and compatibility equation for isotropic material, we are reducing desired equation to integro-differential equation. We have solved these dominant equations by applying simple iteration method.The results for displacement and stresses are computed numerically. One can find the displacement in terms of strains by the integration of Cauchy relations. Using Simple iteration method to find thermal stresses in terms of Volterra- integro differential equations.The calculation to construct the solution can be also useful to solve some optimization problem as well as inverse thermoelasticity problems in terms of stresses.

1. Introduction

Direct integration method plays an important role to solve the boundary value problems. The useful segment for elastic and thermoelastic responses in composite materials exhibit anisotropic features which are depends on the accuracy by considering disparity of material modulie with contract in spatial directions. Hence the method for analysis of isotropic solid will be largely fail when attempted for anisotropic solid. For example, if we consider eigen function method to justify local effects in semi-infinite elastic composites then the convenient solution decomposes when it moves away from the corresponding loaded area. This change disturbs the type with character of corresponding eigenfunctions and involve them by assuring their fade behavior at different points. The another disadvantages restrict the relevance of such type of solutions for the whole spectra of practical anisotropic moduli. The required equilibrium equations can be expressed as in terms of stresses. These do not depend on the material properties as well as on the physical

2020 *Mathematics Subject Classification.* 35B07; 35G30; 35K05; 44A10.

Key words and phrases. Direct integration method, isotropic material, simple iteration method,inhomogeneous strip,Fourier transform.

Submitted March 19, 2024. Revised June 29, 2024.

stress strain relation. In [2] Hamoud et al. studied some numerical techniques to solve integro-differential equations. [3] Kalynyak et al. studied direct and inverse problems of thermomechanics to concern the optimization with identification of the thermal stressed state for deformed solids. [7] Tokovyy and Chien-Ching studied an explict form solution to the plane elasticity and thermoelasticity problems for anisotropic and inhomogeneous solids. Yasinskyy and Ierokhova studied optimization of nonstationary thermal displacements in a given cross section of a half space in the plane strain state [11]. Tokovyy and Chien-Ching provides analytical solutions to the 2D elasticity and thermoelasticity problems for inhomogeneous planes and half-planes [8]. Rychahivskyy and Tokovyy has given correct analytical solutions to the thermoelasticity problems in a semiplane [4]. Rychahivskyy, Tokovyy gives analytic solution of the plane problem of the theory of elasticity for a nonuniform strip [9]. Ghadle and Adhe studied steady state temperature analysis to 2D elasticity and thermoelasticity problems for inhomogeneous solids in half plane with reduction of given heat conduction problem to voltera type integral equations [1].

. In this paper we extend the technique to avoid latter complications to represents solution for an elastic isiotropic material. This method was established by Vigak [10]. This method was already applied to solve some direct and inverse boundary value problems [6]. After integrating the differential equilibrium equations, we can determine the relationship between the stress tensor componennt. With this technique the governing equations are reduced to integro-differential equation for stress tensor component. With application of simple iteration method, derived integral equations has been solved for constructing the solution in explict form expression with interdependance of elastic material modulie.

2. Preliminary

In this section, we collect some basic definitions that will be important to us in the sequel.

2.1. **Definition.** A Fourier transform of function $f(x)$ is defined as: $F(\omega) = \int_{-\infty}^{+\infty} f(x) exp(-i\omega x) dx$

3. Formulation of problem

Consider a plane quasi-static thermoelasticity problem in a rectangular domain $D = \{(x, y) \in (-\infty, \infty) \times [0, \infty)\}$ with the absences of body forces for an isotropic material. The problem is governed by the equilibrium equation [6],

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, (x, y) \in D \tag{1}$$

and compatibility equation in terms of strains [5]

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{2}$$

and physical relation with plane strains

$$\left.\begin{array}{ll} 2Ge_{xx} = & (1-\nu)\sigma - \sigma_{yy} + 2\alpha G(1+\nu)T \\ 2Ge_{yy} = & -\nu\sigma + \sigma_{yy} + 2\alpha G(1+\nu)T \\ e_{xy} = & \frac{\sigma_{xy}}{G} \end{array}\right\} \tag{3}$$

where σ_{xx}, σ_{yy}, σ_{xy}, e_{xx}, e_{yy}, e_{xy} are stress and strain tensor components and G, E, ν are shear modulus, modulus of elasticity and Poission's ratio respectively.
We impose tractions at the boundary,

$$\sigma_{yy}\Big|_{y=0} = -p_1(x), \quad \sigma_{xy}\Big|_{y=0} = q_1(x), \tag{4}$$

