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Research article

Point and Interval Estimation of Reliability and Entropy for Generalized Exponential Distribution under Generalized Type-II Hybrid Censoring Scheme

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Abstract: In this research paper, the estimation of reliability and entropy measures are analyzed within the context of a generalized Type-II hybrid censoring scheme (GT2HCS) The analysis is conducted while considering a generalized exponential distribution as the underlying lifetime distribution. The study specifically emphasizes the calculation of both Shannon and Rényi entropy measures. Maximum likelihood estimation is evaluated for estimating reliability and entropy measures, along with their asymptotic confidence intervals. Bayesian estimation techniques are also considered, utilizing gamma priors for unknown parameters under different loss function. Markov Chain Monte Carlo technique is employed using Metropolis-Hasting algorithm for obtaining Bayesian estimates. Furthermore, credible intervals are determined using the highest posterior density approach. Monte Carlo simulation study is conducted to assess the performance of the proposed estimators. Additionally, a real-life data set from an engineering application is analyzed to demonstrate the GT2HCS under different estimation methods proposed in the study.

Keywords: Generalized exponential distribution, Generalized Type-II hybrid censoring, Shannon and Rényi entropy, Reliability estimation, Maximum likelihood estimation, Bayesian estimation.
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1. Introduction

Entropy is a fundamental concept in statistical distribution that provides important insights into the characteristics of a probability distribution. It is a measure of the uncertainty or randomness associated with the outcomes of a random variable. In other words, it is a measure of the degree of disorder or unpredictability in a system. It is a measure of how much information is needed to describe the probability distributions. The entropy plays a crucial role in a variety of field in science and engineering including in information theory, thermodynamics, statistical physics, signal processing, agricultural products, economics, financial markets and machine learning. Entropy measures help us understand how likely it is for a random variable to take on any given value, how different outcomes of a random variable are distributed, and how likely it is for the outcomes to cluster around certain values. Entropy measures can also be used to assess the amount of information that is lost when a probability distribution is approximated by a simpler model, and can help us understand how different probability distributions compare with each other. Entropy measures can also be used to construct new distributions from existing ones, and to determine the most efficient coding scheme for a given set of probabilities. By combining entropy and statistical reliability, it is possible to gain a better understanding of how data is distributed and how it can be used to draw valid conclusions.

There are several different formulas for calculating entropy, which can be used to measure the degree of disorder or randomness in a system. In this paper we will discuss some of the most common measure of entropy used for statistical life distributions, namely, Shannon entropy (H_{SE}) [1, 2] and Rényi entropy (H^{ρ}_{RE}) [3, 4]. Suppose a non-negative and absolutely continuous random variable X with probability density function (PDF) denoted by $f(x;\omega)$ and cumulative distribution function (CDF) denoted by $F(x;\omega)$, where ω is a vector of parameters. Then, H_{SE} and H^{ρ}_{RE} of X which are defined respectively as follow

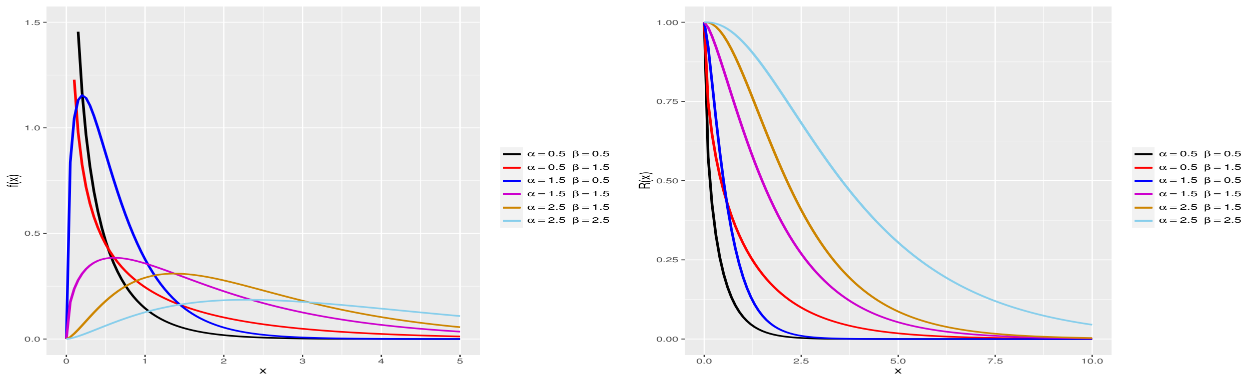
$$H_{SE} = -\int_{-\infty}^{\infty} f(x;\omega)\log\left(f(x;\omega)\right)dx, \tag{1.1}$$

and

$$H^{\rho}_{RE} = \frac{1}{1-\rho}\log\left(\int_{-\infty}^{\infty}(f(x;\omega))^{\rho}dx\right), \qquad \rho \geq 0, \rho \neq 1 \tag{1.2}$$

where ρ is a positive real parameters. Many authors have studied the estimation of entropy for different life distributions by using various methods, see for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The generalized exponential distribution (Gen-Ex distribution) is a flexible probability distribution which can be used for modeling a wide range of data and can be used to generate accurate predictions [18]. It has applications in various fields, including finance, economics, and engineering. In finance, it is used to model stock returns, while economics, it is used to model income distributions. In engineering, it is used to model the failure times of electronic devices. Also, the Gen-Ex distribution is a flexible distribution that can fit data sets including those with heavy tails, skewness, and asymmetry. It is a non-symmetric distribution around zero. The Gen-Ex distribution is a particular member of the exponentiated Weibull distribution [19]. It has a distribution function similar to that of the Weibull distribution, but with an additional parameter that allows for more flexibility in modeling skewed lifetime data [20] and has several properties that are quite similar to the gamma distribution [21]. It was introduced and studied quite extensively by references [22, 23, 24, 25, 26, 27].



(a) PDF of Gen-Ex distribution

(b) RF of Gen-Ex distribution

Figure 1. Graphical representation of PDF and RF of Gen-Ex distribution at different values α and β

The random variable X has the two-parameter α and β, denoted by $Gen-Exdistribution(\alpha,\beta)$, with PDF and CDF are respectively given as

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}e^{\frac{-x}{\beta}}\left(1-e^{\frac{-x}{\beta}}\right)^{\alpha-1}; x \geq 0 \tag{1.3}$$

$$F(x;\alpha,\beta) = \left(1-e^{\frac{-x}{\beta}}\right)^{\alpha}; x \geq 0 \tag{1.4}$$

where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters respectively, and when $\alpha = 1$, it coincides with exponential distribution. Also, the reliability function (RF), denoted by R, is given as

$$R = 1 - F(x;\alpha,\beta) = 1 - \left(1-e^{\frac{-x}{\beta}}\right)^{\alpha} \tag{1.5}$$

The behavior of Gen-Ex distribution at different values of α and β can be shown in Figure 1.

1.1. Expressions of Entropy under Gen-Ex distribution

Shannon entropy (H_{SE}) Let X be a random variable following the Gen-Ex distribution, then using Equations (1.1) and (1.3), the Shannon entropy (H_{SE}) of the random variable X can be obtain based on Equations (1.1) and (1.3) as

$$H_{SE} = -\int_0^{\infty} \frac{\alpha}{\beta}e^{-x/\beta}\left(1-e^{\frac{-x}{\beta}}\right)^{\alpha-1} \log\left(\frac{\alpha}{\beta}e^{\frac{-x}{\beta}}\left(1-e^{\frac{-x}{\beta}}\right)^{\alpha-1}\right) \mathrm{dx}$$

thus, the above integration can be written in a closed form as (for more details see [28])

$$H_{SE} = \psi(\alpha+1) - \psi(1) - \ln\left(\frac{\alpha}{\beta}\right) - \frac{\alpha-1}{\alpha} \tag{1.6}$$

where $\psi(.) = \frac{\Gamma'(.)}{\Gamma(.)}$ is the digamma function, $\Gamma(.)$ is the complete gamma function, and $\Gamma'(.)$ is the ordinary derivative of $\Gamma(.)$.

Rényi entropy (H^{ρ}_{RE}) Similarly to H_{SE}, using equation 1.2 and equation 1.3, the H^{ρ}_{RE} of X can be written as

$$H^{\rho}_{RE} = \frac{1}{1-\rho}\log\left(\int_0^{\infty}\left(\frac{\alpha}{\beta}e^{\frac{-x}{\beta}}\left(1-e^{\frac{-x}{\beta}}\right)^{\alpha-1}\right)^{\rho}\mathrm{dx}\right)$$

thus, the above integration can be rewritten as (for more details see [28])

$$H^{\rho}_{RE} = \frac{1}{1-\rho}[\rho\log\alpha - (\rho-1)\log\beta + \log\Gamma(\rho(\alpha-1)+1) + \log\Gamma(\rho) - \log\Gamma(\alpha\rho+1)] \tag{1.7}$$

with $\rho > 0, \rho \neq 1$.

1.2. Generalized Type-II Hybrid Censoring Scheme (GT2HCS)

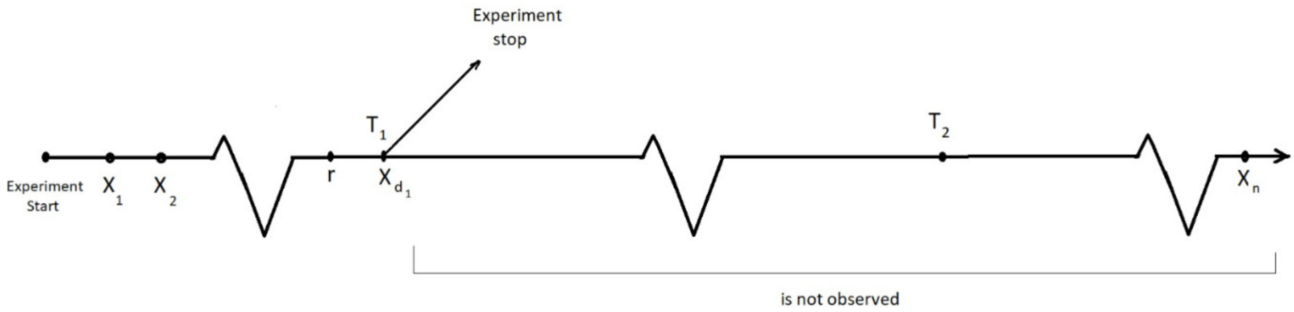
Hybrid censoring schemes (HCS) are a combination of Type-I and Type-II censoring schemes that is used in reliability studies [29, 30, 31, 32, 33, 34, 35]. The HCS was first introduced by Epstein [36]. The experiment is conducted in two stages in Type-I hybrid censoring scheme (T1HCS). In the first stage, an initial sample of units is tested until the first failure is observed. Then the remaining units are tested until the predetermined number of failures are observed. The Type-II hybrid censoring scheme (T2HCS) is a type of censoring scheme in which the experiment is stopped at certain pre-specified times or when a certain number of failures have occurred, whichever comes first. The T1HCS and T2HCS both have some drawbacks [37]. In T1HCS, the number of failures can be very small, leading to inefficient inferential procedures. Additionally, the limited information available makes it difficult to determine the optimal censoring points. In T2HCS, the length of the experiment may be longer than necessary, as it must wait for a certain number of failures before the experiment can be completed. To overcome the previous drawbacks, the generalized type hybrid censoring scheme (GHCS) One such scheme has been developed to handle such situations [37]. This paper concentrates on the generalized Type-II hybrid censoring scheme (GT2HCS). When using this scheme, the experimenter will fix a value for $r \in \{1, 2, \ldots, n\}$. Additionally, two time points are chosen $T_1, T_2 \in (0, \infty)$ and $T_1 < T_2$, with T_2 serves as the absolute maximum time for the experiment. Three cases of GT2HCS can occur:

Case 1: The r^{th} failure can be occur before the time point T_1, then, the experiment is terminated at T_1 and the observed failure items are: $\{x_{(1)} < \ldots < x_{(D_1)}\}$.

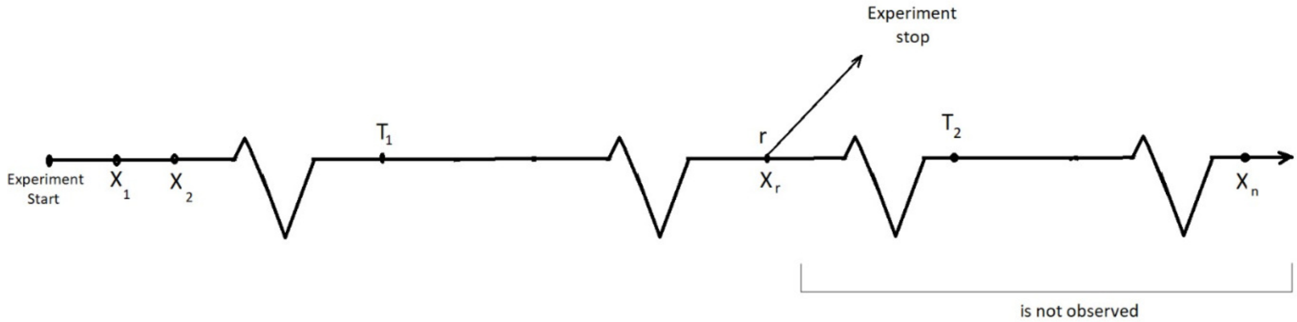
Case 2: The r^{th} failure can be occur between T_1 and T_2, then, the experiment is terminated at $x_{(r)}$ and the observed failure items are: $\{x_{(1)} < \ldots < x_{(r)}\}$.

Case 3: The r^{th} failure can be occur after T_2, then the experiment is terminated at T_2 and the observed failure items are: $\{x_{(1)} < \ldots < x_{(D_2)}\}$.

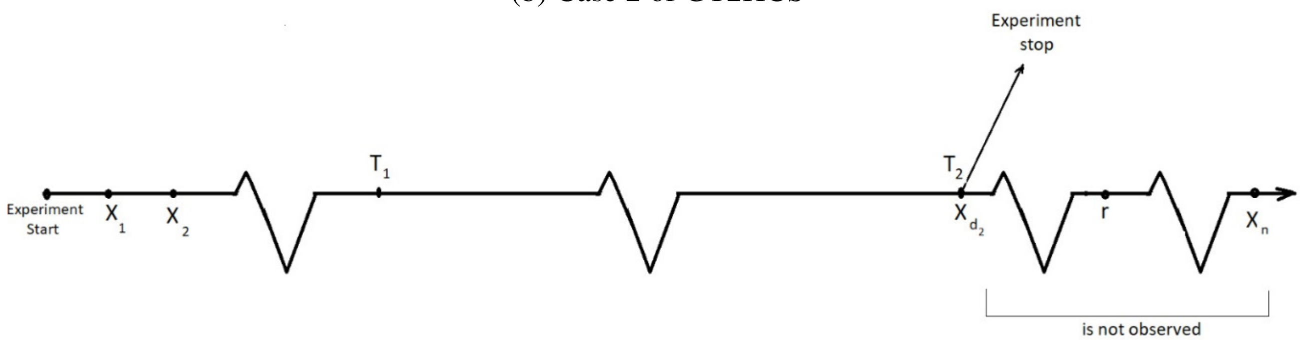
Where D_k indicate the number of failures that occur before time $T_k, k = 1, 2$. Figure 2 represents this GT2HCS strategy.



(a) Case 1 of GT2HCS



(b) Case 2 of GT2HCS



(c) Case 3 of GT2HCS

Figure 2. Schematic representation of GT2HCS

The likelihood functions for three different cases are as follows:

$$L(\omega \mid x) = \begin{cases} \frac{n!}{(n-D_1)!} \prod_{i=1}^{D_1} f\left(x_{(i)}, \omega\right) \left(1 - F\left(T_1, \omega\right)\right)^{n-D_1}, & i = 1, \ldots, D_1, \\ \frac{n!}{(n-r)!} \prod_{i=1}^{r} f\left(x_{(i)}, \omega\right) \left(1 - F\left(x_r, \omega\right)\right)^{n-r}, & i = 1, \ldots, r, \\ \frac{n!}{(n-D_2)!} \prod_{i=1}^{D_2} f\left(x_{(i)}, \omega\right) \left(1 - F\left(T_2, \omega\right)\right)^{n-D_2}, & i = 1, \ldots, D_2. \end{cases} \tag{1.8}$$

equation 1.8 can be rewritten in a general form, for sample $\underline{x} = x_{(1)} < \ldots < X_{(D)}$, as follows:

$$L(\omega \mid \underline{x}) = \frac{n!}{(n-D)!} \prod_{i=1}^{D} f\left(x_{(i)}\right) \left[1 - F(A)\right]^{n-D}, \tag{1.9}$$

where:

$$D = \begin{cases} D_1 & \text{for Case 1} \\ r & \text{for Case 1} \\ D_2 & \text{for Case 3} \end{cases}, \qquad A = \begin{cases} T_1 & \text{for Case 1} \\ x_{(r)} & \text{for Case 2} \\ T_2 & \text{for Case 3} \end{cases}$$

This HCS ensures that the experiment will be completed by time T_2. This ensures that the experiment will be completed by the time T_2, and thus T_2 serves as an absolute maximum time that the experiment can go for. One of the advantages of the GT2HCS is a valuable tool in reliability analysis and can help researchers make more accurate and informed decisions based on partially observed data. Recently, many different life distributions have been studied by using GT2HCS [38, 39, 40].

In this study, our aim is to investigate the estimation of model parameters and delve into crucial aspects of entropy criteria and the RF within the Gen-Ex distribution. By concentrating on RF and entropy measures, such as Shannon entropy (H_{SE}) and Rényi entropy (H^{ρ}_{RE}), we aim to reveal the underlying patterns in the Gen-Ex distribution using GT2HCS data. We employ both maximum likelihood estimation (MLE) and Bayesian estimation (BE) methods to estimate entropy measures and RF. Subsequently, we derive asymptotic confidence intervals (ACIs) for these quantities. The study concludes with a numerical investigation.

The research is structured as follows: In Section 2, we derive MLE and ACIs for entropy and RF under the Gen-Ex distribution in the presence of GT2HCS data. Section 3 discusses the BE procedure under two different loss functions, namely, squared error and linear exponential (LINEX), utilizing Markov chain Monte Carlo. Section 4 covers data analysis and a numerical study of the produced estimators. The study's summary and conclusions are presented in Section 5.

