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Hartmann Flow and Heat Transfer of Two Immiscible Fluids between Two Parallel Porous Plates under Constant Pressure Gradient

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Abstract

Unsteady Hartman flow and heat transfer of two immiscible fluids between two infinite porous parallel plates are studied in this work. Uniform magnetic field is applied on the fluids perpendicular to the plates and Hall Effect under constant pressure gradient are considered. The equations solution is acquired by applying the finite difference technique in both fluid regions. Effects of physical parameters such as viscosity ratio, conductivity ratio, Hartman number, Prandtl number and Eckert number are presented graphically on both the velocity and temperature profile. It was found that, Hall parameter is direct proportional to main component of the velocity and temperature profiles. With the increase of the viscosity ratio, the temperature and velocity profiles decrease. In both regions, the temperature distribution increases with a rise in the Prandtl and Eckert numbers at any given point, whereas it drops with an increase in the thermal conductivity ratio. This research holds significant importance as it focuses on the extensive utilization of The Hartmann fluid flows in cooling electronic components that produce substantial heat during their operation.

Keywords: Hartman flow, Unsteady flow, Immiscible fluids, MHD, Finite-difference solution.

1. Introduction

The study of a multiphase flow or immiscible fluids arises in many engineering applications and technologies include solar energy collectors, heat exchangers, and magnetohydrodynamics (MHD) power generation [1]. Velocity and volumetric flow rates distributions for the co-current laminar flow of two immiscible liquids in rectangular conduits were presented by Charles and Lilleleht [2]. Renardy and Joseph [3] studied the brand problem with two immiscible fluids in layers between two infinite parallel plate. Kuznetsov [4] investigated couette flow of two immiscible fluids in channel with porous

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medium. Steady flow of two viscous, incompressible, and immiscible fluids in an infinitely parallel-plate channel filled with a uniform porous medium was represented by Chamkha [5]. Ngoma and Erchiqui [6] reported the flow of two non-conducting immiscible fluids in a microchannel between two parallel plates in which the effect of viscous shear stress, and the pressure gradient were considered. The two immiscible fluids flow in inclined channel with entropy generation under the influence of a uniform magnetic field was discussed by Nezhad and Shahri [7]. Abbas *et al*. [8] studied the couette flow of two incompressible, viscous, immiscible, with electrically conducting dusty fluids passing through two infinite parallel plates. The unsteady of two immiscible Maxwell fluid flows through two moving parallel plates are listed by Hisham *et al* [9]. Winfred *et al* [10] investigated the flow and heat transfer of two pure and dusty viscous immiscible fluids flows between parallel and vertical wall. The effect of buoyancy and viscous heating for two immiscible fluids flows in isothermal duct was discussed by Umavathia and Anwar [11]. Padma and Srinivas [12] studied the two immiscible incompressible viscous fluid flows with MHD through a channel with a porous medium. The MHD flow between parallel plates of two immiscible nanofluids was caried out by Zeeshan *et al* [13]. Kumar *et al* [14] examined the magnetofluidic actuation-induced interfacial movement of immiscible fluid layers in an inclined fluidic conduit. The hydromagnetic flow of two immiscible pair stress fluids through a homogeneous porous media in a cylindrical conduit with slip effect was caryout by Punnamchandar *et al* [15]. Anandika et al [16] reported the two-layer model of the hybrid nanofluid of MHD flow between two discs with identical radii.