Assume that, as $|x| \to \infty$ the stresses are tending to 0.
From the third relation of (3) and the equilibrium equation (1) representating (2) as follows,

$$\frac{\partial^2}{\partial y^2}\Big(\frac{1-\nu}{2G}\sigma+\alpha(1+\nu)T\Big)+\Big(\frac{1-\nu}{2G}\Big)\frac{\partial^2\sigma}{\partial x^2}+\alpha(1+\nu)\frac{\partial^2 T}{\partial x^2}=\frac{\sigma_y}{2}\frac{d^2}{dy^2}\Big(\frac{1}{G}\Big) \tag{5}$$

To compute the total stress σ=σ_{xx}+σ_{yy} in terms of σ_{yy}. We use the relation,

$$\Delta\sigma_{yy}=\frac{\partial^2\sigma}{\partial x^2} \tag{6}$$

To find out solution for problem (1) to (6), selecting one key stress out of three stress component. To find out the two dimensional stressed state, the equation of continuity for these regions, written for the normal stresses σ_{yy}. Integrating equation (1) as in [5], express the stresses σ_{xy} in terms of σ_{xx}, σ_{yy}.

$$4\sigma_{xy}=q_1-\int_{-\infty}^{\infty}\frac{\partial\sigma_{yy}}{\partial y}sign(x-\eta)d\eta-\int_{0}^{\infty}\frac{\partial\sigma_{xx}}{\partial x}sign(y-\zeta)d\zeta \tag{7}$$

$$sign(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$$

4. Construction of Solution

To find out the key stresses, using integral Fourier transform [8] for (5) - (6) to achieve the following equations,

$$\frac{d^2}{dy^2}\Big(\frac{1-\nu}{2G}\bar{\sigma}+\alpha(1+\nu)\bar{T}\Big)-\omega^2\Big(\frac{1-\nu}{2G}\bar{\sigma}+\alpha(1+\nu)\bar{T}\Big)=\frac{\bar{\sigma}_{yy}}{2}\frac{d^2}{dy^2}\Big(\frac{1}{G}\Big) \tag{8}$$

$$\frac{d^2\bar{\sigma}_{yy}}{dy^2}-\omega^2\bar{\sigma}_{yy}=-\omega^2\bar{\sigma} \tag{9}$$

$$\bar{\sigma}_{yy}\Big|_{y=0}=-\bar{p}_1,\ \bar{\sigma}_{xy}\Big|_{y=0}=-\bar{q}_1,$$

This key stress σ_{yy} should satisfy the boundary condition

$$\frac{\partial\sigma_{yy}}{\partial y}\Bigg|_{y=0}=-i\omega\bar{q}_1, \tag{10}$$

Here, ω deontes integral transform parameter, i=$\sqrt{-1}$
Solving (8)-(9), we obtain the expression for $\bar{\sigma}_{yy}$ as,

$$\begin{aligned}\bar{\sigma}_{yy}=&-\bar{p}_1cosh(\omega(1+y))-i\bar{q}_1sinh(\omega(1+y))\\&+\frac{|\omega|}{2}\int_0^y\bar{\sigma}sinh(\omega(y-\zeta))d\zeta\end{aligned} \tag{11}$$

with two integral conditions

$$\int_0^\infty\bar{\sigma}sinh(\omega\zeta)d\zeta=(\bar{q}_1)\frac{isinh\omega}{(\omega)}-(\bar{p}_1)\frac{cosh(\omega)}{\omega}, \tag{12}$$

$$\int_0^\infty\bar{\sigma}cosh(\omega\zeta)d\zeta=(-\bar{q}_1)\frac{icosh\omega}{\omega}-(\bar{p}_1)\frac{sinh(\omega)}{\omega}$$

From eq. (11), eq. (8) can be written as,

$$\begin{aligned}\bar{\sigma}=&\frac{2G}{1-\nu}\Big\{Acosh(\omega y)+Bsinh(\omega y)+P_1\bar{p_1}+Q_1\bar{q_1}-\alpha(1+\nu)\bar{T}\Big\}-\\&\frac{1}{2}\int_0^y\frac{d^2}{d\zeta^2}\left(\frac{1}{G(\zeta)}\right)sinh(\omega(y-\zeta))\int_0^\zeta\bar{\sigma}(\bar{\eta})sinh(\omega(\zeta-\eta))d\eta d\zeta\end{aligned} \tag{13}$$

where,

$$P_1 = -\frac{1}{2\omega}\int_0^y \frac{d^2}{d\zeta^2}\big(\frac{1}{G(\zeta)}\big)cosh\big(\omega(1+\zeta)\big)sinh\big(\omega(y-\zeta)\big)d\zeta$$

$$Q_1 = -\frac{i}{2\omega}\int_0^y \frac{d^2}{d\zeta^2}\big(\frac{1}{G(\zeta)}\big)sinh\big(\omega(1+\zeta)\big)sinh\big(\omega(y-\zeta)\big)d\zeta$$