2. Maximum Likelihood Estimation

In this section, the maximum likelihood (ML) approach is used to obtain the point and interval estimates of the unknown parameters of the Gen-Ex distribution as well as the reliability and proposed entropy measures based on a GT2HCS.

2.1. MLEs for α and β

The likelihood function for the unknown parameters α and β of the Gen-Ex distribution under the general case of GT2HCS, as given in Equation (1.9), is as follows:

$$L(\alpha,\beta|\underline{x}) = \frac{n!}{(n-D)} \prod_{i=1}^{D} \left[\frac{\alpha}{\beta} e^{\frac{-x_{(i)}}{\beta}} \left(1 - e^{\frac{-x_{(i)}}{\beta}}\right)^{\alpha-1}\right] \left[1 - \left(1 - e^{\frac{-A}{\beta}}\right)^{\alpha}\right]^{n-D} \tag{2.1}$$

The nature logarithm of equation 2.1, $\ell(\alpha,\beta) = \log L(\alpha,\beta|\underline{x})$, can be given as

$$\ell(\alpha,\beta) \propto D\log\alpha - D\log\beta - \sum_{i=1}^{D} \frac{x_{(i)}}{\beta} + (\alpha-1)\sum_{i=1}^{D} \log\left(1 - e^{\frac{-x_{(i)}}{\beta}}\right) + (n-D)\log\left(1 - \left(1 - e^{\frac{-A}{\beta}}\right)^{\alpha}\right)$$

put: $z_{(i)} = 1 - e^{\frac{-x_{(i)}}{\beta}}$ and $z_A = 1 - e^{\frac{-A}{\beta}}$, thus, $\ell(\alpha,\beta)$ can be rewritten as:

$$\ell(\alpha,\beta) \propto D\log\alpha - D\log\beta - \sum_{i=1}^{D} \frac{x_{(i)}}{\beta} + (\alpha-1)\sum_{i=1}^{D} \log\left(z_{(i)}\right) + (n-D)\log\left(1 - (z_A)^{\alpha}\right) \tag{2.2}$$

The first partial derivative of Equation (2.2) with regard to the unknown parameters are given as follows

$$\frac{\partial\ell(\alpha,\beta)}{\partial\alpha} = \frac{D}{\alpha} + \sum_{i=1}^{D} \log\left(z_{(i)}\right) - \frac{(n-D)\log\left(z_A\right)}{(1 - z_A)^{\alpha}} \tag{2.3}$$

$$\frac{\partial \ell(\alpha,\beta)}{\partial \beta} = \frac{-D}{\beta} - \sum_{i=1}^{D} \frac{-x_{(i)}}{\beta^2} - \frac{(\alpha-1)x_{(i)}e^{-\frac{x_{(i)}}{\beta}}}{(z_{(i)})\beta^2} + \frac{(n-D)\alpha A\,(z_A)^{\alpha-1}\,e^{\frac{-A}{\beta}}}{(1-(z_A)^{\alpha})\beta^2} \tag{2.4}$$

The maximum likelihood estimates (MLEs) of α and β, denoted by $\hat{\alpha}$ and $\hat{\beta}$, can be computed by setting the first partial derivative of the log-likelihood function with respect to α and β (as given by Equations (2.3) and (2.4)) equal to zero, and then solving the resulting system of two normal equations simultaneously.

2.2. *MLEs for R, H_{SE} and H^{ρ}_{RE}*

Based on invariance property of MLEs, the corresponding MLE of R and entropy's measures H_{SE} and H^{ρ}_{RE}, for the Gen-Ex distribution in presences of GT2HCS can be directly obtained on substituting of $\hat{\alpha}$ and $\hat{\beta}$ in equation 1.5, 1.6, and 1.7, respectively obtained as shown in the following:

$$\hat{R}(x) = 1 - \left(1 - e^{\frac{-x}{\hat{\beta}}}\right)^{\hat{\alpha}}, \tag{2.5}$$

$$\hat{H}_{SE} = \psi(\hat{\alpha}+1) - \psi(1) - \ln\left(\frac{\hat{\alpha}}{\hat{\beta}}\right) - \frac{\hat{\alpha}-1}{\hat{\alpha}}, \tag{2.6}$$

and

$$\hat{H}^{\rho}_{RE} = \frac{1}{1-\rho}[\rho \log\hat{\alpha} - (\rho-1)\log\hat{\beta} + \log\Gamma(\rho(\hat{\alpha}-1)+1) + \log\Gamma(\rho) - \log\Gamma(\hat{\alpha}\rho+1)], \tag{2.7}$$

2.3. *Confidence Interval Estimation*

To obtain the confidence intervals of the Gen-Ex distribution (α,β), reliability estimates (R) and measures of entropy's H_{SE} and H^{ρ}_{RE}, we need to obtain the approximate asymptotic variance-covariance matrix of MLEs of α and β. The approximate asymptotic variance-covariance can be obtained using the inverse of the observed Fisher information matrix, denoted by $I(.)$, as demonstrated below:

$$\hat{I}^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) \\ \mathrm{cov}(\hat{\beta},\hat{\alpha}) & \mathrm{var}(\hat{\beta}) \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha\partial\beta} & -\frac{\partial^2\ell(\alpha,\beta)}{\partial\beta^2} \end{pmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \tag{2.8}$$

The elements of (2.8) are given by

$$\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha^2} = -\frac{D}{\alpha^2} - (n-D)\frac{(z_A)^{\alpha}\,(\log(z_A))^2}{(1-(z_A)^{\alpha})^2}$$

$$\frac{\partial^2\ell(\alpha,\beta)}{\partial\beta^2} = \frac{D}{\alpha^2} - 2\sum_{i=1}^{D}\frac{x_{(i)}}{\beta^3} - (\alpha-1)\sum_{i=1}^{D}\frac{\left(\frac{x_{(i)}}{\beta^2}\right)e^{-\frac{x_{(i)}}{\beta}}\left[\frac{x_{(i)}}{\beta^2} - \frac{2}{\beta} + \frac{2}{\beta}e^{-\frac{x_{(i)}}{\beta}}\right]}{(z_{(i)})^2}$$

$$+\,\alpha(n-D)\frac{(z_A)^{\alpha}e^{-\frac{A}{\beta}}\left(\frac{A}{\beta^2}\right)\left[\frac{A}{\beta^2} - \frac{2}{\beta}\right] + (z_A)^{\alpha-1}e^{-\frac{2A}{\beta}}\left(\frac{A}{\beta^2}\right)^2\left[\alpha - 1 - \alpha(z_A)^{\alpha-1}\right]}{(1-(z_A))^2}$$

$$\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha\partial\beta} = -\sum_{i=1}^{D}\frac{\left(\frac{x_{(i)}}{\beta^2}\right)e^{-\frac{x_{(i)}}{\beta}}}{(z_{(i)})}$$

$$+ (n-D)\frac{(z_A)^{2\alpha-1}e^{-\frac{A}{\beta}}\left(\frac{A}{\beta^2}\right)[\alpha\log(z_A)-1] + (z_A)^{\alpha-1}e^{-\frac{A}{\beta}}\left(\frac{A}{\beta^2}\right)[\alpha\log(z_A)+1]}{(1-(z_A)^{\alpha})^2}$$

Asymptotic Confidence Interval (Asy-CI) of the MLE

The Asy-CI is constructed based on the fact that the MLEs, $\hat{\alpha}$ and $\hat{\beta}$, exhibit asymptotic normality under certain assumptions. This implies that their joint distribution can be approximated by a bivariate normal distribution with a mean vector of (α, β) and a covariance matrix $\hat{I}^{-1}\left((\hat{\alpha}, \hat{\beta}\right)$. So, the $(1-\gamma)100\%$ Asy-CIs for α and β are

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\alpha})} \qquad \text{and} \qquad \hat{\beta} \pm Z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\beta})}$$

where $0 < \gamma < 1$ and $Z_{\gamma/2}$ is the percentile of the standard normal distribution with probability $\gamma/2$. To obtain Asy-CI for R, for this purpose, we use the delta method [10, 28] as well. Let

$$\tau'_{R(x)} = \left(\frac{\partial R(x)}{\partial \alpha}, \frac{\partial R(x)}{\partial \beta}\right)$$

where

$$\frac{\partial R(x)}{\partial \alpha} = -\left[1-e^{\frac{-x}{\beta}}\right]^{\alpha}\log\left[1-e^{\frac{-x}{\beta}}\right]$$
$$\frac{\partial R(x)}{\partial \beta} = \alpha x\left[1-e^{\frac{-x}{\beta}}\right]^{\alpha} e^{\frac{-x}{\beta}}$$

Thus, using Delta method, the variances of $\hat{R}(x)$ can be approximated by

$$\widehat{\mathrm{Var}}(\hat{R}(x)) = \left[\tau'_{R(x)}\hat{I}^{-1}(\hat{\alpha}, \hat{\beta})\tau_{R(x)}\right]$$

Further, $\frac{\hat{R}(x)-R(x)}{\widehat{\mathrm{Var}}(\hat{R}(x))}$ follow standard normal distribution asymptotically Therefore, the $100(1-\gamma)\%$ Asy-CIs for R is given by

$$\hat{R}(x) \pm Z_{\gamma/2}\sqrt{\widehat{\mathrm{Var}}(\hat{R}(x))}$$

Also, to create the Asy-CIs of the entropy measures: H_{SE} and H^{ρ}_{RE}, one can use the Delta approach to obtain the approximate variances of the entropy measures H_{SE} and H^{ρ}_{RE}, respectively, as

$$\widehat{\mathrm{Var}}\left(\widehat{H}_{SE}\right) = \left[\tau'_{H_{SE}}\hat{I}^{-1}(\hat{\alpha}, \hat{\beta})\tau_{H_{SE}}\right] \qquad \text{and} \qquad \widehat{\mathrm{Var}}\left(\widehat{H}^{\rho}_{RE}\right) = \left[\tau'_{H^{\rho}_{RE}}\hat{I}^{-1}(\widehat{\alpha}, \hat{\beta})\tau_{H^{\rho}_{RE}}\right]$$

where

$$\tau'_{H_{SE}} = \left(\frac{\partial H_{SE}}{\partial \alpha}, \frac{\partial H_{SE}}{\partial \beta}\right) \qquad \text{and} \qquad \tau'_{H^{\rho}_{RE}} = \left(\frac{\partial H^{\rho}_{RE}}{\partial \alpha}, \frac{\partial H^{\rho}_{RE}}{\partial \beta}\right)$$

and

$$\frac{\partial H_{SE}}{\partial \alpha} = \psi(\alpha+1) - \frac{\alpha+1}{\alpha^2}$$
$$\frac{\partial H_{SE}}{\partial \beta} = \psi(\alpha+1) - \frac{\alpha+1}{\alpha^2}$$

$$\frac{\partial H_{RE}^{\rho}}{\partial \alpha} = \frac{\rho}{1-\rho}\left[\psi(\rho(\alpha-1)+1) - \psi(\rho\alpha+1) + \frac{1}{\alpha}\right]$$
$$\frac{\partial H_{RE}^{\rho}}{\partial \beta} = \frac{1}{\beta}$$

Further, $\frac{\widehat{H}_{SE}-H_{SE}}{\widehat{\text{Var}}(\widehat{H}_{RS})}$ and $\frac{\widehat{H}_{RE}^{\rho}-H_{RE}^{\rho}}{\widehat{\text{Var}}(\widehat{H}_{RE}^{\rho})}$ follow standard normal distribution asymptotically. Therefore, the $100(1-\gamma)\%$ Asy-CIs for H_{SE} and H_{RE}^{ρ} are respectively given by

$$\widehat{H}_{SE} \pm Z_{\gamma/2}\sqrt{\widehat{\text{Var}}\left(\widehat{H}_{SE}\right)} \qquad \text{and} \qquad \widehat{H}_{RE}^{\rho} \pm Z_{\gamma/2}\sqrt{\widehat{\text{Var}}\left(\widehat{H}_{RE}^{\rho}\right)}$$

Normal Approximation of the Log-Transformed MLE

One limitation of the $100(1-\gamma)\%$ Asy-CIs, as discussed earlier, is that they can yield negative lower bounds for parameters that are inherently positive. In such cases, substituting a negative value with zero might be considered. To address this shortcoming in the accuracy of the normal approximation, Meeker and Escobar [41] proposed a solution in the form of log-transformed MLE-based Asy-CIs. Their findings suggest that this type of confidence interval offers improved coverage probability. The NA-CI, which is a $100(1-\gamma)\%$ normal approximate confidence interval, is calculated for log-transformed MLEs and can be expressed as follows:

$$\log(\hat{\alpha}) \pm Z_{\gamma/2}\sqrt{\tau_{11}} \qquad \text{and} \qquad \log\left(\hat{\beta}\right) \pm Z_{\gamma/2}\sqrt{\tau_{22}}$$

where τ_{11} and τ_{22} are the estimated variance of $\log(\hat{\alpha})$ and $\log\left(\hat{\beta}\right)$, respectively. Thus, based on log-transformed, $100(1-\gamma)\%$ NA-CI for α and β are respectively given by

$$\hat{\alpha} \times e^{\pm\frac{Z_{\gamma/2}\sqrt{\tau_{11}}}{\hat{\alpha}}} \qquad \text{and} \qquad \hat{\beta} \times e^{\pm\frac{Z_{\gamma/2}\sqrt{\tau_{11}}}{\hat{\beta}}}$$

Using similar approach, $100(1-\gamma)\%$ NA-CI for R, H_{SE} and H_{RE}^{ρ} is given by

$$\hat{R}(x) \times e^{\pm\frac{Z_{\gamma/2}\sqrt{\text{var}(\hat{R}(x))}}{\hat{R}(x)}}$$

and for each H_{SE} and H_{RE}^{ρ} are respectively given by

$$\hat{H}_{SE} \times e^{\pm\frac{Z_{\gamma/2}\sqrt{\widehat{\text{Var}}(\hat{H}_{SE})}}{\hat{H}_{SE}}} \qquad \text{and} \qquad \hat{H}_{RE}^{\rho} \times e^{\pm\frac{Z_{\gamma/2}\sqrt{\widehat{\text{Var}}(\hat{H}_{RE}^{\rho})}}{\hat{H}_{RE}^{\rho}}}.$$

3. Bayesian Estimation

The Bayes estimation (BE) of the R and entropy measures H_{SE} and H_{RE}^{ρ} for Gen-Ex distribution under the GT2HCS will be covered in this section. Different loss functions, such as the squared error (SE) and Linear exponential (LINEX), can be taken into consideration for the BE. For informative prior (IP) distribution of the parameters α and β of the Gen-Ex distribution, we can suggest using independent gamma priors having PDFs

$$\pi(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha} \qquad \alpha > 0, a_1 > 0, b_1 > 0,$$

$$\pi(\beta) \propto \beta^{a_2-1}\mathrm{e}^{-b_2\beta} \qquad \beta > 0, a_2 > 0, b_2 > 0$$

where the hyper-parameters a_1, b_1, a_2 and b_2 are chosen to represent the prior knowledge about the unknown parameters. Thus, the joint prior density can be given as

$$\pi(\alpha,\beta) \propto \alpha^{a_1}\beta^{a_2}\mathrm{e}^{-b_1\alpha-b_2\beta} \tag{3.1}$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(D)})$ is given by:

$$\pi(\alpha,\beta \mid \mathbf{x}) = \frac{\pi(\alpha,\beta)L(\alpha,\beta \mid \mathbf{x})}{\int_0^\infty \int_0^\infty \pi(\alpha,\beta)L(\alpha,\beta \mid \mathbf{x})d\alpha d\beta} \propto \pi(\alpha,\beta)L(\alpha,\beta \mid \mathbf{x}).$$

As a result from Equations (2.1) and (3.1), the posterior density function can be written as follows:

$$\pi(\alpha,\beta \mid \mathbf{x}) \quad \propto \quad \alpha^{a_1+1}\beta^{a_2-1}\mathrm{e}^{-b_1\alpha-b_2\beta}\prod_{i=1}^{D}\left[e^{\frac{-x_{(i)}}{\beta}}\left(1-e^{\frac{-x_{(i)}}{\beta}}\right)^{\alpha-1}\right]\left[1-\left(1-e^{\frac{-A}{\beta}}\right)^{\alpha}\right]^{n-D}. \tag{3.2}$$

The BEs of any function of parameters, say $g(\alpha,\beta)$, with respect to SE loss function can given by:

$$\hat{g}_{SE} = E\left[g(\alpha,\beta) \mid \mathbf{x}\right] = \int_0^\infty \int_0^\infty g(\alpha,\beta)\pi(\alpha,\beta \mid \mathbf{x})d\alpha d\beta. \tag{3.3}$$

The SE loss function exhibits asymmetry by attributing the same significance to both underestimation and overestimation. However, practical scenarios often involve situations where the severity of underestimation is more significant than that of overestimation, or vice versa. To address this, an alternative approach is to consider a LINEX loss function, represented as LN, which offers an alternative to the SE loss. This can be defined using the following equation:

$$(g(\alpha,\beta), \hat{g}(\alpha,\beta)) = e^{\hat{g}(\alpha,\beta)-g(\alpha,\beta)} - v\left(\hat{g}(\alpha,\beta) - g(\alpha,\beta)\right) - 1,$$

where $v \neq 0$. In this context, when $v > 1$, it indicates a situation where overestimation carries more substantial consequences compared to underestimation. Conversely, for $v < 0$, the emphasis is on the severity of underestimation. As v approaches 0, the behavior of the BE converges to that of the BE under the SE loss function. For further insight into this matter, additional information can be found in references [42] and [43].

The BEs of $g(\alpha,\beta)$ concerning this particular loss function can be calculated using the following formula:

$$\hat{g}_{LN} = E\left[e^{-vg(\alpha,\beta))} \mid \mathbf{x}\right] = -\frac{1}{v}\log\left[\int_0^\infty \int_0^\infty e^{-vg(\alpha,\beta)}\pi(\alpha,\beta \mid \mathbf{x})d\alpha d\beta\right]. \tag{3.4}$$

Subsequently, the BEs for the parameters of the Gen-Ex distribution, namely α and β, as well as the reliability measure R, and the entropy measures H_{SE} and H^{ρ}_{RE}, under the GT2HCS scheme in relation to the SE and LN loss functions, can be acquired by substituting the values of α, β, R, H_{SE}, and H^{ρ}_{RE} in place of $g(\alpha,\beta)$ within equations (3.3) and (3.4), correspondingly.