MHD) flow is of interest of many researchers for its importance in many engineering applications such as MHD generator, cooling of nuclear reactors, MHD pump, plasma physics, and petroleum industries [17]. Malashetty and Leela [18] studied analytically the immiscible MHD flow within a channel horizontally. Navier-Stock equations for MHD flow are solved exactly by Andersson [19]. Cortell [20] investigated numerically the flow of an electrically conducting power-law fluid with a uniform magnetic field. Mekheimer et al. [21] studied the influence of heat transfer of MHD flow in a vertical annulus for Newtonian fluid on the peristaltic flow with a zero Reynolds number. The effects of heat transfer on a peristaltic flow of a MHD Newtonian fluid in a porous horizontal tube were studied by Nadeem et al. [22-23]. The effects of heat transfer and MHD with peristaltic flow of incompressible Newtonian fluid with a porous medium in a vertical tube are investigated by Vasudev et al. [24]. Abdeen *et al* [25] and Joseph *et al* [26] studied the heat transfer of Couette fluid flow passing between two parallel porous plates, under effect of Hall current. The two immiscible fluids flow along a horizontal channel having two porous media with an oscillating lateral wall mass flux during transient Hartmann magnetohydrodynamic flow was investigated by *Bég et al* [27]. Chandrawat and Josi [28] studied the MHD unsteady Couette flow of two immiscible fluids flow on horizontal channel with heat transfer. Abbas *et al* [29] focused on the effects of thermal radiations and chemical reactions on MHD nanofluid flow with mass and heat transfer around a vertical cone in porous media is conducted. Through a curved corrugated channel, two layered immiscible time-dependent flow in the presence of a magnetic field has been statistically analyzed by Goyal and Srinivas [30]. Enamul and Surender [31] studied the hybrid nanofluid flow bounded by double-revolving disks.

The flow along the annular channel between two non-conducting cylinders with high Hartmann number was studied by Todd [32]. Hartmann flow of a transverse magnetic field with an interface at the surface of the permeable bed was investigated by Rudraiah et al. [33]. Nagy and Demendy [34] investigated the effect of Hall currents and rotation on a generalized Hartmann flow and heat transfer. Attia and Lotfy [35] studied the impact of heat transport and temperature on the transient Hartmann flow. Attia [36] investigated the Hartmann flow and thermal transfer of an electrically transmitted incompressible non-Newtonian fluid between two parallel plates. Umavathi et al. [37] discussed an unsteady Hartmann flow of two immiscible fluids passing through a horizontal channel with oscillatory time-dependent wall transpiration velocity. The MHD non-Darcy flow of Hartmann fluid with heat transfer passing through two infinite parallel insulating porous plates was investigated by Attia *et al* [38]. The impact of a magnetic field oblique to the channel walls on MHD friction factors for a range of aspect ratios and channel wall conductivities at varying Hartmann values by Kamble et al [39]. Megahed et al [40] investigated the unstable stretched sheet with extended heat flux that causes the boundary layer laminar flow and heat transfer for MHD fluid. The Unsteady MHD Dusty fluid flow between parallel plates with heat and mass transfer through a Porous Media was studied by Abbas *et al* [41]. The effects of viscous dissipation, the slip velocity phenomena, and Joule heating on the heat transfer mechanism of a non-Newtonian Powell-Eyring fluid that flows due to a stretched sheet is investigated by Abbas *et al* [42-43]. The movement of two immiscible, incompressible, electrically conducting viscous fluids of different viscosities in two distinct layers of equal width within a channel filled with porous media under the influence of a transverse magnetic field and a steady pressure gradient was reported by Ansari and Deo [44]. Srivastava and Deo [45] examined the fully developed flow in a channel of an incompressible, electrically conducting viscous fluid under the application of a transverse uniform magnetic field through a porous media with varying permeability.There is a vast amount of information accessible on the Physical problems involving MHD fluids flow [46–53].

The present study aims to address this gap by investigate an unsteady Hartman flow and heat transfer of two immiscible fluids passing through a horizontal two porous parallel plates with considering Hall Effect, therefore. The novelty of this work lies in its both fluids are acted upon by continuous gradient of pressure and a constant electromagnetic field that is normal on plates, a uniform suction from above and a uniform injection from below. A problem is presented, numerically sorted out, and the relevant results are further discussed in depth graphically to explore the impact of different fluid parameters. Finally, this type of immiscible fluids can find application in real life in manufacturing processes such as power generators, reactors and industrial cooling systems. Additionally, their heightened thermal conductivity makes them suitable for cooling systems applications, enhancing heat dissipation.