From (12), we can determine the constants A and B. eq. (13) yields the following integral equation,

$$\begin{aligned}\bar{\sigma} =& \frac{2G}{1-\nu}\Big\{Acosh(\omega y) + Bsinh(\omega y) + P_1\bar{p_1} + Q_1\bar{q_1} - \alpha(1+\nu)\bar{T}\Big\} \\ &- \frac{1}{2}\int_0^y \bar{\sigma}(\zeta)K(\zeta,\eta,y)d\zeta,\end{aligned} \tag{14}$$

where

$$K(\zeta,\eta,y) = \int_\zeta^y \frac{d^2}{d\zeta^2}\big(\frac{1}{G(\zeta)}\big)sinh\big(\omega(y-\zeta)\big)sinh\big(\omega(\zeta-\eta)\big)d\zeta$$

We can solve (14), by simple iteration method [5] as follows

$$\begin{aligned}\bar{\sigma}_n =& \frac{2G}{1-\nu}\Big\{A_n cosh(\omega y) + B_n sinh(\omega y) + P_1\bar{p_1} + Q_1\bar{q_1} \\ &- \alpha(1+\nu)\bar{T}\Big\} - \frac{1}{2}\int_0^y \bar{\sigma}_n(\zeta)K(\zeta,\eta,y)d\zeta,\end{aligned} \tag{15}$$

After $\bar{\sigma}$ we can find $\bar{\sigma}_{yy}$ from (11). The constantA_1, B_1 can be determined by using $\bar{\sigma}_0 = 0$. Applying the inverse Fourier transform [6] we can determine σ, σ_{yy} and from (7) we can determine the shear stress σ_{xy}. If $\frac{1}{G}$ is linear in y then we can find exact solution for Eq. (15)

$$\bar{\sigma} = \frac{2G}{1-\nu}\Big\{Acosh(\omega y) + Bsinh(\omega y) - \alpha(1+\nu)\bar{T}\Big\} \tag{16}$$

$$\begin{aligned}\bar{\sigma}_{yy} = & -\bar{p}_1 cosh(\omega(1+y)) - i\bar{q}_1 sinh(\omega(1+y)) - 2A\omega\int_0^y \frac{G(\zeta)cosh(\omega\zeta)sinh(\omega(y-\zeta))}{1-\nu}d\zeta \\ &- 2B\omega\int_0^y \frac{G(\zeta)sinh(\omega\zeta)sinh(\omega(y-\zeta))}{1-\nu}d\zeta \\ &- \omega\int_0^y \frac{\alpha(\zeta)E(\zeta)\bar{T}(\zeta) - G(\zeta)H(\zeta))}{1-\nu(\zeta)}sinh(\omega(y-\zeta))d\zeta\end{aligned}$$

where

$$A = \frac{I_2\phi_1 - I_1\phi_2}{I_3I_2 - I_1^2}, \quad B = \frac{I_3\phi_2 - I_1\phi_1}{I_3I_2 - I_1^2}, \quad I_1 = \int_0^\infty \frac{G(\zeta)}{1-\nu(\zeta)}sinh(\omega\zeta)cosh(\omega\zeta)d\zeta,$$

$$I_2 = \int_0^\infty \frac{G(\zeta)}{1-\nu(\zeta)}sinh^2(\omega\zeta)d\omega, \quad I_3 = \int_0^\infty \frac{G(\zeta)}{1-\nu(\zeta)}cosh^2(\omega\zeta)d\omega$$

$$\phi_1 = \frac{1}{2}\int_0^\infty \left(\bar{\sigma}cosh(\omega\zeta) - \frac{G(\zeta)H(\zeta)}{1-\nu(\zeta)}cosh(\omega\zeta) + \frac{\alpha E\bar{T}}{2(1-\nu(\zeta))}\cosh(\omega\zeta)\right)d\zeta$$

$$\phi_2 = \frac{1}{2}\int_0^\infty \left(\bar{\sigma}sinh(\omega\zeta) - \frac{G(\zeta)H(\zeta)}{1-\nu(\zeta)}sinh(\omega\zeta) + \frac{\alpha E\bar{T}}{2(1-\nu(\zeta))}sinh(\omega\zeta)\right)d\zeta$$