It can be observed that the estimates provided by equations (3.3) and (3.4) cannot be simplified into closed-form expressions. Hence, we proceed to employ the Markov chain Monte Carlo (MCMC) approach and utilize the Metropolis-Hastings (MH) Algorithm to generate a posterior sample. This will allow us to obtain the desired estimates of $g(\alpha,\beta)$ under the BE framework.

3.1. MH Algorithm

In order to execute the MH algorithm for the BEs of Gen-Ex distribution, a suggested distribution and initial values of the unknown parameters α and β have to specify. For the proposal distribution, a normal distribution will be taken into account, that is $q\left(\xi'|\xi\right) \equiv N\left(\xi, S_{\xi}\right)$, where $\xi = (\alpha, \beta)$ and S_{ξ} represent the variance-covariance matrix of ξ. For the initial values, the MLE for ξ is considered, that is $\xi^{(0)} = \left(\hat{\alpha}, \hat{\beta}\right)$. The choice of S_{ξ} is thought to be the asymptotic variance-covariance $I^{-1}\left(\hat{\alpha}, \hat{\beta}\right)$ given in equation 2.8. In this manner, the MH method uses the stages listed below to draw a sample from the posterior density provided by (3.2)

Step 1. Set initial value of ξ as $\xi = \left(\alpha^{(0)}, \beta^{(0)}\right) = \left(\hat{\alpha}, \hat{\beta}\right)$.

Step 2. For $i = 1, 2, \ldots, M$ repeat the following steps:

2.1: Set $\xi = \xi^{(i-1)}$.

2.2: Generate a new candidate parameter value, ξ', by sampling from $N_2(\xi, S_{\xi})$, where $N_2(.)$ denotes a bivariate normal distribution with a mean vector of ξ and a variance-covariance matrix of S_{ξ}.

2.3: Compute the formula $\delta = \frac{\pi(\xi'|\mathbf{x})}{\pi(\xi|\mathbf{x})}$, where $\pi(\cdot)$ is the posterior density in equation 3.2.

2.4: Generate a sample u from the uniform $U(0, 1)$ distribution.

2.5: Accept or reject the new candidate ξ'

$$\begin{cases} \text{If} \quad u \le \delta \quad \text{put} \quad \xi^{(i)} = \xi' \\ \text{elsewhere} \quad \text{put} \quad \xi^{(i)} = \xi \end{cases}$$

thus, one can obtain the following MCMC samples of (α, β) as:

$$\xi^{(i)} = \left(\alpha^{(i)}, \beta^{(i)}\right), \qquad i = 1, 2, \ldots, M.$$

Hence, one can compute $R(x), H_{SE}$ and H^{ρ}_{RE} by substituting $\xi^{(i)}$ in equation 1.5, equation 1.6, and equation 1.7, receptively. Eventually, a portion of the initial samples can indeed be removed (burn-in), and the remaining samples can be used to calculate BEs using random samples of size M drawn from the posterior density. The BEs of a parametric function $g(\alpha, \beta)$ under SE and LN are respectively given by

$$\hat{g}_{SE}(\alpha, \beta) = \frac{1}{M - l_B} \sum_{i=l_B}^{M} g(\alpha^{(i)}, \beta^{(i)}), \tag{3.5}$$

and

$$\hat{g}_{LN}(\alpha, \beta) = \frac{-1}{v} \log\left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \mathrm{e}^{-v g(\alpha^{(i)}, \beta^{(i)})}\right], \tag{3.6}$$

where l_B represents the number of burn-in samples. Substituting $g(\alpha, \beta) = \left(\alpha, \beta, R(x), H_{SE}, H^{\rho}_{RE}\right)$ in the above equations, one can easily obtain respective BEs for $\left(\alpha, \beta, R(x), H_{SE}, H^{\rho}_{RE}\right)$ with respect to SE and LN loss functions.

3.2. Hyper-Parameters Elicitation

The informative priors are employed to select the hyper-parameters. Such informative priors are derived from the MLEs for (α,β) via equating the mean and variance of $(\hat{\alpha}^j,\hat{\beta}^j)$ along with the mean and variance of the considered priors (gamma priors), where $j = 1, 2, \ldots, k$, k is the available samples number from the Gen-Ex distribution. On equating the mean and variance of $(\hat{\alpha}^j,\hat{\beta}^j)$ and the mean and variance of α and β priors [44], we get

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j = \frac{a_1}{b_1} \quad \& \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2 = \frac{a_1}{b_1^2}$$
$$\frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j = \frac{a_2}{b_2} \quad \& \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j\right)^2 = \frac{a_2}{b_2^2}.$$

Note that, by selecting the hyper-parameters $a1 = b1 = a_2 = b_2 = 0$ and using the same MH algorithm, the BEs obtained in this case are called BE non-informative prior (BE-NIF).

The estimated hyper-parameters can therefore be written and illustrated by solving the two equations.

$$\begin{aligned}
a_1 &= \frac{(\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2}\\
b_1 &= \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2}\\
a_2 &= \frac{(\frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j\right)^2}\\
b_2 &= \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\beta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\beta}^j\right)^2}.
\end{aligned} \tag{3.7}$$

3.3. Highest Posterior Density

According to the technique of Chen and Shao [45], one can establish the highest posterior density (HPD) intervals for the unknown parameters ξ of the Gen-Ex distribution under GT2HCS using the samples obtained by the suggested MH algorithm in the preceding paragraph, where

$$\xi = \left(\alpha, \beta, R(x), H_{SE}, H_{RE}^{\rho}\right).$$

Considering $\xi^{(\gamma)}$ be the γth quantile of ξ, that is,

$$\xi^{(\gamma)} = \inf\{\xi : \Pi(\xi \mid \mathbf{x}) \geq \gamma\},$$

where $0 < \gamma < 1$ and $\Pi(\cdot)$ is the posterior function of ξ. It should be noted that for a given ξ^*, an accurate estimator based on simulation of $\pi(\xi \mid \mathbf{x})$ might well be computed as

$$\Pi\{\xi \mid \mathbf{x}\} = \frac{1}{M - l_B} \sum_{i=l_B}^{M} I_{\xi \leq \xi^*}$$

Here $I_{\xi \leq \xi^*}$ is the indicator function. The proper estimate is then determined as

$$\hat{\Pi}(\xi^* \mid \mathbf{x}) = \begin{cases} 0 & \text{if } \xi^* < \xi_{l_B} \\ \sum_{j=l_B}^{i} \omega_j & \text{if } (\xi_i < \xi^* < \xi_{i+1} \\ 1 & \text{if } \xi^* > \xi_M \end{cases}$$

where $\omega_j = \frac{1}{M - l_B}$ and ξ^j are the ordered values of $\xi^{(j)}$. Now, for $i = l_B, \ldots, M$, $\xi^{(\gamma)}$ may be estimated by

$$\widetilde{\xi}^{(\gamma)} = \begin{cases} \xi_{l_B} & \text{if} \quad \gamma = 0 \\ \xi_i & \text{if } \sum_{j=l_B}^{i-1} \omega_j < \gamma < \sum_{j=l_B}^{i} \omega_j. \end{cases}$$

Furthermore, let us determine a $100(1-\gamma)\%$ HPD credible interval for ξ as

$$HPD_j^{\xi} = \left(\widetilde{\xi}^{(\frac{j}{M})}, \quad \widetilde{\xi}^{(\frac{j+(1-\gamma)M}{M})} \right)$$

for $j = l_B, \ldots, [\gamma M]$, here $[a]$ represents indicates the largest integer that $\leq a$. Need to choose HPD_{j^*} from one of many HPD_j^{ξ}s with the narrowest width.

4. Data Analysis and Simulation Study

This section's goal is to examine the behavior of the suggested reliability and entropy measures estimators for the Gen-Ex distribution under the GT2HCS that was addressed in earlier parts. For demonstration purposes, we investigate an actual data set. Furthermore, we conducted a simulation study to investigate the behavior of the proposed approaches and evaluate the estimates' performance under various GT2HCS. For calculations, we utilized *R*, a statistical programming language. In addition, the *bbmle* and *HDInterval* packages may be used to compute MLEs and HPD intervals in *R*-language.

4.1. Simulation Study

In this part, we use a Monte Carlo simulation analysis to assess the effectiveness of estimate techniques, specifically MLE and BEs using MCMC, for Gen-Ex distribution under the GT2HCS. We generate 1000 random data set from the Gen-Ex distribution based on the following assumptions:

1. The parameters of the Gen-Ex distribution are set to: $(\alpha, \beta) = (0.5, 1.5), (1, 0.5)$, and $(2.5, 2.5)$.
2. The constant term of H_{RE}^{ρ} is assumed to take values: $\rho = 0.30$ and 0.60.
3. The parameters of the GT2HCS are chosen as follows:

(a) The sample size and the number of failure items are assumed to be $(n, r) = (30, 15), (60, 30)$, and $(120, 60)$.

(b) The censoring times $T_j, j = 1, 2$ are determined as quantiles from the distribution $F^{-1}(u_{kj}, \alpha, \beta)$, where $u_{kj} = (0.20, 0.60), u_{kj} = (0.40, 0.60)$ for $k = 1, 2, 3$. Here, k corresponds to the assumed case of the Gen-Ex distribution with parameters (α, β). Consequently, the time of censoring can be computed and is displayed in Table 1.

4. The true reliability value can be calculated based on Equation (1.5) with the parameters $(x = 0.5, \alpha, \beta)$. Furthermore, the entropy metrics, namely H_{SE} and H^{ρ}_{RE}, are determined utilizing Equations (1.7) and (1.6), correspondingly, by employing the values of α and β. The complete set of authentic reliability values and diverse entropy metrics can be showcased in Table 1.

Table 1. True values of reliability, different entropy measures and times of censoring for GT2HCS

Parm.	$R_{x=0.5}$	H_{SE}	H^{ρ}_{RE}		Times of censoring	
(α, β)			$\rho = 0.30$	$\rho = 0.60$	(u_{k1}, u_{k2})	(T_1, T_2)
(0.5, 1.5)	0.4676	2.7123	1.9251	1.2839	(0.20, 0.60)	(0.06, 0.67)
					(0.40, 0.80)	(0.26, 1.53)
(1, 0.5)	0.3679	0.3069	1.0268	0.5839	(0.20, 0.60)	(0.11, 0.46)
					(0.40, 0.80)	(0.26, 0.80)
(2.5, 2.5)	0.9860	1.0804	2.8215	2.4771	(0.20, 0.60)	(1.86, 4.22)
					(0.40, 0.80)	(2.95, 6.15)

Steps of Monte Carlo simulation:

Step.1: Generating n random data from Gen-Ex distribution characterized by parameters α and β, and subsequently applying the GT2HCS, involves specifying values for r, T_1, and T_2. This process leads to the definition of the variable D and determines the particular censoring scenario to be implemented.

Step.2: Estimating the MLEs for α and β involves calculating their respective variances using $I^{-1}(\hat{\alpha}, \hat{\beta})$. Subsequently, this information is utilized to compute the Asy-CIs and NA-CIs.

Step.3: Calculating the MLEs of R, H_{SE}, and H^{ρ}_{RE} involves substituting the estimated values $\hat{\alpha}$ and $\hat{\beta}$ obtained from Equation (2.5), Equation (2.6), and Equation (2.7) respectively. Afterward, confidence intervals, both Asy-CIs and NA-CIs, can be computed for these estimates.

Step.4: Compute BEs using the MH algorithm as follows:

1. Assuming two cases for prior distributions: the first being an informative prior (IF), where the hyper-parameter values are computed from Equation 3.7. Specifically, we generate 1000 complete samples, each consisting of 60 data points, from the Gen-Ex(α, β) distribution as past samples and compute their MLEs $(\hat{\theta}, \hat{\alpha})$. Subsequently, by utilizing Equation 3.7, we can determine the hyper-parameter values as follows: for $(\alpha, \beta) = (0.5, 1.5)$ we obtain the hyper-parameter values as $a_1 = 39.45$, $b_1 = 75.92$, $a_2 = 23.56$, and $b_2 = 15.95$. For $(\alpha, \beta) = (1, 0.5)$, we obtain $a_1 = 32.55$, $b_1 = 31.20$, $a_2 = 35.24$, and $b_2 = 71.32$. Finally, for $(\alpha, \beta) = (2.5, 2.5)$, we obtain $a_1 = 22.33$, $b_1 = 8.38$, $a_2 = 48$, and $b_2 = 19.36$.

2. Consider the second case involves non-informative prior (NIF), where the values of the hyper-parameter values are $a_1 = b_1 = a_2 = b_2 = 0$, thus $\pi(\alpha) = \frac{1}{\alpha}$ and $\pi(\beta) = \frac{1}{\beta}$.
3. Generating 10000 MCMC samples of α and β for the above two cases (IF and NIF) using the initial of MLEs and its variance-covariance matrix and given GT2HCS data $x = (x_{(1)}, (x_{(2)}, \ldots, (x_{(D)})$.
4. Removing the first 2000 burn-in samples from the total 10000 samples created by the posterior density.
5. Computing BEs of α and β under different loss function: SE, LN_1 wit $v = -1.5$, and LN_2 with $v = 1.5$ using respectively 3.5 and 3.6 and hence computing there HPD.
6. Computing BEs of of R, H_{SE} and H^{ρ}_{RE} under different loss function using respectively 3.5 and 3.6, and hence computing there HPD intervals.

Step.5: Repeating Step.1 to Step.4 number of times 1000 and saving all estimates.

Step.6: Computing statistical measures of performance for point estimates: average (Avg.) estimate and mean square error (MSE) estimate. These measures can be computed as:

$$Avg.(\theta) = \frac{1}{1000}\sum_{l=1}^{1000}\hat{\theta}_l \qquad \& \qquad MSE(\theta) = \frac{1}{1000}\sum_{l=1}^{1000}\left(\hat{\theta}_l - \theta\right)^2$$

where θ referees to the parameter and $\hat{\theta}$ referees to the estimate value of the given parameter.

Step.7: Computing statistical measures of performance for interval estimates: average interval length (AIL) and coverage probability (CP) in %.

For point estimates, the results of Avg. and MSE of estimates under different GT2HCS scenarios for the Gen-Ex distribution with parameters ($\alpha = 0.5, \beta = 1.5$) are presented in Table 2, Table 3, and Table 4 for sample sizes of $n = 30$, $n = 60$, and $n = 120$, respectively. The corresponding results for the Gen-Ex distribution with parameters ($\alpha = 1, \beta = 0.5$) are provided in Table 5, Table 6, and Table 7 for the same sample sizes, and similarly, for the Gen-Ex distribution with parameters ($\alpha = 2.5, \beta = 2.5$), the results are reported in Table 8, Table 9, and Table 10 for the respective sample sizes of $n = 30$, $n = 60$, and $n = 120$.

For interval estimates: Asy-CI, NA-CI, and HPD interval, lower and upper limits of CIs, AILs and CPs under different GT2HCS for Gen-Ex distribution ($\alpha = 0.5, \beta = 1.5$) are reported in Table 11, Table 12 and Table 13 for $n = 30, 60$ and 120, respectively, and for Gen-Ex distribution ($\alpha = 1, \beta = 0.5$) are reported in Table 14, Table 15 and Table 16 for $n = 30, 60$ and 120, and finally for Gen-Ex distribution ($\alpha = 2.5, \beta = 2.5$) are reported in Table 17, Table 18 and Table 19 for $n = 30, 60$ and 120.

Based on the numerical outputs presented in Table (2) through Table (19), several overarching conclusions can be inferred:

- Across various settings, the results of the simulation study indicate that the BE-IF$_{LN2}$ method has outperformed other methods in terms of Avg. and MSE for estimating the parameters α and β, as well as for estimating H_{SE} and H^{ρ}_{RE}.
- As n and r increase, the MSE consistently decreases across all tables of the simulation study, aligning with our expectations.

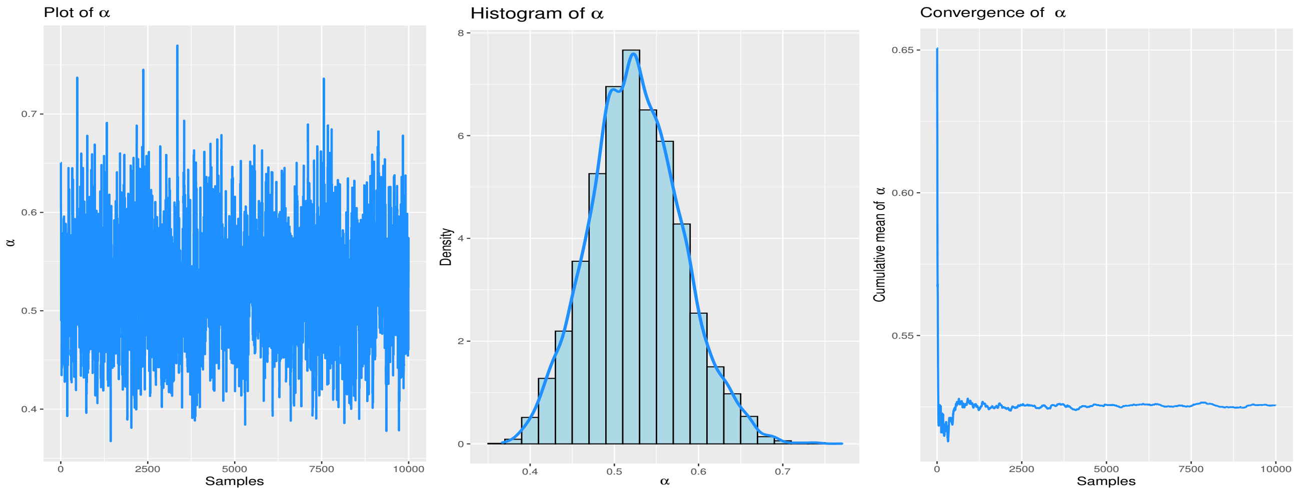
- It is noteworthy that an increase in the value of α coupled with a decrease in β results in higher MSE values across all simulation tables. Conversely, when both α and β increase together, a simultaneous decrease in MSE values is observed for all estimation approaches.
- The simulation results reveal that as n, r, T_1, and T_2 increase, the MSE values consistently decrease across all simulation tables. Similarly, an increase in the parameter ρ is associated with a decrease in MSE for the estimation approaches.
- In most scenarios, when both n and r increase, along with the increase of T_1 and T_2, the confidence intervals constructed using Asy-CI, NA-CI, and HPD-CI methods tend to shrink for parameters such as α, β, R, H_{SE}, and $H_{RE}^{(\rho=0.3,0.6)}$.
- Additionally, for each simulation scenario, the AIL tends to decrease as n and r increase. It's important to note that the simulation study reveals that the HPD-CI: NIF method consistently yields the largest confidence interval lengths among the approaches considered.

Finally, the convergence of MCMC estimates using the MH algorithm can be visualized through scatter plots, histograms, and cumulative means for each estimated parameter: α, β, R, H_{SE}, and H_{RE}^{ρ} as showed in Figure 3. These visualizations are based on the Gen-Ex distribution with $(\alpha = 0.5, \beta = 1.5)$ under the GT2HCS with $n = 60, r = 30, T_1 = 0.06$, and $T_2 = 0.67$, where IF priors are employed. These graphical representations demonstrate the normality of the generated posterior samples and provide insight into the convergence behavior of the BEs. Notably, these graphs illustrate how the BEs converge toward the true values of the parameters.

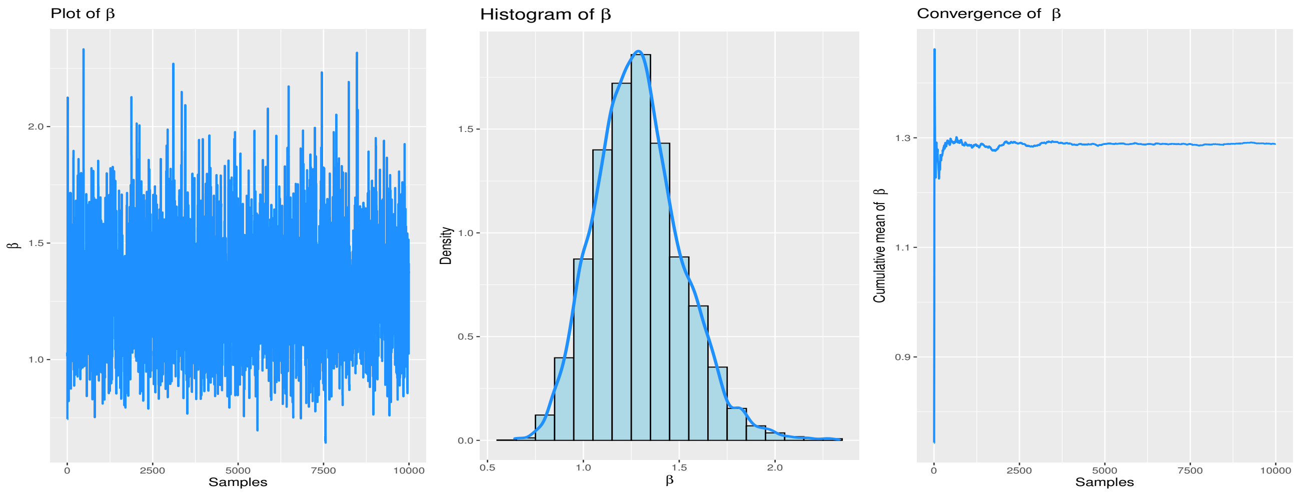
4.2. Real Data Analysis

A real data set is analyzed for illustrative purposes as well as to assess the statistical performances of the MLE and BEs for reliability and entropy measures of the Gen-Ex distribution under different GT2HCS.

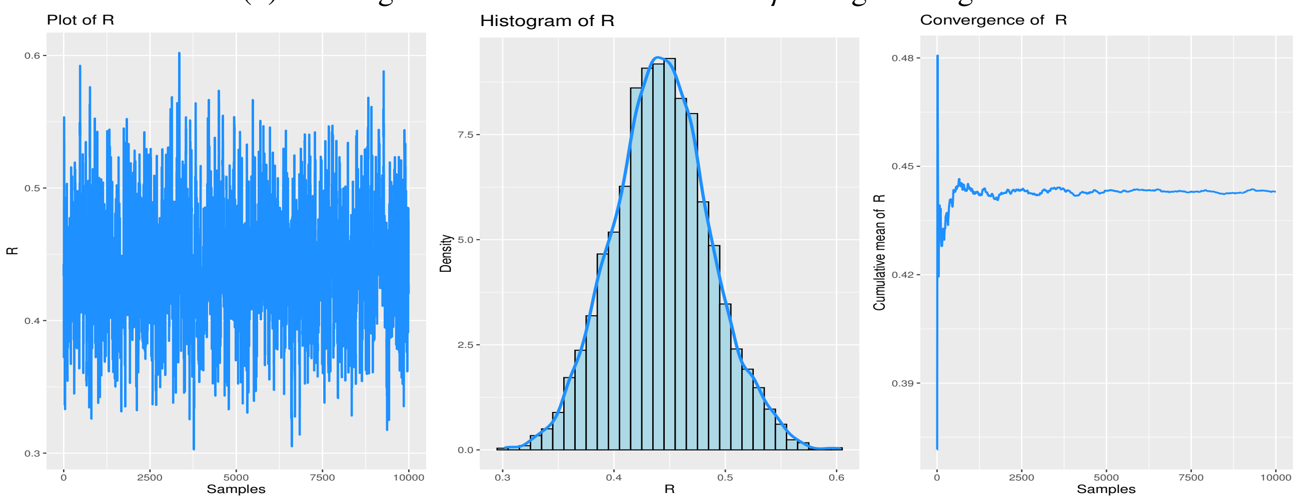
Time between Failures: The following data, which labeled in Table 20, shows the time between failures for 30 repairable items and has been provided by [46]. The data are listed in Table 20.



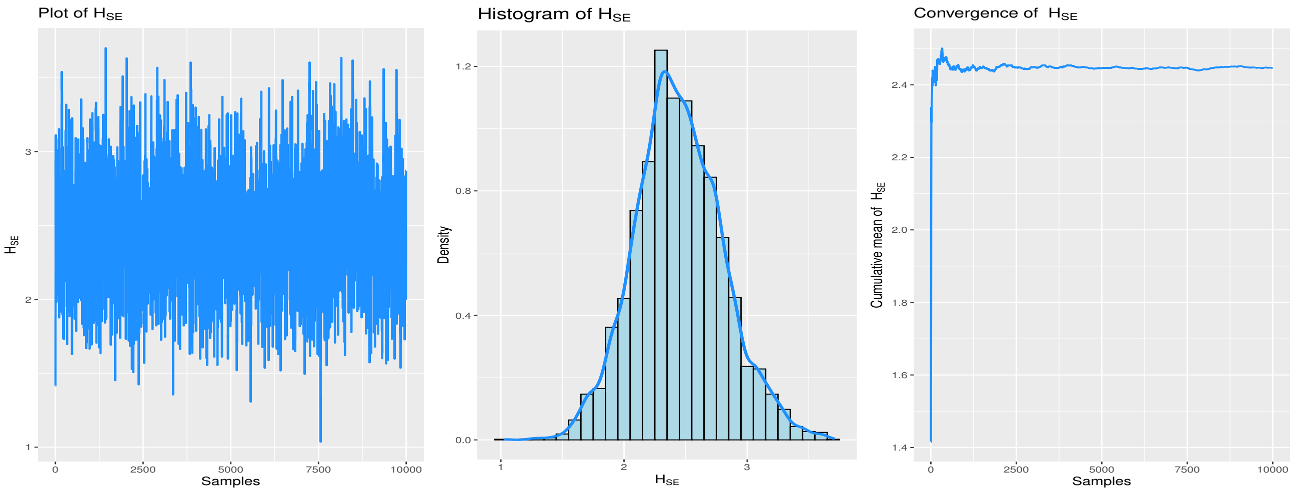
(a) Convergence of MCMC estimates for α using MH algorithm



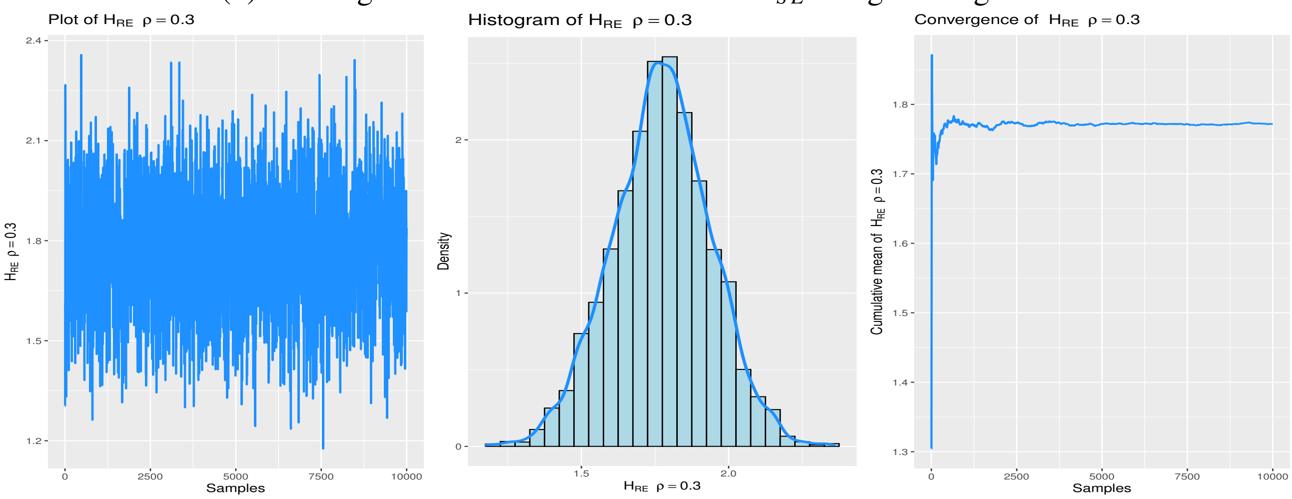
(b) Convergence of MCMC estimates for β using MH algorithm



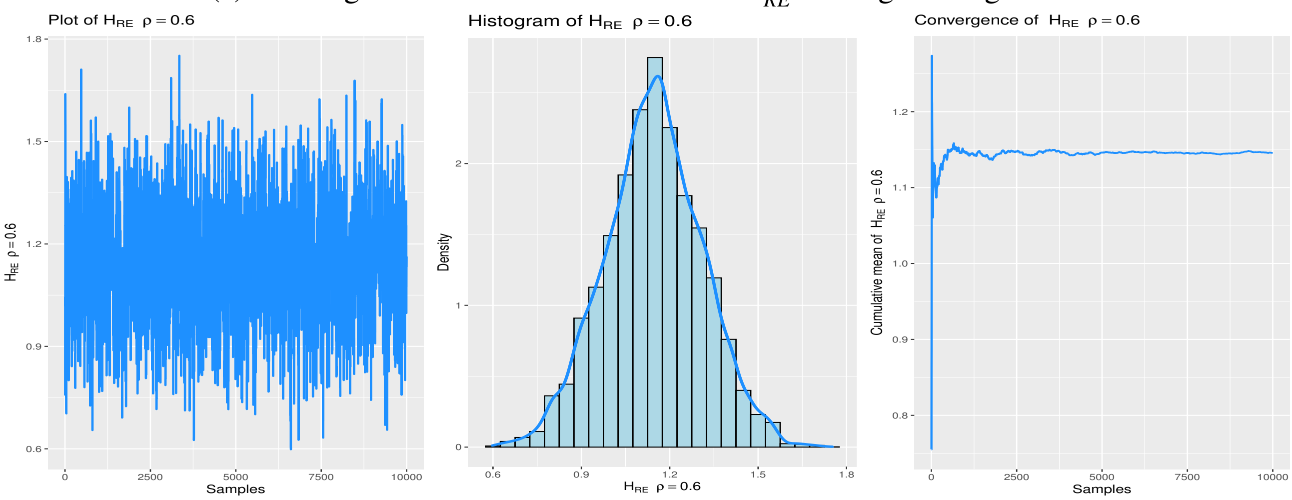
(c) Convergence of MCMC estimates for R using MH algorithm



(d) Convergence of MCMC estimates for H_{SE} using MH algorithm



(e) Convergence of MCMC estimates for $H_{RE}^{\rho=0.3}$ using MH algorithm



(f) Convergence of MCMC estimates for $H_{RE}^{\rho=0.6}$ using MH algorithm

Figure 3. Convergence of MCMC estimates for Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) under GT2HCS for $n = 60$, $r = 30$, and $T_1 = 0.06$, $T_2 = 0.67$

Table 2. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) at $n = 30, r = 15$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H_{RE}^{ρ}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(0.06, 0.67)	MLE	Avg.	0.6003	1.5107	0.4346	2.2577	1.7197	1.1115
		MSE	0.0526	1.9144	0.0140	1.8915	0.4146	0.3271
	BE-NIF$_{SE}$	Avg.	0.6670	1.7119	0.4448	2.3585	1.7267	1.1058
		MSE	0.1580	3.3434	0.0265	2.5598	0.4767	0.4269
	BE-NIF$_{LN1}$	Avg.	0.6500	1.5824	0.4418	2.2176	1.7060	1.0780
		MSE	0.1358	2.4263	0.0264	2.4101	0.4869	0.4405
	BE-NIF$_{LN2}$	Avg.	0.6852	1.8828	0.4477	2.5151	1.7474	1.1330
		MSE	0.1853	5.0259	0.0266	2.8431	0.4681	0.4162
	BE-IF$_{SE}$	Avg.	0.5111	1.4238	0.4556	2.6396	1.8649	1.2255
		MSE	0.0024	0.0166	0.0021	0.0403	0.0130	0.0174
	BE-IF$_{LN1}$	Avg.	0.5076	1.3733	0.4532	2.5230	1.8375	1.1967
		MSE	0.0022	0.0303	0.0021	0.0684	0.0180	0.0226
	BE-IF$_{LN2}$	Avg.	0.5148	1.4805	0.4581	2.7659	1.8918	1.2537
		MSE	0.0028	0.0117	0.0020	0.0430	0.0098	0.0141
(0.26, 1.53)	MLE	Avg.	0.5969	1.4514	0.4386	2.2776	1.7299	1.1204
		MSE	0.0508	1.1754	0.0112	1.6911	0.3329	0.2578
	BE-NIF$_{SE}$	Avg.	0.6645	1.6492	0.4486	2.3844	1.7354	1.1122
		MSE	0.1596	2.4214	0.0240	2.4739	0.4255	0.3810
	BE-NIF$_{LN1}$	Avg.	0.6473	1.5311	0.4457	2.2426	1.7147	1.0844
		MSE	0.1368	1.7532	0.0239	2.3216	0.4353	0.3944
	BE-NIF$_{LN2}$	Avg.	0.6829	1.8076	0.4516	2.5418	1.7561	1.1395
		MSE	0.1878	3.8132	0.0241	2.7593	0.4175	0.3706
	BE-IF$_{SE}$	Avg.	0.5115	1.4245	0.4556	2.6427	1.8659	1.2262
		MSE	0.0021	0.0159	0.0016	0.0395	0.0117	0.0160
	BE-IF$_{LN1}$	Avg.	0.5082	1.3740	0.4532	2.5261	1.8389	1.1978
		MSE	0.0020	0.0258	0.0016	0.0663	0.0161	0.0208
	BE-IF$_{LN2}$	Avg.	0.5149	1.4806	0.4578	2.7693	1.8925	1.2541
		MSE	0.0024	0.0109	0.0016	0.0423	0.0088	0.0130

Table 3. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) at $n = 60, r = 30$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H^{ρ}_{RE}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
$(0.06, 0.67)$	MLE	Avg.	0.5462	1.4737	0.4508	2.4753	1.8175	1.1937
		MSE	0.0173	0.5487	0.0059	0.8678	0.1840	0.1448
	BE-NIF$_{SE}$	Avg.	0.6047	1.6454	0.4603	2.6019	1.8264	1.1871
		MSE	0.0876	1.1806	0.0192	1.7445	0.2758	0.2699
	BE-NIF$_{LN1}$	Avg.	0.5918	1.5423	0.4574	2.4527	1.8055	1.1586
		MSE	0.0772	0.8823	0.0191	1.5683	0.2809	0.2789
	BE-NIF$_{LN2}$	Avg.	0.6183	1.7762	0.4631	2.7682	1.8471	1.2149
		MSE	0.1002	1.7226	0.0194	2.0702	0.2722	0.2637
	BE-IF$_{SE}$	Avg.	0.5134	1.4264	0.4570	2.6275	1.8694	1.2322
		MSE	0.0017	0.0239	0.0014	0.0441	0.0131	0.0163
	BE-IF$_{LN1}$	Avg.	0.5112	1.3743	0.4555	2.5261	1.8454	1.2089
		MSE	0.0016	0.0334	0.0015	0.0699	0.0170	0.0199
	BE-IF$_{LN2}$	Avg.	0.5156	1.4860	0.4584	2.7345	1.8931	1.2553
		MSE	0.0018	0.0197	0.0014	0.0406	0.0105	0.0139
$(0.26, 1.53)$	MLE	Avg.	0.5459	1.4584	0.4514	2.4751	1.8174	1.1936
		MSE	0.0169	0.4587	0.0054	0.8309	0.1683	0.1314
	BE-NIF$_{SE}$	Avg.	0.6047	1.6290	0.4610	2.6013	1.8262	1.1869
		MSE	0.0879	1.0711	0.0188	1.6996	0.2595	0.2562
	BE-NIF$_{LN1}$	Avg.	0.5917	1.5278	0.4581	2.4522	1.8053	1.1585
		MSE	0.0773	0.7939	0.0187	1.5251	0.2646	0.2652
	BE-NIF$_{LN2}$	Avg.	0.6183	1.7579	0.4639	2.7676	1.8469	1.2148
		MSE	0.1007	1.5833	0.0189	2.0247	0.2559	0.2499
	BE-IF$_{SE}$	Avg.	0.5135	1.4264	0.4570	2.6278	1.8694	1.2322
		MSE	0.0017	0.0237	0.0014	0.0435	0.0130	0.0162
	BE-IF$_{LN1}$	Avg.	0.5113	1.3745	0.4555	2.5265	1.8455	1.2089
		MSE	0.0016	0.0332	0.0015	0.0692	0.0168	0.0198
	BE-IF$_{LN2}$	Avg.	0.5158	1.4860	0.4584	2.7350	1.8931	1.2552
		MSE	0.0018	0.0196	0.0014	0.0400	0.0103	0.0137