If E, G, ν are constants then from eq. (16) gives the similar equation for σ and σ_{yy}. Reducing (14) to obtain the following expression,

$$\bar{\sigma} = E\Big\{Acosh(\omega y) + Bsinh(\omega y) + P_1\bar{p_1} + Q_1\bar{q_1} - \alpha(1+\nu)\bar{T}\Big\} - \frac{1}{2}\int_0^y \bar{\sigma}(\zeta)K(\zeta,\eta,y)d\zeta.$$

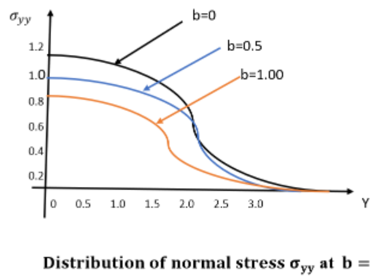

FIGURE 1. Distribution of σ_{yy} for 0, 0.5, 1

5. NUMERICAL RESULTS

EXAMPLE 5.1

Consider an inhomogeneous strip D which is infinite. Let

$$p_1 = \frac{exp(-x^4)}{2}, \quad q_1 = 0, \quad \nu = \frac{2}{3 - by} \quad G = a = const \quad and \quad b = const.$$

From (16)

$$\bar{\sigma} = \frac{2a}{1-\nu}\Big\{Acosh(\omega y) + Bsinh(\omega y) - \alpha(1+\nu)\bar{T}\Big\},$$

$$\begin{aligned}\bar{\sigma}_{yy} =& -\bar{p}_1 cosh(\omega(1+y)) - i\bar{q}_1 sinh(\omega(1+y))\\ & - 2A\omega\int_0^y \frac{G(\zeta)cosh(\omega\zeta)sinh(\omega(y-\zeta))}{1-\nu}d\zeta\\ & - 2B\omega\int_0^y \frac{G(\zeta)sinh(\omega\zeta)sinh(\omega(y-\zeta))}{1-\nu}d\zeta\end{aligned}$$

where A and B can be determined as follows,

$$\begin{aligned}A =& -\frac{1}{2\omega}\Big[i(-\bar{p_1})cosh(\omega)\Big]\int_0^y \frac{sinh(\omega\zeta)cosh(\omega\zeta)}{1-\nu}d\zeta\\ & + (\bar{p_1})sinh(\omega)\int_0^y \frac{sinh^2(\omega)}{1-\nu}d\zeta\\ B =& -\frac{1}{2\omega}\Big[i(-\bar{p_1})sinh(\omega)\Big]\int_0^y \frac{sinh(\omega\zeta)cosh(\omega\zeta)}{1-\nu}d\zeta\\ & - (\bar{p_1})cosh(\omega)\int_0^y \frac{cosh^2(\omega)}{1-\nu}d\zeta\end{aligned}$$

EXAMPLE 5.2

Consider example to find distribution of the normal stress σ_{yy} and σ.

$$E = b_1 f(y), f(y) = \begin{cases} \frac{1}{1+y} & \\ \frac{1}{2}, & y > \frac{1}{d}\end{cases}$$

$b_1 = const$, $0 < d = const$

Assume Possion's ratioν=0.2

We can represent relation between Young and Shear modulus in the form $G = G_0 f(y)$ where G_0=$\frac{b_1}{1-\nu}$

To obtain the relative stress, consider the 1-d temperature field $T(x) = x^3$.The normal stress computed from the formulae(11) and (16) for $y = 2, 3, 4$ as shown in figure-3.

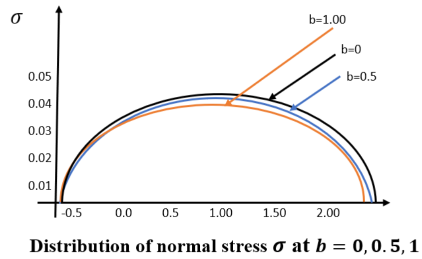

Figure 2. Distribution of σ for 0, 0.5, 1

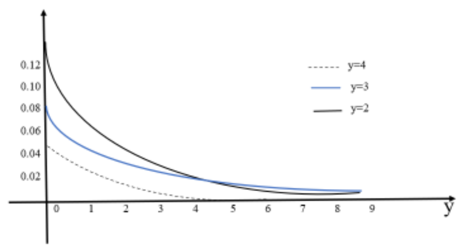

Figure 3. Distribution of $\frac{\sigma_{yy}}{\sigma}$ at y= 2, 3, 4

Here, FIGURE-1 indicates the distribution of normal stess σ_{yy} at b=0, 0.5, 1. FIGURE-2 indicates the distribution of normal stress σ at b=0, o.5, 1 and FIGURE-3 indicates the relationship between the stresses σ_{yy} and σ.