Table 4. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) at $n = 120, r = 60$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H_{RE}^{ρ}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(0.06, 0.67)	MLE	Avg.	0.5233	1.4901	0.4603	2.5874	1.8734	1.2421
		MSE	0.0066	0.2614	0.0027	0.4364	0.0914	0.0712
	BE-NIF$_{SE}$	Avg.	0.5797	1.6495	0.4688	2.7211	1.8805	1.2326
		MSE	0.0700	0.7322	0.0159	1.3230	0.1765	0.1903
	BE-NIF$_{LN1}$	Avg.	0.5677	1.5529	0.4659	2.5683	1.8596	1.2038
		MSE	0.0608	0.5463	0.0158	1.1434	0.1793	0.1964
	BE-NIF$_{LN2}$	Avg.	0.5925	1.7731	0.4717	2.8917	1.9013	1.2608
		MSE	0.0821	1.0991	0.0161	1.6564	0.1750	0.1868
	BE-IF$_{SE}$	Avg.	0.5138	1.4432	0.4592	2.6064	1.8671	1.2325
		MSE	0.0015	0.0277	0.0011	0.0532	0.0129	0.0139
	BE-IF$_{LN1}$	Avg.	0.5122	1.3984	0.4582	2.5256	1.8496	1.2164
		MSE	0.0015	0.0334	0.0012	0.0764	0.0156	0.0161
	BE-IF$_{LN2}$	Avg.	0.5155	1.4942	0.4602	2.6902	1.8847	1.2486
		MSE	0.0016	0.0261	0.0011	0.0435	0.0109	0.0121
(0.26, 1.53)	MLE	Avg.	0.5233	1.4879	0.4603	2.5867	1.8730	1.2418
		MSE	0.0066	0.2541	0.0027	0.4327	0.0898	0.0697
	BE-NIF$_{SE}$	Avg.	0.5797	1.6470	0.4688	2.7204	1.8801	1.2323
		MSE	0.0700	0.7220	0.0158	1.3203	0.1745	0.1885
	BE-NIF$_{LN1}$	Avg.	0.5678	1.5506	0.4659	2.5677	1.8592	1.2034
		MSE	0.0608	0.5379	0.0157	1.1408	0.1773	0.1946
	BE-NIF$_{LN2}$	Avg.	0.5926	1.7702	0.4717	2.8911	1.9009	1.2604
		MSE	0.0821	1.0856	0.0160	1.6534	0.1730	0.1849
	BE-IF$_{SE}$	Avg.	0.5139	1.4432	0.4592	2.6065	1.8671	1.2325
		MSE	0.0015	0.0275	0.0011	0.0529	0.0129	0.0138
	BE-IF$_{LN1}$	Avg.	0.5122	1.3984	0.4582	2.5257	1.8496	1.2164
		MSE	0.0014	0.0333	0.0011	0.0761	0.0156	0.0160
	BE-IF$_{LN2}$	Avg.	0.5155	1.4942	0.4602	2.6905	1.8847	1.2485
		MSE	0.0016	0.0258	0.0011	0.0433	0.0108	0.0121

Table 5. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 30, r = 15$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H_{RE}^{ρ}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(0.11, 0.46)	MLE	Avg.	1.2220	0.4808	0.3352	0.0144	0.8917	0.4627
		MSE	0.2998	0.0640	0.0133	0.7822	0.1749	0.1424
	BE-NIF$_{SE}$	Avg.	1.4266	0.5613	0.3540	0.1388	0.9311	0.4882
		MSE	1.1306	0.1985	0.0287	1.0560	0.2339	0.2132
	BE-NIF$_{LN1}$	Avg.	1.2996	0.5345	0.3502	0.0805	0.9117	0.4662
		MSE	0.6726	0.1476	0.0283	1.0018	0.2383	0.2188
	BE-NIF$_{LN2}$	Avg.	1.5953	0.5921	0.3578	0.2013	0.9504	0.5100
		MSE	2.1896	0.2812	0.0291	1.1429	0.2307	0.2092
	BE-IF$_{SE}$	Avg.	1.0299	0.4818	0.3533	0.2451	0.9747	0.5320
		MSE	0.0088	0.0017	0.0023	0.0192	0.0108	0.0125
	BE-IF$_{LN1}$	Avg.	1.0150	0.4778	0.3510	0.1912	0.9598	0.5170
		MSE	0.0077	0.0018	0.0024	0.0280	0.0129	0.0147
	BE-IF$_{LN2}$	Avg.	1.0454	0.4858	0.3556	0.3020	0.9896	0.5469
		MSE	0.0105	0.0016	0.0022	0.0166	0.0092	0.0109
(0.26, 0.80)	MLE	Avg.	1.2118	0.4736	0.3371	0.0234	0.8964	0.4671
		MSE	0.2788	0.0445	0.0113	0.7071	0.1467	0.1181
	BE-NIF$_{SE}$	Avg.	1.4161	0.5530	0.3555	0.1391	0.9299	0.4870
		MSE	1.1001	0.1732	0.0270	0.9966	0.2151	0.1977
	BE-NIF$_{LN1}$	Avg.	1.2904	0.5275	0.3517	0.0811	0.9105	0.4649
		MSE	0.6521	0.1274	0.0266	0.9466	0.2194	0.2034
	BE-NIF$_{LN2}$	Avg.	1.5837	0.5823	0.3594	0.2013	0.9493	0.5088
		MSE	2.1617	0.2487	0.0274	1.0773	0.2120	0.1937
	BE-IF$_{SE}$	Avg.	1.0294	0.4820	0.3539	0.2453	0.9748	0.5320
		MSE	0.0087	0.0015	0.0020	0.0181	0.0098	0.0116
	BE-IF$_{LN1}$	Avg.	1.0146	0.4781	0.3516	0.1919	0.9601	0.5172
		MSE	0.0076	0.0016	0.0020	0.0266	0.0118	0.0136
	BE-IF$_{LN2}$	Avg.	1.0449	0.4861	0.3562	0.3016	0.9894	0.5468
		MSE	0.0104	0.0014	0.0019	0.0155	0.0084	0.0101

Table 6. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 60, r = 30$

(T_1, T_2)	Method		Parm. α	Parm. β	R	H_{SE}	H^{ρ}_{RE} $\rho = 0.3$	H^{ρ}_{RE} $\rho = 0.6$
(0.11, 0.46)	MLE	Avg.	1.0993	0.4851	0.3501	0.1505	0.9547	0.5196
		MSE	0.0833	0.0247	0.0062	0.3629	0.0792	0.0642
	BE-NIF$_{SE}$	Avg.	1.2800	0.5615	0.3664	0.2366	0.9684	0.5205
		MSE	0.6518	0.1212	0.0231	0.6759	0.1633	0.1602
	BE-NIF$_{LN1}$	Avg.	1.1782	0.5388	0.3626	0.1774	0.9487	0.4979
		MSE	0.3872	0.0952	0.0227	0.6372	0.1663	0.1648
	BE-NIF$_{LN2}$	Avg.	1.4125	0.5870	0.3702	0.2996	0.9882	0.5427
		MSE	1.2670	0.1600	0.0235	0.7364	0.1616	0.1572
	BE-IF$_{SE}$	Avg.	1.0299	0.4844	0.3568	0.2435	0.9826	0.5412
		MSE	0.0080	0.0015	0.0016	0.0203	0.0086	0.0097
	BE-IF$_{LN1}$	Avg.	1.0186	0.4810	0.3552	0.1962	0.9706	0.5298
		MSE	0.0071	0.0016	0.0016	0.0280	0.0100	0.0109
	BE-IF$_{LN2}$	Avg.	1.0416	0.4879	0.3584	0.2926	0.9947	0.5526
		MSE	0.0091	0.0014	0.0016	0.0172	0.0075	0.0087
(0.26, 0.80)	MLE	Avg.	1.0979	0.4841	0.3503	0.1500	0.9545	0.5195
		MSE	0.0814	0.0227	0.0058	0.3479	0.0735	0.0593
	BE-NIF$_{SE}$	Avg.	1.2784	0.5601	0.3667	0.2333	0.9664	0.5186
		MSE	0.6489	0.1176	0.0227	0.6605	0.1587	0.1563
	BE-NIF$_{LN1}$	Avg.	1.1769	0.5375	0.3629	0.1742	0.9466	0.4961
		MSE	0.3853	0.0923	0.0223	0.6223	0.1618	0.1611
	BE-NIF$_{LN2}$	Avg.	1.4104	0.5852	0.3705	0.2962	0.9861	0.5409
		MSE	1.2615	0.1553	0.0231	0.7206	0.1569	0.1532
	BE-IF$_{SE}$	Avg.	1.0299	0.4844	0.3568	0.2445	0.9832	0.5417
		MSE	0.0079	0.0015	0.0016	0.0201	0.0083	0.0094
	BE-IF$_{LN1}$	Avg.	1.0187	0.4810	0.3552	0.1972	0.9711	0.5303
		MSE	0.0071	0.0016	0.0016	0.0277	0.0097	0.0107
	BE-IF$_{LN2}$	Avg.	1.0416	0.4879	0.3585	0.2936	0.9952	0.5531
		MSE	0.0091	0.0014	0.0016	0.0170	0.0073	0.0084

Table 7. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 120, r = 60$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H_{RE}^{ρ}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(0.11, 0.46)	MLE	Avg.	1.0472	0.4943	0.3600	0.2266	0.9929	0.5545
		MSE	0.0328	0.0130	0.0030	0.1819	0.0390	0.0311
	BE-NIF$_{SE}$	Avg.	1.2195	0.5724	0.3743	0.3060	0.9992	0.5477
		MSE	0.5355	0.1137	0.0198	0.5076	0.1239	0.1280
	BE-NIF$_{LN1}$	Avg.	1.1284	0.5488	0.3705	0.2454	0.9792	0.5247
		MSE	0.3166	0.0862	0.0194	0.4708	0.1260	0.1316
	BE-NIF$_{LN2}$	Avg.	1.3353	0.5987	0.3781	0.3706	1.0192	0.5702
		MSE	1.0231	0.1550	0.0202	0.5659	0.1231	0.1260
	BE-IF$_{SE}$	Avg.	1.0293	0.4875	0.3604	0.2450	0.9928	0.5526
		MSE	0.0068	0.0018	0.0011	0.0279	0.0080	0.0080
	BE-IF$_{LN1}$	Avg.	1.0210	0.4846	0.3594	0.2032	0.9832	0.5441
		MSE	0.0062	0.0019	0.0011	0.0344	0.0089	0.0087
	BE-IF$_{LN2}$	Avg.	1.0378	0.4904	0.3615	0.2880	1.0025	0.5611
		MSE	0.0077	0.0018	0.0011	0.0249	0.0074	0.0075
(0.26, 0.80)	MLE	Avg.	1.0470	0.4942	0.3600	0.2262	0.9927	0.5543
		MSE	0.0326	0.0127	0.0029	0.1803	0.0384	0.0306
	BE-NIF$_{SE}$	Avg.	1.2192	0.5722	0.3743	0.3057	0.9990	0.5475
		MSE	0.5344	0.1130	0.0197	0.5062	0.1232	0.1273
	BE-NIF$_{LN1}$	Avg.	1.1282	0.5486	0.3705	0.2451	0.9790	0.5245
		MSE	0.3161	0.0857	0.0193	0.4693	0.1252	0.1309
	BE-NIF$_{LN2}$	Avg.	1.3348	0.5985	0.3781	0.3702	1.0190	0.5701
		MSE	1.0206	0.1540	0.0201	0.5644	0.1223	0.1253
	BE-IF$_{SE}$	Avg.	1.0292	0.4876	0.3605	0.2450	0.9928	0.5526
		MSE	0.0068	0.0018	0.0011	0.0277	0.0080	0.0079
	BE-IF$_{LN1}$	Avg.	1.0209	0.4847	0.3594	0.2032	0.9832	0.5441
		MSE	0.0062	0.0019	0.0011	0.0343	0.0088	0.0086
	BE-IF$_{LN2}$	Avg.	1.0377	0.4905	0.3615	0.2880	1.0025	0.5611
		MSE	0.0076	0.0018	0.0011	0.0247	0.0073	0.0074

Table 8. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 2.5, \beta = 2.5$) at $n = 30, r = 15$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H^{ρ}_{RE}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(1.86, 4.22)	MLE	Avg.	3.3980	2.3344	0.9861	0.8946	2.7175	2.3754
		MSE	3.9056	0.6939	0.0002	0.3068	0.0890	0.0812
	BE-NIF$_{SE}$	Avg.	3.7921	2.6054	0.9737	0.9340	2.7241	2.3773
		MSE	8.4230	1.9954	0.0015	0.4560	0.1782	0.1704
	BE-NIF$_{LN1}$	Avg.	3.3691	2.3942	0.9733	0.9070	2.7048	2.3576
		MSE	4.7864	1.3937	0.0016	0.4541	0.1838	0.1762
	BE-NIF$_{LN2}$	Avg.	4.3236	2.8800	0.9740	0.9618	2.7434	2.3970
		MSE	15.1843	3.4087	0.0015	0.4626	0.1739	0.1659
	BE-IF$_{SE}$	Avg.	2.6245	2.4345	0.9834	1.0397	2.7879	2.4432
		MSE	0.0932	0.0274	0.0001	0.0072	0.0057	0.0061
	BE-IF$_{LN1}$	Avg.	2.4913	2.3723	0.9833	1.0166	2.7795	2.4351
		MSE	0.0631	0.0388	0.0001	0.0096	0.0065	0.0068
	BE-IF$_{LN2}$	Avg.	2.7848	2.5032	0.9835	1.0635	2.7962	2.4513
		MSE	0.1822	0.0239	0.0001	0.0059	0.0051	0.0055
(2.95, 6.15)	MLE	Avg.	3.3612	2.3249	0.9860	0.8983	2.7202	2.3781
		MSE	3.5595	0.5965	0.0002	0.2789	0.0776	0.0705
	BE-NIF$_{SE}$	Avg.	3.7664	2.5947	0.9735	0.9379	2.7267	2.3799
		MSE	8.3480	1.8781	0.0015	0.4243	0.1655	0.1586
	BE-NIF$_{LN1}$	Avg.	3.3456	2.3856	0.9732	0.9110	2.7073	2.3602
		MSE	4.6931	1.3075	0.0016	0.4227	0.1708	0.1642
	BE-NIF$_{LN2}$	Avg.	4.2942	2.8672	0.9739	0.9657	2.7460	2.3996
		MSE	15.1651	3.2502	0.0015	0.4302	0.1615	0.1544
	BE-IF$_{SE}$	Avg.	2.6229	2.4348	0.9834	1.0410	2.7887	2.4440
		MSE	0.0903	0.0261	0.0001	0.0060	0.0051	0.0055
	BE-IF$_{LN1}$	Avg.	2.4900	2.3731	0.9833	1.0179	2.7804	2.4360
		MSE	0.0609	0.0373	0.0001	0.0083	0.0058	0.0062
	BE-IF$_{LN2}$	Avg.	2.7837	2.5030	0.9835	1.0647	2.7970	2.4520
		MSE	0.1796	0.0227	0.0001	0.0048	0.0046	0.0050

Table 9. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 2.5, \beta = 2.5$) at $n = 60, r = 30$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H^{ρ}_{RE}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(1.86, 4.22)	MLE	Avg.	2.8514	2.4158	0.9862	0.9787	2.7656	2.4227
		MSE	0.8906	0.3090	0.0001	0.1436	0.0404	0.0367
	BE-NIF$_{SE}$	Avg.	3.3071	2.7798	0.9727	1.0181	2.7709	2.4231
		MSE	4.8126	2.5035	0.0015	0.2812	0.1279	0.1252
	BE-NIF$_{LN1}$	Avg.	2.8182	2.4128	0.9723	0.9910	2.7517	2.4035
		MSE	2.0763	1.2848	0.0016	0.2791	0.1312	0.1289
	BE-NIF$_{LN2}$	Avg.	3.9920	3.2653	0.9731	1.0458	2.7902	2.4427
		MSE	12.6081	5.7529	0.0014	0.2865	0.1258	0.1228
	BE-IF$_{SE}$	Avg.	2.6093	2.4413	0.9849	1.0280	2.7884	2.4447
		MSE	0.0712	0.0246	3.90E-5	0.0084	0.0045	0.0046
	BE-IF$_{LN1}$	Avg.	2.5095	2.3910	0.9848	1.0086	2.7819	2.4386
		MSE	0.0500	0.0323	3.94E-5	0.0108	0.0050	0.0051
	BE-IF$_{LN2}$	Avg.	2.7252	2.4967	0.9849	1.0477	2.7949	2.4508
		MSE	0.1230	0.0220	3.86E-5	0.0068	0.0041	0.0042
(2.95, 6.15)	MLE	Avg.	2.8457	2.4140	0.9862	0.9783	2.7655	2.4226
		MSE	0.8638	0.2913	1.04E-4	0.1376	0.0380	0.0345
	BE-NIF$_{SE}$	Avg.	3.3008	2.7762	0.9727	1.0176	2.7709	2.4231
		MSE	4.7701	2.4611	0.0015	0.2741	0.1252	0.1227
	BE-NIF$_{LN1}$	Avg.	2.8141	2.4107	0.9724	0.9905	2.7516	2.4035
		MSE	2.0587	1.2631	0.0016	0.2720	0.1285	0.1264
	BE-NIF$_{LN2}$	Avg.	3.9824	3.2589	0.9731	1.0453	2.7901	2.4427
		MSE	12.5048	5.6531	0.0014	0.2793	0.1231	0.1203
	BE-IF$_{SE}$	Avg.	2.6088	2.4418	0.9849	1.0279	2.7882	2.4446
		MSE	0.0716	0.0242	3.90E-5	0.0084	0.0045	0.0046
	BE-IF$_{LN1}$	Avg.	2.5090	2.3916	0.9848	1.0085	2.7818	2.4384
		MSE	0.0504	0.0318	3.94E-5	0.0108	0.0050	0.0051
	BE-IF$_{LN2}$	Avg.	2.7245	2.4970	0.9849	1.0477	2.7947	2.4507
		MSE	0.1232	0.0217	3.86E-5	0.0067	0.0041	0.0042

Table 10. Avg.and MSE estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 2.5, \beta = 2.5$) at $n = 120, r = 60$

(T_1, T_2)	Method		Parm.		R	H_{SE}	H_{RE}^{ρ}	
			α	β			$\rho = 0.3$	$\rho = 0.6$
(1.86, 4.22)	MLE	Avg.	2.6652	2.4633	0.9860	1.0296	2.7953	2.4517
		MSE	0.3338	0.1644	0.0001	0.0715	0.0195	0.0176
	BE-NIF$_{SE}$	Avg.	3.0922	2.8353	0.9705	1.0698	2.7984	2.4496
		MSE	3.6174	2.5344	0.0016	0.2058	0.1044	0.1039
	BE-NIF$_{LN1}$	Avg.	2.6661	2.4583	0.9701	1.0425	2.7791	2.4299
		MSE	1.5862	1.2039	0.0017	0.2030	0.1067	0.1066
	BE-NIF$_{LN2}$	Avg.	3.6759	3.3304	0.9709	1.0976	2.8176	2.4692
		MSE	9.3317	6.0205	0.0016	0.2114	0.1033	0.1025
	BE-IF$_{SE}$	Avg.	2.6117	2.4468	0.9853	1.0434	2.7992	2.4553
		MSE	0.0729	0.0290	2.28E-5	0.0120	0.0042	0.0041
	BE-IF$_{LN1}$	Avg.	2.5333	2.4063	0.9853	1.0253	2.7939	2.4503
		MSE	0.0529	0.0339	2.29E-5	0.0136	0.0045	0.0043
	BE-IF$_{LN2}$	Avg.	2.7006	2.4910	0.9853	1.0621	2.8046	2.4603
		MSE	0.1122	0.0275	2.27E-5	0.0110	0.0040	0.0038
(2.95, 6.15)	MLE	Avg.	2.6642	2.4632	0.9860	1.0293	2.7952	2.4516
		MSE	0.3306	0.1620	0.0001	0.0709	0.0193	0.0173
	BE-NIF$_{SE}$	Avg.	3.0910	2.8348	0.9705	1.0696	2.7983	2.4495
		MSE	3.6023	2.5253	0.0016	0.2050	0.1041	0.1036
	BE-NIF$_{LN1}$	Avg.	2.6654	2.4581	0.9701	1.0424	2.7790	2.4298
		MSE	1.5823	1.2000	0.0017	0.2023	0.1063	0.1063
	BE-NIF$_{LN2}$	Avg.	3.6738	3.3294	0.9709	1.0974	2.8175	2.4691
		MSE	9.2913	5.9998	0.0016	0.2107	0.1029	0.1022
	BE-IF$_{SE}$	Avg.	2.6118	2.4469	0.9853	1.0436	2.7993	2.4554
		MSE	0.0729	0.0287	0.0000	0.0119	0.0042	0.0040
	BE-IF$_{LN1}$	Avg.	2.5334	2.4064	0.9853	1.0255	2.7940	2.4505
		MSE	0.0528	0.0336	2.30E-5	0.0135	0.0045	0.0043
	BE-IF$_{LN2}$	Avg.	2.7006	2.4909	0.9853	1.0622	2.8047	2.4603
		MSE	0.1122	0.0272	2.27E-5	0.0109	0.0039	0.0038

Table 11. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) at $n = 30, r = 15$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(0.06, 0.67)	α	(0.1895, 0.9870) 0.7974 / 95.2	(0.2987, 1.1586) 0.8599 / 98.3	(0.2267, 1.2590) 1.0323 / 96.0	(0.4306, 0.5986) 0.1680 / 97.3
	β	(0.0000, 3.8554) 3.8554 / 95.4	(0.3202, 7.1305) 6.8104 / 99.2	(0.1752, 4.1695) 3.9943 / 95.3	(1.2335, 1.6136) 0.3800 / 96.2
	R	(0.2161, 0.6544) 0.4383 / 98.8	(0.2631, 0.7201) 0.4571 / 99.9	(0.1632, 0.7828) 0.6196 / 98.1	(0.3824, 0.5317) 0.1493 / 96.3
	H_{SE}	(0.0000, 4.8589) 4.8589 / 96.6	(0.7841, 6.9071) 6.1230 / 99.8	(0.0000, 5.5550) 5.5550 / 96.8	(2.2719, 2.9744) 0.7024 / 96.4
	$H_{RE}^{\rho=0.3}$	(0.5506, 2.9337) 2.3831 / 97.2	(0.8791, 3.4524) 2.5733 / 99.2	(0.4936, 3.1446) 2.6510 / 97.5	(1.7272, 2.0564) 0.3292 / 98.2
	$H_{RE}^{\rho=0.6}$	(0.0653, 2.1899) 2.1246 / 97.1	(0.4396, 2.8927) 2.4531 / 99.8	(0.0528, 2.5717) 2.5189 / 98.6	(1.0291, 1.4645) 0.4354 / 97.9
(0.26, 1.53)	α	(0.1953, 0.9731) 0.7778 / 95.2	(0.3002, 1.1368) 0.8366 / 98.3	(0.2086, 1.2058) 0.9973 / 95.7	(0.4347, 0.6003) 0.1656 / 97.9
	β	(0.0000, 3.4446) 3.4446 / 95.7	(0.3864, 5.6164) 5.2301 / 99.5	(0.2444, 3.8805) 3.6361 / 95.4	(1.2300, 1.6169) 0.3869 / 96.7
	R	(0.2449, 0.6339) 0.3889 / 98.5	(0.2823, 0.6840) 0.4017 / 99.9	(0.1718, 0.7558) 0.5840 / 97.2	(0.3927, 0.5381) 0.1454 / 97.8
	H_{SE}	(0.0000, 4.7128) 4.7128 / 96.8	(0.8559, 6.4326) 5.5767 / 99.9	(0.0000, 5.2915) 5.2915 / 95.8	(2.2596, 2.9733) 0.7137 / 96.0
	$H_{RE}^{\rho=0.3}$	(0.7049, 2.7989) 2.0940 / 97.1	(0.9637, 3.1846) 2.2209 / 99.6	(0.6580, 2.9663) 2.3083 / 97.6	(1.7213, 2.0485) 0.3272 / 98.1
	$H_{RE}^{\rho=0.6}$	(0.2115, 2.0608) 1.8493 / 97.7	(0.5035, 2.5637) 2.0603 / 99.9	(0.0112, 2.2461) 2.2349 / 96.9	(1.0176, 1.4506) 0.4330 / 97.4

Table 12. CIs estimates under different GT2HCS based on Gen-Ex distribution at (α = 0.5, β = 1.5) at $n = 60, r = 30$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(0.06, 0.67)	α	(0.1895, 0.9870) 0.7974 / 95.2	(0.2987, 1.1586) 0.8599 / 98.3	(0.2267, 1.2590) 1.0323 / 96.0	(0.4306, 0.5986) 0.1680 / 97.3
	β	(0.0000, 3.8554) 3.8554 / 95.4	(0.3202, 7.1305) 6.8104 / 99.2	(0.1752, 4.1695) 3.9943 / 95.3	(1.2335, 1.6136) 0.3800 / 96.2
	R	(0.2161, 0.6544) 0.4383 / 98.8	(0.2631, 0.7201) 0.4571 / 99.9	(0.1632, 0.7828) 0.6196 / 98.1	(0.3824, 0.5317) 0.1493 / 96.3
	H_{SE}	(0.0000, 4.8589) 4.8589 / 96.6	(0.7841, 6.9071) 6.1230 / 99.8	(0.0000, 5.5550) 5.5550 / 96.8	(2.2719, 2.9744) 0.7024 / 96.4
	$H_{RE}^{\rho=0.3}$	(0.5506, 2.9337) 2.3831 / 97.2	(0.8791, 3.4524) 2.5733 / 99.2	(0.4936, 3.1446) 2.6510 / 97.5	(1.7272, 2.0564) 0.3292 / 98.2
	$H_{RE}^{\rho=0.6}$	(0.4827, 1.9156) 1.4329 / 97.5	(0.6598, 2.1795) 1.5197 / 99.6	(0.1911, 2.1846) 1.9936 / 97.5	(1.0333, 1.4693) 0.4361 / 97.7
(0.26, 1.53)	α	(0.1953, 0.9731) 0.7778 / 95.2	(0.3002, 1.1368) 0.8366 / 98.3	(0.2086, 1.2058) 0.9973 / 95.7	(0.4347, 0.6003) 0.1656 / 97.9
	β	(0.0000, 3.4446) 3.4446 / 95.7	(0.3864, 5.6164) 5.2301 / 99.5	(0.2444, 3.8805) 3.6361 / 95.4	(1.2300, 1.6169) 0.3869 / 96.7
	R	(0.2449, 0.6339) 0.3889 / 98.5	(0.2823, 0.6840) 0.4017 / 99.9	(0.1718, 0.7558) 0.5840 / 97.2	(0.3927, 0.5381) 0.1454 / 97.8
	H_{SE}	(0.0000, 4.7128) 4.7128 / 96.8	(0.8559, 6.4326) 5.5767 / 99.9	(0.0000, 5.2915) 5.2915 / 95.8	(2.2596, 2.9733) 0.7137 / 96.0
	$H_{RE}^{\rho=0.3}$	(0.7049, 2.7989) 2.0940 / 97.1	(0.9637, 3.1846) 2.2209 / 99.6	(0.6580, 2.9663) 2.3083 / 97.6	(1.7213, 2.0485) 0.3272 / 98.1
	$H_{RE}^{\rho=0.6}$	(0.5169, 1.8844) 1.3674 / 97.7	(0.6793, 2.1219) 1.4426 / 99.7	(0.1146, 2.0487) 1.9341 / 96.0	(1.0468, 1.4687) 0.4219 / 97.8

Table 13. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 0.5, \beta = 1.5$) at $n = 120, r = 60$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(0.06, 0.67)	α	(0.3693, 0.6696) 0.3003 / 95.5	(0.3891, 0.6936) 0.3045 / 97.3	(0.2437, 0.9378) 0.6942 / 96.1	(0.4348, 0.5743) 0.1395 / 96.3
	β	(0.4936, 2.5092) 2.0156 / 95.4	(0.7673, 2.9378) 2.1704 / 98.8	(0.5183, 3.0415) 2.5232 / 95.9	(1.1748, 1.7215) 0.5467 / 96.8
	R	(0.3601, 0.5588) 0.1987 / 98.4	(0.3701, 0.5704) 0.2003 / 98.9	(0.2382, 0.7387) 0.5005 / 97.3	(0.4056, 0.5220) 0.1164 / 97.3
	H_{SE}	(1.3360, 3.8874) 2.5513 / 97.1	(1.6025, 4.2565) 2.6540 / 99.3	(0.6952, 4.7370) 4.0417 / 96.3	(2.1982, 3.0117) 0.8135 / 96.4
	$H_{RE}^{\rho=0.3}$	(1.2908, 2.4659) 1.1751 / 97.4	(1.3738, 2.5681) 1.1944 / 98.9	(1.0525, 2.6515) 1.5990 / 96.7	(1.7015, 2.0763) 0.3748 / 97.7
	$H_{RE}^{\rho=0.6}$	(0.7266, 1.7622) 1.0356 / 97.7	(0.8208, 1.8865) 1.0657 / 99.2	(0.3006, 2.0173) 1.7167 / 96.2	(1.0556, 1.4588) 0.4032 / 98.1
(0.26, 1.53)	α	(0.3695, 0.6693) 0.2998 / 95.5	(0.3892, 0.6932) 0.3040 / 97.3	(0.2437, 0.9378) 0.6942 / 96.1	(0.4348, 0.5743) 0.1395 / 96.3
	β	(0.5053, 2.4952) 1.9899 / 95.2	(0.7729, 2.9119) 2.1390 / 98.9	(0.5439, 3.0425) 2.4987 / 96.1	(1.1846, 1.7274) 0.5428 / 97.3
	R	(0.3616, 0.5576) 0.1959 / 98.5	(0.3714, 0.5688) 0.1974 / 99.2	(0.2405, 0.7387) 0.4982 / 97.3	(0.4047, 0.5218) 0.1171 / 97.3
	H_{SE}	(1.3440, 3.8806) 2.5367 / 97.1	(1.6075, 4.2450) 2.6375 / 99.3	(0.6952, 4.7315) 4.0363 / 96.2	(2.1982, 3.0117) 0.8135 / 96.4
	$H_{RE}^{\rho=0.3}$	(1.2978, 2.4594) 1.1616 / 97.2	(1.3790, 2.5592) 1.1802 / 98.9	(1.1009, 2.6932) 1.5922 / 97.2	(1.6985, 2.0707) 0.3722 / 97.2
	$H_{RE}^{\rho=0.6}$	(0.7334, 1.7559) 1.0225 / 97.5	(0.8254, 1.8769) 1.0515 / 99.4	(0.3117, 2.0086) 1.6969 / 96.1	(1.0556, 1.4588) 0.4032 / 98.1

Table 14. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 30, r = 15$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
$(0.11, 0.46)$	α	(0.2729, 2.1625) 1.8896 / 95.0	(0.5605, 2.6455) 2.0850 / 98.0	(0.3853, 2.6270) 2.2417 / 95.4	(0.8718, 1.2236) 0.3519 / 97.5
	β	(0.0000, 0.9755) 0.9755 / 95.9	(0.1719, 1.3452) 1.1733 / 98.9	(0.1099, 1.1151) 1.0052 / 95.5	(0.4132, 0.5569) 0.1437 / 99.0
	R	(0.1234, 0.5498) 0.4264 / 98.4	(0.1787, 0.6342) 0.4555 / 99.9	(0.0506, 0.6940) 0.6434 / 97.0	(0.2715, 0.4420) 0.1704 / 96.9
	H_{SE}	(0.0000, 1.6554) 1.6554 / 96.4	(0.0000, 1.7435) 1.7435 / 98.0	(0.0000, 1.9697) 1.9697 / 96.5	(0.0000, 0.4750) 0.4750 / 98.3
	$H_{RE}^{\rho=0.3}$	(0.1320, 1.6788) 1.5469 / 97.1	(0.3854, 2.1273) 1.7420 / 99.7	(0.0526, 1.9826) 1.9301 / 98.4	(0.8161, 1.1497) 0.3336 / 97.9
	$H_{RE}^{\rho=0.6}$	(0.0000, 1.1705) 1.1705 / 97.0	(0.1085, 2.0646) 1.9560 / 99.9	(0.0000, 1.4606) 1.4606 / 98.5	(0.3622, 0.7354) 0.3732 / 98.2
$(0.26, 0.80)$	α	(0.2972, 2.1179) 1.8207 / 94.6	(0.5682, 2.5663) 1.9981 / 98.2	(0.3901, 2.6185) 2.2284 / 95.4	(0.8533, 1.2018) 0.3485 / 96.4
	β	(0.0632, 0.8840) 0.8208 / 95.2	(0.1991, 1.1266) 0.9275 / 98.8	(0.1259, 1.0382) 0.9124 / 95.3	(0.4127, 0.5483) 0.1356 / 98.0
	R	(0.1434, 0.5339) 0.3905 / 98.5	(0.1902, 0.6027) 0.4125 / 96.8	(0.0602, 0.6786) 0.6185 / 96.4	(0.2723, 0.4397) 0.1674 / 96.5
	H_{SE}	(0.0000, 1.5827) 1.5827 / 96.7	(0.0000, 1.6535) 1.6535 / 97.6	(0.0000, 1.8138) 1.8138 / 95.8	(0.0000, 0.4460) 0.4460 / 97.4
	$H_{RE}^{\rho=0.3}$	(0.2145, 1.6056) 1.3911 / 96.9	(0.4238, 1.9543) 1.5306 / 99.8	(0.0101, 1.7841) 1.7739 / 96.9	(0.8292, 1.1543) 0.3251 / 98.4
	$H_{RE}^{\rho=0.6}$	(0.0000, 1.1010) 1.1010 / 97.4	(0.1296, 1.7611) 1.6315 / 98.3	(0.0000, 1.3659) 1.3659 / 97.6	(0.3662, 0.7257) 0.3595 / 97.7

Table 15. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 60, r = 30$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(0.11, 0.46)	α	(0.5680, 1.6306) 1.0626 / 95.2	(0.6780, 1.7825) 1.1045 / 97.8	(0.4535, 2.1340) 1.6804 / 95.8	(0.8852, 1.2123) 0.3271 / 97.8
	β	(0.1783, 0.7919) 0.6136 / 95.5	(0.2577, 0.9131) 0.6553 / 98.5	(0.1696, 0.9656) 0.7960 / 95.6	(0.4123, 0.5507) 0.1383 / 97.0
	R	(0.2034, 0.4978) 0.2944 / 97.4	(0.2304, 0.5335) 0.3031 / 99.7	(0.0880, 0.6724) 0.5844 / 96.5	(0.2935, 0.4433) 0.1498 / 98.0
	H_{SE}	(0.0000, 1.3010) 1.3010 / 97.0	(0.0002, 123.7396) 123.7394 / 96.8	(0.0000, 1.7726) 1.7726 / 97.0	(0.0068, 0.5056) 0.4987 / 98.4
	$H_{RE}^{\rho=0.3}$	(0.4338, 1.4887) 1.0548 / 97.0	(0.5553, 1.6639) 1.1085 / 99.6	(0.1690, 1.7191) 1.5502 / 97.4	(0.8509, 1.1632) 0.3123 / 98.2
	$H_{RE}^{\rho=0.6}$	(0.0510, 0.9981) 0.9471 / 97.3	(0.2127, 1.2938) 1.0811 / 99.9	(0.0000, 1.2784) 1.2784 / 97.2	(0.4062, 0.7408) 0.3346 / 98.6
(0.26, 0.80)	α	(0.5725, 1.6233) 1.0508 / 95.3	(0.6804, 1.7717) 1.0913 / 97.8	(0.4623, 2.1396) 1.6773 / 96.0	(0.8852, 1.2149) 0.3296 / 97.9
	β	(0.1901, 0.7780) 0.5879 / 95.3	(0.2637, 0.8885) 0.6247 / 98.6	(0.1810, 0.9606) 0.7797 / 95.9	(0.4191, 0.5537) 0.1346 / 97.6
	R	(0.2087, 0.4933) 0.2846 / 97.7	(0.2340, 0.5264) 0.2924 / 99.5	(0.0948, 0.6683) 0.5735 / 96.4	(0.2914, 0.4362) 0.1449 / 97.4
	H_{SE}	(0.0000, 1.2807) 1.2807 / 97.1	(0.0003, 103.7431) 103.7428 / 96.8	(0.0000, 1.7726) 1.7726 / 97.0	(0.0081, 0.4933) 0.4852 / 98.1
	$H_{RE}^{\rho=0.3}$	(0.4532, 1.4709) 1.0177 / 97.3	(0.5669, 1.6327) 1.0658 / 99.7	(0.1556, 1.6923) 1.5368 / 96.6	(0.8537, 1.1535) 0.2998 / 97.9
	$H_{RE}^{\rho=0.6}$	(0.0696, 0.9810) 0.9114 / 97.1	(0.2206, 1.2507) 1.0301 / 99.9	(0.0000, 1.2311) 1.2311 / 96.6	(0.4104, 0.7323) 0.3219 / 98.3