6. ACKNOWLEDGEMENT

The authors are thankful to Council of Scientific and Industrial Research **(CSIR-HRDG)**, Delhi for awarding the research fellowship **09/809-0028(2021)-EMR-I**.

7. Conclusion

This paper construct an analytical approach to solve the two dimensional problems of elasticity and thermoelasticity in terms of stresses for isotropic material in an inhomogeneous strip which is infinite. This approach is placed on the direct integration of differential equilibrium equations. This technique permits to construct analytical solution for intederpendance between the elastic modulie of an isotropic material. We reduce the governing integro-differential equations with variable coefficients in accordance with compatibility and equilibrium equations. The calculation for constructing the solution can be also applied to solve optimization problems, comparable inverse thermoelasticity problems in terms of stresses. In this method we can easily calculate the stressed state in an infinite strip, as compare to solving such problem in terms of displacement. With the help of simple iteration method, we have solved these governing equations. This method gives exact analytical solutions if the shear modulus is reciprocal of linear function in catresian coordinate system for corresponding problems. Direct integration method is very useful technique to solve the boundary value problems. Since, application of this method depend on the direct integration of the equilibrium equations for effiicient analysis of inhomogeneous solids.

References

[1] K. Ghadle and A. Adhe, Steady state temperature analysis to 2D elasticity and thermoelasticity problems for inhomogeneous solids in half plane. J. Korean Soc. Ind. App.(JKSIAM) , Vol. 24, pp. 93-102, 2020.

[2] A. Hamoud, N. Mohammed, K. Ghadle and S. Dhondge, Solving integro-differential equations by using numerical techniques. Int. J. App. Eng. Res. , Vol. 14, pp. 3219-3225, 2019.

[3] B. Kalynyak, Y. Tokovyy and A. Yasinskyy, Direct and inverse problems of thermomechanics concerning the optimization and identification of the thermal stressed state of deformed solids. J. Math. Sci. , Vol. 236, pp. 21-34, 2019.

[4] A. Rychahivskyy and Y. Tokovyy, Correct analytical solutions to the thermoelasticity problems in a semiplane. J. Therm. Stresses, Vol. 31, pp-1125-1145, 2008.

[5] Y. Tokovyy and A. Rychahivskyyi, Reduction of plane thermoelasticity problem in inhomogeneous strip to integral voltera type equation. Math. Model. Anal. , Vol. 10, pp. 91-100, 2005.

[6] Y. Tokovyy and Ma. Chien-Ching, Elastic analysis of inhomogeneous solids. J. Mech. , Vol. 35, pp. 613-626, 2019.

[7] Y. Tokovyy and Ma. Chien-Ching, An explict form solution to the plane elasticity and thermoelasticity problems for anisotropic and inhomogeneous solids. Int. J. Solids Struct. , Vol. 46, pp. 3850-3859, 2009.

[8] Y. Tokovyy and Ma. Chien-Ching, Analytical solutions to the 2D elasticity and thermoelasticity problems for inhomogeneous planes and half-planes. Arch. Appl. Mech., Vol. 79, pp. 441–456, 2009.

[9] Y. Tokovyy and A. Rychahivskyy, Analytic solution of the plane problem of the theory of elasticity for a nonuniform strip. Mater. Sci. , Vol. 41, pp. 114-116, 2005.

[10] V. Vigak, Correct solutions of plane elastic problems for a half-plane. Int.Appl. Mech. , Vol. 40, pp. 283-289, 2004.

[11] A. Yasinskyy and O. Ierokhova, Optimization of nonstationary thermal displacements in a given cross section of a half space in the plane strain state. J. Math. Sci. , Vol. 223, pp. 140-147, 2017.

S. D. Nirde
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar, India
Email address: nirdeshamald@gmail.com

K. P. Ghadle
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Chhatrapati Sambhajinagar, India
Email address: drkp.ghadle@gmail.com