Table 16. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 1, \beta = 0.5$) at $n = 120, r = 60$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
$(0.11, 0.46)$	α	(0.7045, 1.3900) 0.6856 / 95.6	(0.7549, 1.4528) 0.6979 / 97.5	(0.4764, 1.9053) 1.4290 / 95.9	(0.8845, 1.1821) 0.2977 / 97.4
	β	(0.2711, 0.7175) 0.4463 / 95.7	(0.3147, 0.7764) 0.4617 / 98.6	(0.1967, 0.9191) 0.7224 / 95.6	(0.4119, 0.5630) 0.1511 / 97.2
	R	(0.2547, 0.4646) 0.2099 / 98.4	(0.2686, 0.4815) 0.2129 / 99.0	(0.1124, 0.6602) 0.5478 / 96.2	(0.2955, 0.4215) 0.1261 / 96.5
	H_{SE}	(0.0000, 1.0683) 1.0683 / 97.0	(0.0079, 7.3793) 7.3714 / 96.8	(0.0000, 1.5272) 1.5272 / 96.6	(0.0000, 0.5159) 0.5159 / 97.2
	$H_{RE}^{\rho=0.3}$	(0.6122, 1.3802) 0.7681 / 97.7	(0.6775, 1.4648) 0.7872 / 99.1	(0.3222, 1.6916) 1.3694 / 97.3	(0.8564, 1.1531) 0.2967 / 97.8
	$H_{RE}^{\rho=0.6}$	(0.2138, 0.8990) 0.6852 / 97.5	(0.3006, 1.0299) 0.7294 / 99.7	(0.0000, 1.2204) 1.2204 / 96.5	(0.4096, 0.7142) 0.3046 / 97.6
$(0.26, 0.80)$	α	(0.7052, 1.3888) 0.6836 / 95.7	(0.7554, 1.4512) 0.6958 / 97.5	(0.4764, 1.9053) 1.4290 / 95.9	(0.8845, 1.1821) 0.2977 / 97.4
	β	(0.2731, 0.7153) 0.4422 / 95.7	(0.3159, 0.7730) 0.4571 / 98.7	(0.2030, 0.9098) 0.7068 / 95.6	(0.4126, 0.5620) 0.1494 / 97.1
	R	(0.2558, 0.4635) 0.2077 / 98.4	(0.2695, 0.4801) 0.2106 / 99.2	(0.1128, 0.6602) 0.5474 / 96.2	(0.2955, 0.4200) 0.1246 / 96.3
	H_{SE}	(0.0000, 1.0638) 1.0638 / 97.1	(0.0081, 7.2095) 7.2014 / 96.8	(0.0000, 1.5272) 1.5272 / 96.6	(0.0000, 0.5306) 0.5306 / 98.1
	$H_{RE}^{\rho=0.3}$	(0.6162, 1.3766) 0.7604 / 97.7	(0.6803, 1.4593) 0.7790 / 99.2	(0.3294, 1.6818) 1.3524 / 97.3	(0.8564, 1.1531) 0.2967 / 97.8
	$H_{RE}^{\rho=0.6}$	(0.2176, 0.8955) 0.6779 / 97.4	(0.3027, 1.0232) 0.7205 / 99.9	(0.0000, 1.2648) 1.2648 / 97.3	(0.4123, 0.7178) 0.3055 / 97.8

Table 17. CIs estimates under different GT2HCS based on Gen-Ex distribution at (α = 2.5, β = 2.5) at $n = 30, r = 15$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(1.86, 4.22)	α	(0.0000, 6.9701) 6.9701 / 95.8	(1.1122, 9.9477) 8.8355 / 98.7	(0.9193, 8.0184) 7.0991 / 95.4	(2.1526, 3.1355) 0.9828 / 97.4
	β	(0.7604, 3.9637) 3.2033 / 95.7	(1.1989, 4.6535) 3.4545 / 98.4	(0.6947, 4.7531) 4.0583 / 95.3	(2.1980, 2.7105) 0.5125 / 97.8
	R	(0.9582, 1.0128) 0.0545 / 96.8	(0.9586, 1.0132) 0.0545 / 96.8	(0.9052, 1.0000) 0.0948 / 96.8	(0.9727, 0.9967) 0.0240 / 98.7
	H_{SE}	(0.0000, 1.9386) 1.9386 / 96.6	(0.3020, 2.7905) 2.4885 / 99.8	(0.0000, 2.3062) 2.3062 / 97.7	(0.8955, 1.1550) 0.2594 / 96.7
	$H_{RE}^{\rho=0.3}$	(2.1798, 3.2742) 1.0944 / 97.2	(2.2312, 3.3330) 1.1018 / 98.4	(1.9159, 3.5051) 1.5892 / 97.1	(2.6878, 2.9219) 0.2341 / 97.9
	$H_{RE}^{\rho=0.6}$	(1.8632, 2.9050) 1.0417 / 97.4	(1.9162, 2.9662) 1.0500 / 98.5	(1.6575, 3.2086) 1.5510 / 98.5	(2.3381, 2.5847) 0.2465 / 97.9
(2.95, 6.15)	α	(0.0000, 6.7345) 6.7345 / 95.5	(1.1470, 9.3951) 8.2481 / 98.9	(0.7185, 7.6843) 6.9659 / 95.2	(2.1270, 3.1062) 0.9792 / 96.9
	β	(0.9161, 3.7837) 2.8677 / 95.1	(1.2766, 4.3256) 3.0490 / 98.6	(0.9096, 4.6231) 3.7135 / 95.9	(2.1944, 2.7073) 0.5129 / 97.8
	R	(0.9583, 1.0126) 0.0543 / 96.8	(0.9587, 1.0130) 0.0543 / 96.8	(0.9052, 1.0000) 0.0948 / 96.8	(0.9737, 0.9971) 0.0234 / 99.5
	H_{SE}	(0.0000, 1.8821) 1.8821 / 96.7	(0.3251, 2.6129) 2.2878 / 99.9	(0.0000, 2.1667) 2.1667 / 96.8	(0.8955, 1.1495) 0.2540 / 96.6
	$H_{RE}^{\rho=0.3}$	(2.2271, 3.2324) 1.0052 / 97.7	(2.2707, 3.2816) 1.0109 / 98.6	(1.9915, 3.5051) 1.5136 / 97.4	(2.6826, 2.9131) 0.2305 / 97.7
	$H_{RE}^{\rho=0.6}$	(1.9093, 2.8645) 0.9552 / 97.9	(1.9541, 2.9157) 0.9616 / 98.7	(1.6207, 3.1065) 1.4859 / 97.0	(2.3317, 2.5747) 0.2431 / 97.8

Table 18. CIs estimates under different GT2HCS based on Gen-Ex distribution at ($\alpha = 2.5, \beta = 2.5$) at $n = 60, r = 30$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
(1.86, 4.22)	α	(1.1339, 4.5688) 3.4350 / 94.5	(1.5612, 5.2077) 3.6465 / 98.2	(1.0107, 5.7372) 4.7266 / 95.4	(2.1620, 3.0902) 0.9282 / 97.1
	β	(1.3383, 3.4933) 2.1550 / 95.8	(1.5465, 3.7737) 2.2271 / 98.2	(1.0180, 4.4901) 3.4722 / 95.8	(2.1992, 2.7339) 0.5347 / 97.7
	R	(0.9662, 1.0055) 0.0393 / 96.8	(0.9664, 1.0057) 0.0393 / 96.8	(0.9021, 1.0000) 0.0979 / 96.8	(0.9754, 0.9958) 0.0204 / 99.2
	H_{SE}	(0.2838, 1.7005) 1.4168 / 97.0	(0.4859, 2.0261) 1.5402 / 99.9	(0.0000, 1.9287) 1.9287 / 96.4	(0.8791, 1.1770) 0.2979 / 96.5
	$H_{RE}^{\rho=0.3}$	(2.3978, 3.1445) 0.7467 / 96.7	(2.4218, 3.1708) 0.7489 / 98.1	(2.1106, 3.4564) 1.3458 / 97.9	(2.6810, 2.9087) 0.2277 / 96.7
	$H_{RE}^{\rho=0.6}$	(2.0731, 2.7824) 0.7093 / 97.3	(2.0978, 2.8096) 0.7119 / 98.2	(1.6971, 3.0469) 1.3498 / 96.5	(2.3547, 2.5904) 0.2357 / 98.6
(2.95, 6.15)	α	(1.1541, 4.5374) 3.3833 / 94.4	(1.5705, 5.1566) 3.5861 / 98.2	(1.0107, 5.7178) 4.7071 / 95.4	(2.1534, 3.0829) 0.9295 / 96.9
	β	(1.3691, 3.4588) 2.0897 / 95.7	(1.5659, 3.7214) 2.1555 / 98.3	(1.0146, 4.4458) 3.4312 / 95.7	(2.2056, 2.7249) 0.5194 / 97.9
	R	(0.9662, 1.0055) 0.0394 / 96.8	(0.9664, 1.0057) 0.0394 / 96.8	(0.9033, 1.0000) 0.0967 / 96.8	(0.9757, 0.9960) 0.0203 / 99.5
	H_{SE}	(0.2984, 1.6874) 1.3890 / 97.2	(0.4933, 1.9984) 1.5051 / 96.8	(0.0778, 1.9707) 1.8928 / 96.9	(0.8856, 1.1818) 0.2962 / 96.8
	$H_{RE}^{\rho=0.3}$	(2.4090, 3.1342) 0.7252 / 97.1	(2.4318, 3.1590) 0.7272 / 98.4	(2.1027, 3.4372) 1.3345 / 97.6	(2.6955, 2.9130) 0.2175 / 97.4
	$H_{RE}^{\rho=0.6}$	(2.0841, 2.7724) 0.6883 / 97.4	(2.1074, 2.7980) 0.6906 / 98.4	(1.7302, 3.0660) 1.3358 / 96.9	(2.3458, 2.5727) 0.2269 / 97.2

Table 19. CIs estimates under different GT2HCS based on Gen-Ex distribution at (α = 2.5, β = 2.5) at $n = 120, r = 60$

(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI: NIF	HPD-CI: IF
		(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP	(Lower, Upper) AIL / CP
$(1.86, 4.22)$	α	(1.5795, 3.7508) 2.1712 / 95.3	(1.7734, 4.0052) 2.2318 / 97.4	(0.9907, 4.8185) 3.8278 / 95.2	(2.1650, 3.0427) 0.8777 / 96.9
	β	(1.6716, 3.2550) 1.5834 / 96.3	(1.7862, 3.3971) 1.6108 / 98.4	(1.1673, 4.4240) 3.2568 / 95.9	(2.1639, 2.7497) 0.5859 / 97.2
	R	(0.9716, 1.0003) 0.0287 / 96.8	(0.9717, 1.0004) 0.0287 / 96.8	(0.8991, 1.0000) 0.1009 / 96.8	(0.9783, 0.9951) 0.0168 / 99.5
	H_{SE}	(0.5179, 1.5586) 1.0407 / 97.7	(0.6290, 1.7138) 1.0848 / 99.6	(0.2092, 1.8667) 1.6575 / 97.0	(0.8720, 1.2323) 0.3603 / 97.9
	$H_{RE}^{\rho=0.3}$	(2.5265, 3.0692) 0.5428 / 97.7	(2.5392, 3.0828) 0.5436 / 98.1	(2.1586, 3.4149) 1.2562 / 96.8	(2.6957, 2.9167) 0.2210 / 97.8
	$H_{RE}^{\rho=0.6}$	(2.1968, 2.7110) 0.5141 / 97.7	(2.2099, 2.7249) 0.5151 / 98.1	(1.8049, 3.0643) 1.2594 / 96.8	(2.3509, 2.5652) 0.2144 / 97.3
$(2.95, 6.15)$	α	(1.5836, 3.7447) 2.1610 / 95.3	(1.7759, 3.9967) 2.2207 / 97.5	(1.2061, 5.0287) 3.8226 / 96.4	(2.1650, 3.0427) 0.8777 / 96.9
	β	(1.6773, 3.2491) 1.5718 / 96.3	(1.7903, 3.3890) 1.5986 / 98.5	(1.1673, 4.4240) 3.2568 / 95.9	(2.1639, 2.7497) 0.5859 / 97.2
	R	(0.9716, 1.0003) 0.0287 / 96.8	(0.9717, 1.0004) 0.0287 / 96.8	(0.9003, 1.0000) 0.0997 / 96.8	(0.9783, 0.9951) 0.0168 / 99.5
	H_{SE}	(0.5212, 1.5557) 1.0345 / 97.7	(0.6311, 1.7089) 1.0778 / 99.5	(0.2033, 1.8489) 1.6456 / 96.7	(0.8746, 1.2323) 0.3577 / 97.9
	$H_{RE}^{\rho=0.3}$	(2.5289, 3.0671) 0.5381 / 97.6	(2.5415, 3.0804) 0.5390 / 98.1	(2.1586, 3.4052) 1.2465 / 96.7	(2.6969, 2.9167) 0.2198 / 97.8
	$H_{RE}^{\rho=0.6}$	(2.1992, 2.7088) 0.5096 / 97.7	(2.2120, 2.7225) 0.5105 / 98.0	(1.8178, 3.0733) 1.2555 / 97.0	(2.3546, 2.5680) 0.2133 / 97.6

Table 20. Time between failures for 30 repairable items data set

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74	1.23	0.94	4.36	0.40	1.74
4.73	2.23	0.45	0.70	1.06	1.46	0.30	1.82	2.37	0.63	1.23	1.24	1.97	1.86	1.17

It is important to begin by assessing whether the Gen-Ex distribution is suitable for analyzing the provided data set. The MLEs are computed for the parameters (α, β), and various goodness-of-fit criteria are evaluated. These criteria include the negative log-likelihood criterion (NLC), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the Kolmogorov-Smirnov (K-S) test statistic along with its associated p-value. These criteria are used to assess the adequacy of the Gen-Ex distribution when compared to other distributions such as Weibull, Generalized Inverted Exponential (GIE), Burr XII, Exponential, and Gamma distributions. In Table 21, the estimated parameters and goodness-of-fit statistics are presented. Lower values of these criteria along with larger p-values indicate a better fit. The values presented in Table 21 suggest that the Gen-Ex distribution provides a

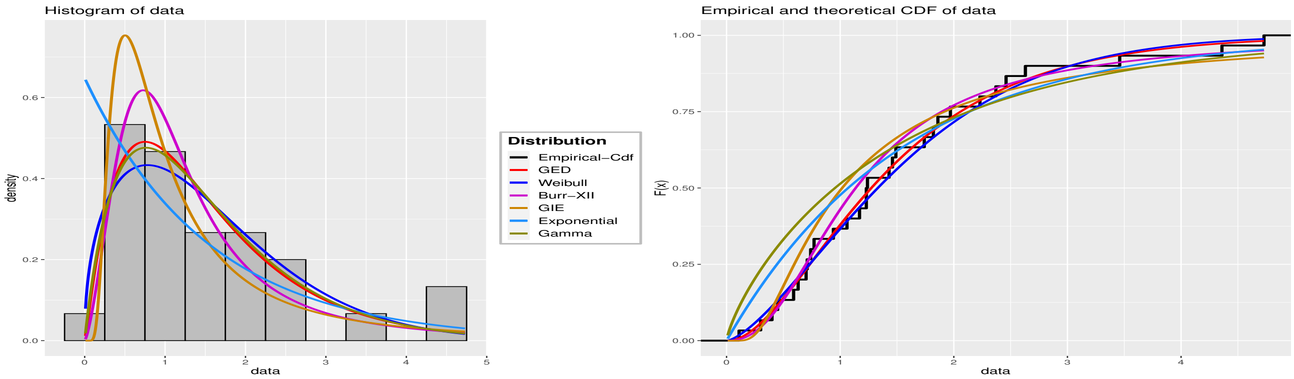


Figure 4. The density and empirical CDF for given real data set with corresponding distributions

suitable model for the given data set when compared to the other distributions. As a result, the data set can be effectively analyzed using the Gen-Ex distribution, with MLEs of $\hat{\alpha} = 2.1235$ and $\hat{\beta} = 0.9967$.

Table 21. Goodness of fit for given data set

PDF	Estimates (St.Er)		NLC	AIC	BIC	K-S	P-value
Gen-Ex	2.1235 (0.5875)	0.9967 (0.2001)	39.6143	83.2287	86.0310	0.0650	0.9996
Weibull	1.4633 (0.2029)	1.7101 (0.2254)	39.9103	83.8207	86.6231	0.0749	0.9959
Burr XII	2.3705 (0.4217)	0.8097 (0.1724)	40.9061	85.8123	88.6147	0.1161	0.8133
GIE	1.6678 (0.4722)	1.0975 (0.2480)	44.9656	93.9313	96.7337	0.1631	0.4014
Exponential	1.5425 (0.2815)	—	43.0053	88.0107	89.4119	0.1845	0.2588
gamma	1.9761 (0.4734)	0.7806 (0.2126)	39.6295	83.2592	86.0615	0.2367	0.0692

To visually assess the fit of the Gen-Ex distribution to the given data set, graphical representations can be used. The empirical CDF of the data set can be plotted alongside the corresponding fitted CDFs for the Weibull, GIE, Burr XII, Exponential, and Gamma distributions. Additionally, a histogram of the data set can be plotted, accompanied by the corresponding fitted PDF lines for the same distributions. Figure 4 displays the fitted lines for both the CDFs and PDFs of the given data set and the respective distributions. These figures provide a visual comparison, illustrating that the Gen-Ex distribution offers a better fit than the other distributions, at least for this specific data set.

Table 22 provides the outcomes of the Gen-Ex distribution analysis under different GT2HCS with a total sample size of $n = 30$ and failure sample sizes of $r = 10$ and $r = 20$ from the original real data set. Various censoring times are taken into account for computational purposes. The table includes estimates and standard errors for the MLEs of the Gen-Ex distribution parameters $\hat{\alpha}$ and $\hat{\beta}$. Subsequently, MLEs of R, H_{SE}, and H_{RE} (with ($\rho = 0.3$ and 0.6)) are computed by substituting the estimated values of $\hat{\alpha}$ and $\hat{\beta}$. Furthermore, BEs are calculated using the MH algorithm with the NIF prior. In generating samples from the posterior distribution via MH, the initial values of (α, β) are set as $(\alpha^{(0)}, \beta^{(0)}) = (\hat{\alpha}, \hat{\beta})$. The first 2000 samples are discarded as burn-in from a total of 10,000 samples generated from the posterior density. It's worth noting that BEs are computed using different loss functions: SE, LN_1 with $v = -1.5$, and LN_2 with $v = 1.5$. The BEs of R, H_{SE}, and H_{RE} (with ($\rho = 0.3$ and 0.6)) are obtained using equations 3.5 and 3.6. Additionally, Table 23 presents the lower and upper bounds of confidence intervals for the parameters α, β, R, H_{SE}, and H_{RE} (with ($\rho = 0.3$ and 0.6))

using different interval estimation methods: Asy-CI, NA-CI, and HPD.

Table 22. Estimate values and standard error under GT2HCS for given real data set

r	(T_1, T_2)	Method	Parm.				R	H_{SE}	H_{RE}^{ρ}	
			α		β				$\rho = 0.3$	$\rho = 0.6$
10	$(0.65, 1.50)$	MLE	2.7428	(1.1935)	0.7164	(0.2778)	0.8486	-0.2204	1.5853	1.2450
		BE_{SE}	3.6079	(0.8141)	0.5601	(0.0794)	0.8505	-0.5931	1.3742	1.0426
		BE_{LN1}	3.1429	—	0.5549	—	0.8056	-0.5428	1.3482	1.0128
		BE_{LN2}	4.1929	—	0.5653	—	0.8928	-0.6402	1.3998	1.0713
	$(1.20, 2.25)$	MLE	1.8973	(0.7375)	1.1578	(0.4654)	0.8630	0.4918	2.0059	1.6448
		BE_{SE}	1.5747	(0.4490)	1.5652	(0.5007)	0.8702	0.9453	2.2716	1.8940
		BE_{LN1}	1.4646	—	1.4081	—	0.8294	0.9063	2.1509	1.7653
		BE_{LN2}	1.6876	—	1.8230	—	0.9101	1.0382	2.4379	2.0669
	$(1.25, 4.75)$	MLE	2.1994	(0.7920)	0.9595	(0.3063)	0.8622	0.2010	1.8436	1.4924
		BE_{SE}	2.6695	(0.4934)	0.8513	(0.1818)	0.8854	-0.0334	1.7539	1.4125
		BE_{LN1}	2.5127	—	0.8345	—	0.8650	-0.0198	1.7251	1.3809
		BE_{LN2}	2.8730	—	0.8692	—	0.9070	-0.0509	1.7851	1.4465
20	$(0.65, 1.50)$	MLE	2.2268	(0.7484)	0.9471	(0.2645)	0.8625	0.1799	1.8326	1.4821
		BE_{SE}	2.3261	(0.7285)	0.9338	(0.1769)	0.8710	0.1385	1.8256	1.4775
		BE_{LN1}	2.1031	—	0.9128	—	0.8373	0.1808	1.7863	1.4323
		BE_{LN2}	2.7506	—	0.9552	—	0.9153	0.0658	1.8733	1.5331
	$(1.20, 2.25)$	MLE	2.1295	(0.6849)	0.9957	(0.2628)	0.8618	0.2593	1.8753	1.5221
		BE_{SE}	2.0579	(0.3189)	1.0931	(0.1926)	0.8728	0.3759	1.9628	1.6074
		BE_{LN1}	1.9941	—	1.0622	—	0.8582	0.3693	1.9286	1.5710
		BE_{LN2}	2.1275	—	1.1289	—	0.8877	0.3855	2.0007	1.6475
	$(1.25, 4.75)$	MLE	2.1295	(0.6849)	0.9957	(0.2628)	0.8618	0.2593	1.8753	1.5221
		BE_{SE}	1.6257	(0.4352)	1.3210	(0.3523)	0.8470	0.7478	2.1084	1.7339
		BE_{LN1}	1.4908	—	1.2375	—	0.8064	0.7604	2.0255	1.6420
		BE_{LN2}	1.7756	—	1.4328	—	0.8858	0.7559	2.2067	1.8402

Table 23. CIs estimates under different under GT2HCS for given real data set

r	(T_1, T_2)	Parm	Asy-CI	NA-CI	HPD-CI
			(Lower, Upper)	(Lower, Upper)	(Lower, Upper)
10	(0.65, 1.50)	α	(0.4036, 0.1720)	(1.1690, 0.3351)	(2.0195, 4.9508)
		β	(5.0820, 1.2608)	(6.4355, 1.5317)	(0.4219, 0.7225)
		R	(0.7427, 0.9546)	(0.7490, 0.9615)	(0.7388, 0.9380)
		H_{SE}	(-1.1912, 0.7505)	(-18.0543, -0.0027)	(-0.9738, -0.1559)
		$H_{RE}^{\rho=0.3}$	(0.4038, 2.7669)	(0.7524, 3.3404)	(1.1186, 1.6058)
		$H_{RE}^{\rho=0.6}$	(-0.0756, 2.5655)	(0.4310, 3.5960)	(0.7938, 1.2727)
	(1.20, 2.25)	α	(0.4518, 0.2458)	(0.8856, 0.5267)	(0.8527, 2.4533)
		β	(3.3429, 2.0699)	(4.0647, 2.5454)	(0.8772, 2.6514)
		R	(0.7620, 0.9641)	(0.7677, 0.9703)	(0.7448, 0.9489)
		H_{SE}	(-0.2009, 1.1844)	(0.1202, 2.0112)	(0.1357, 1.9685)
		$H_{RE}^{\rho=0.3}$	(0.9440, 3.0679)	(1.1814, 3.4060)	(1.8066, 2.7396)
		$H_{RE}^{\rho=0.6}$	(0.4568, 2.8327)	(0.7988, 3.3867)	(1.4495, 2.3119)
	(1.25, 4.75)	α	(0.6471, 0.3591)	(1.0859, 0.5132)	(1.6056, 3.5805)
		β	(3.7517, 1.5599)	(4.4548, 1.7938)	(0.5877, 1.2154)
		R	(0.7612, 0.9632)	(0.7669, 0.9693)	(0.7864, 0.9463)
		H_{SE}	(-0.1199, 0.5218)	(0.0407, 0.9919)	(-0.5430, 0.5672)
		$H_{RE}^{\rho=0.3}$	(0.9306, 2.7566)	(1.1236, 3.0251)	(1.4314, 2.0934)
		$H_{RE}^{\rho=0.6}$	(0.4622, 2.5227)	(0.7483, 2.9764)	(1.0964, 1.7428)
20	(0.65, 1.50)	α	(0.7599, 0.4286)	(1.1524, 0.5478)	(1.4405, 3.8354)
		β	(3.6937, 1.4655)	(4.3030, 1.6372)	(0.6232, 1.2411)
		R	(0.7617, 0.9633)	(0.7674, 0.9695)	(0.7850, 0.9315)
		H_{SE}	(-0.1261, 0.4859)	(0.0329, 0.9855)	(-0.5479, 0.6746)
		$H_{RE}^{\rho=0.3}$	(1.0178, 2.6474)	(1.1749, 2.8586)	(1.4789, 2.0550)
		$H_{RE}^{\rho=0.6}$	(0.5586, 2.4056)	(0.7948, 2.7637)	(1.1372, 1.6818)
	(1.20, 2.25)	α	(0.7872, 0.4807)	(1.1338, 0.5936)	(1.4688, 2.6320)
		β	(3.4718, 1.5107)	(3.9998, 1.6702)	(0.7566, 1.5189)
		R	(0.7607, 0.9629)	(0.7664, 0.9691)	(0.7662, 0.9226)
		H_{SE}	(-0.0386, 0.5572)	(0.0822, 0.8180)	(-0.0383, 0.8944)
		$H_{RE}^{\rho=0.3}$	(1.1145, 2.6360)	(1.2499, 2.8135)	(1.6803, 2.3066)
		$H_{RE}^{\rho=0.6}$	(0.6578, 2.3864)	(0.8627, 2.6857)	(1.3055, 1.9100)
	(1.25, 4.75)	α	(0.7872, 0.4807)	(1.1338, 0.5936)	(0.9908, 2.4050)
		β	(3.4718, 1.5107)	(3.9998, 1.6702)	(0.8524, 2.0502)
		R	(0.7607, 0.9629)	(0.7664, 0.9691)	(0.7291, 0.9201)
		H_{SE}	(-0.0386, 0.5572)	(0.0822, 0.8180)	(0.0884, 1.6349)
		$H_{RE}^{\rho=0.3}$	(1.1145, 2.6360)	(1.2499, 2.8135)	(1.7170, 2.4621)
		$H_{RE}^{\rho=0.6}$	(0.6578, 2.3864)	(0.8627, 2.6857)	(1.3615, 2.0523)

5. Concluding Remarks

In this paper, we conducted an analysis of the reliability and two distinct entropy measures – Shannon entropy and Rényi entropy – for the Gen-Ex distribution within the framework of GT2HCS. We also assessed its performance in diverse scenarios. The study included the examination of parameter estimation, reliability assessment, and entropy measurement for Gen-Ex distribution under GT2HCS. We investigated MLE and its corresponding interval estimates, including Asy-CI and NA-CI. Additionally, BEs were examined, and their credible intervals (HPD-CI) were obtained using MCMC techniques under IF and NIF priors. This exploration was conducted under two distinct loss functions, specifically SE and LN. To compare the performance of the proposed estimation methods, Monte Carlo simulation and real data analysis have been studied. The main conclusion of our research is that: BEs under IF priors has the best performance among the different method of estimation. The future work can be extended to the study of the optimal censoring times and other censoring schemes can be employed. Also, different loss functions of BEs can be used.

Data Availability

The data is available in this article.

References

1 Amigó, J. M., Balogh, S. G., & Hernández, S. (2018). A brief review of generalized entropies. Entropy, 20(11), 813.

2 Shannon, C. E. (1948). A mathematical theory of communication. Bell System Technical Journal, 27(3), 379-423.

3 Rényi, A. (1961). On measures of entropy and information. In Proceedings of the fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the theory of Statistics. University of California Press.

4 Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. Journal of Statistical Physics, 52(1), 479–487.

5 Amal, S. H., & Ahmed, N. Z. (2019). Estimation of entropy for inverse Weibull distribution under multiple censored data. Journal of Taibah University for Science, 13(1), 331-337.

6 Bantan, A. R., Elgarhy, M., Chesneau, C., & Jamal, F. (2020). Estimation of entropy for inverse Lomax distribution under multiple censored data. Entropy, 22(6), 601.

7 Cho, Y., Sun, H., & Lee, K. (2014). An estimation of the entropy for a Rayleigh distribution based on doubly generalized type-II hybrid censored samples. Entropy, 16, 3655-3669.

8 Cho, Y., Sun, H., & Lee, K. (2015). Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. Entropy, 17, 102-122.

9 Harte, D., & Vere-Jones, D. (2005). The entropy score and its uses in earthquake forecasting. Pure and Applied Geophysics, 162(6-7), 1229-1253.

10 Hassan, O., & Mazen, N. (2022). Product of spacing estimation of entropy for inverse Weibull distribution under progressive type-II censored data with applications. Journal of Taibah University for Science, 16(1), 259-269.

11 Lee, K. (2017). Estimation of entropy of the inverse Weibull distribution under generalized progressive hybrid censored data. Journal of Korean Data & Information Science Society, 28(3), 659-668.

12 Liu, S., & Gui, W. (2019). Estimating the entropy for Lomax distribution based on generalized progressively hybrid censoring. Symmetry, 11(10), 1219.

13 Mahmoud, R. M., Ahmad, M. A., & Mohamed, B. S. (2021). Estimating the Entropy of a Lomax Distribution under Generalized Type-I Hybrid Censoring. Journal of University of Shanghai for Science and Technology, 23(7), 470-481.

14 Xu, R., & Gui, W. H. (2019). Entropy estimation of inverse Weibull Distribution under adaptive Type-II progressive hybrid censoring schemes. Symmetry, 11, 1463.

15 Shi, X., Shi, Y., & Zhou, K. (2021). Estimation for Entropy and Parameters of Generalized Bilal Distribution under Adaptive Type II Progressive Hybrid Censoring Scheme. Entropy, 23, 206.

16 Shrahili, M., Ahmed, R. E. S., Amal, S. H., Ibrahim, E., & Elgarhy, M. (2022). Estimation of Entropy for Log-Logistic Distribution under Progressive Type II Censoring. Journal of Nanomaterials.

17 Yu, J., Gui, W. H., & Shan, Y. Q. (2019). Statistical inference on the Shannon entropy of inverse Weibull distribution under the progressive first-failure censoring. Entropy, 21(12), 1209.

18 Gupta, R. D., & Kundu, D. (1999). Generalized exponential distribution. Australian & New Zealand Journal of Statistics, 41, 173-188.

19 Gupta, R. D., & Kundu, D. (2003). Discriminating between the Weibull and Generalized exponential distribution. Computational Statistics and Data Analysis, 43, 179-196.

20 Alizadeh, M., & Amiri, S. (2015). Generalized exponential distribution: Properties and applications. Journal of Statistical Theory and Practice, 9(2), 303-320.

21 Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family; an alternative to gamma and Weibull distributions. Biometrical Journal, 43, 117-130.

22 Gupta, R. D., & Kundu, D. (2001). Generalized exponential distribution: different methods of estimation. Journal of Statistical Computation and Simulation, 69, 315-337.

23 Gupta, R. D., & Kundu, D. (2002). Generalized exponential distribution: statistical inferences. Journal of Statistical Theory and Applications, 1, 101-118.

24 Kundu, D., & Gupta, R. D. (2009). Bivariate generalized exponential distribution. Journal of Multivariate Analysis, 100, 581-593.

25 Gupta, R. D., & Kundu, D. (2007). Generalized exponential distribution: Existing results and some recent developments. Journal of Statistical Planning and Inference, 137(11), 3537-3547.

26 Raqab, M. Z. (2002). Inference for generalized exponential distribution based on record statistics. Journal of Statistical Planning and Inference, 104, 339-350.

27 Rather, N. A., & Rather, T. A. (2017). New Generalizations of Exponential Distribution with Applications. Journal of Probability and Statistics, Article ID 2106748, 9 pages.

28 Maiti, K., Kayal, S., & Kundu, D. (2022). Statistical inference on the Shannon and Rényi entropy measures of generalized exponential distribution under progressive censoring. SN Computer Science, 3, 317.

29 Childs, A., Chandrasekar, B., Balakrishnan, N., & Kundu, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. Annals of the Institute of Statistical Mathematics, 55(2), 319-330.

30 Giri, N. C., & Shaked, M. (2011). Hybrid censoring schemes: Some models and applications. Statistical Science, 26(3), 316-331.

31 Gupta, R. D., & Kundu, D. (1998). Hybrid censoring schemes with exponential failure distribution. Communications in Statistics - Theory and Methods, 27(12), 3065-3083.

32 Helmy, B. A., Hassan, A. S., & El-Kholy, A. K. (2022). Analysis of uncertainty measure using unified hybrid censored data with applications. Journal of Taibah University for Science, 15, 1130-1143.

33 Helmy, B. A., Hassan, A. S., El-Kholy, A. K., Bantan, R. A. R., & Elgarhy, M. (2023). Analysis of information measures using generalized type-I hybrid censored data. AIMS Mathematics, 8(9), 20283-20304.

34 Kundu, D., & Pradhan, B. (2009). Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring. Communications in Statistics - Theory and Methods, 38(12), 2030-2041.

35 Al Mutairi, A., Khashab, R. H., Almetwally, E. M., Abo-Kasem, O. E., & Ibrahim, G. M. (2023). Bayesian and non-Bayesian inference for inverse Weibull model based on jointly type-II hybrid censoring samples with modeling to physics data. AIP Advances, 13(10).

36 Epstein, B. (1954). Truncated life-tests in the exponential case. Annals of Mathematical Statistics, 25, 555-564.

37 Balakrishnan, N., & Kundu, D. (2013). Hybrid censoring: Models, inferential results, and applications. Computational Statistics and Data Analysis, 57, 166-209.

38 Mahmoud, M. A. W., & Ghazal, M. G. M. (2017). Estimation from the exponentiated Rayleigh distribution based on generalized Type-II hybrid censored data. Journal of the Egyptian Mathematical Society, 25(1), 71-78.

39 Mahmoud, R. M., Moshera, A. M., & Badiaa, S. Kh. Mohamed. (2021). Estimating the Entropy and Residual Entropy of a Lomax Distribution under Generalized Type-II Hybrid Censoring. Mathematics and Statistics, 9(5), 780-791.

40 Moshera, A. M. (2021). Estimation of the Residuals Entropy Function of Inverse Weibull Distribution Based on Generalized Type-II Hybrid Censored Samples. Asian Research Journal of Mathematics, 17(3), 21-34.

41 Meeker, W. Q.,& Escobar, L. A. (1998). Statistical methods for reliability data. Wiley.

42 Doostparast, M., Akbari, M. G., & Balakrishna, N. (2011). Bayesian analysis for the two-parameter Pareto distribution based on record values and times. Journal of Statistical Computation and Simulation, 81, 1393-1403.

43 Varian, H. R. (2000). A Bayesian approach to real estate assessment. In H. R. Varian (Ed.), Variants in Economic Theory: Selected Works of H. R. Varian (pp. 144-155). Edward Elgar Publishing.

44 Dey, S., Singh, S., Tripathi, Y. M., & Asgharzadeh, A. (2016). Estimation and prediction for a progressively censored generalized inverted exponential distribution. Statistical Methodology, 132, 185-202.

45 Chen, M. H., & Shao, Q. M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. Journal of Computational and Graphical Statistics, 8, 69-92.

46 Murthy, D. N. P., Xie, M., & Jiang, R. (2004). Weibull models. Series in Probability and Statistics. Wiley.