Formulation of the problem

We consider that an unsteady Hartman flow of two immiscible fluids flowing between two infinite porous horizontal parallel plates located at y = ±h planes and extending in the x and z directions from 0 to infinite, as shown in figure 1. Region-I which extended from $y \geq -h$ to $y \leq 0$ is filled with a viscous fluid having density ρ_1, thermal conductivity k_1, dynamic viscosity μ_1, and specific heat at constant pressure C_{p1}. Region-II extended from $y \geq 0$ to $y \leq h$ it is stuffed by an additional viscous fluid that has a density of ρ_2, thermal conductivity k_2, dynamic viscosity μ_2, and specific heat at constant pressure C_{p2}. Both the plates are stationary maintained at different T_{w1} at bottom plate at y=-h and T_{w2} at lower plate y= h, in which$T_{w2} > T_{w1}$. A uniform magnetic field $B = (0, B_o, 0)$ is applied normal to the planes of the plates (parallel to y-axis) in positive y-axis. It is taken into account how Hall current affected the speed in z-direction. Due to both plates are porous consistent upward suction and downward injection are utilized in contrast the speed in y-axis v_o is taken to be uniform. The flow in both regions is assumed to be fully developed and acted upon a constant pressure gradient $(-\partial p/\partial x = constant)$ and electrical conductivity $(\sigma_1 = \sigma_2 = \sigma)$.

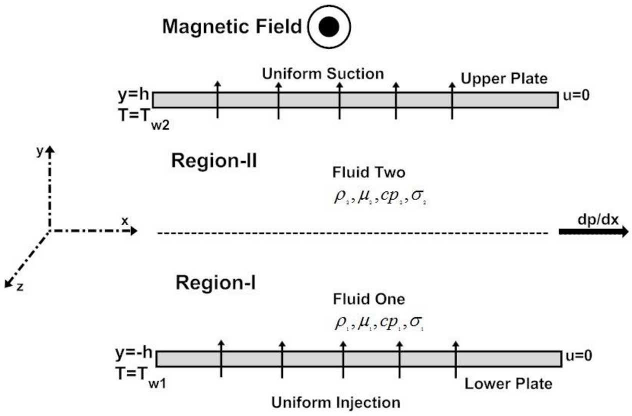


Fig. 1 Schematic diagram of the problem

The generalized Ohm's law including Hall current is given in the form [54]:

$$\vec{J} = \sigma[\vec{E} + \vec{v} \times \vec{B} - \beta\,(\vec{J} \times \vec{B})] \qquad (1)$$

Where $\vec{J}$ vector of electric current density, σ the electric conductivity of the fluid, $\vec{E}$ vector of intensity of the electric field, $\vec{v}$ vector of velocity where $\vec{v}(y,t) = u(y,t)\vec{\imath} + v_0\vec{\jmath} + w(y,t)\vec{k}$, $\vec{B}$ vector of induced magnetic, and β is the Hall factor. By ignoring polarization influence, we obtain no electric field ($\vec{E}$=0). Equation (1) may be solved in $\vec{J}$ to yield [22]:

$$\vec{J} \times \vec{B} = \frac{\sigma B_0}{1+m^2}(mu - w)\vec{\imath} + \frac{\sigma B_0}{1+m^2}(u + mw)\vec{k} \qquad (2)$$

Where, the Hall parameter, $m = \beta\,\sigma B_0$.

The effect of Joule and viscous dissipations are included in the model. Under these assumptions the basics equations of the two immiscible fluids are:

Region-I

$$\frac{\partial v_1}{\partial t} = 0 \qquad (3)$$

$$\rho_1\,\frac{\partial u_1}{\partial t} + \rho_1 v_{o1}\frac{\partial u_1}{\partial y} = -\,\frac{\partial p_1}{\partial x} + \mu_1\,\frac{\partial^2 u_1}{\partial y^2} - \frac{\sigma_1\beta_o^2}{1+m^2}\,(u_1 + m\,w_1) \qquad (4)$$

$$\rho_1\,\frac{\partial w_1}{\partial t} + \rho_1 v_{o1}\frac{\partial w_1}{\partial y} = \mu_1\,\frac{\partial^2 w_1}{\partial y^2} - \frac{\sigma_1\beta_o^2}{1+m^2}\,(w_1 - m\,u_1) \qquad (5)$$

$$\rho_1 c_{p1}\,\frac{\partial T_1}{\partial t} + \rho_1 c_{p1} v_{o1}\frac{\partial T_1}{\partial y} = k_1\frac{\partial^2 T_1}{\partial y^2} + \mu_1\left[\left(\frac{\partial u_1}{\partial y}\right)^2 + \left(\frac{\partial w_1}{\partial y}\right)^2\right] + \frac{\sigma_1\beta_o^2}{1+m^2}\,({u_1}^2 + {w_1}^2) \qquad (6)$$

Region-II

$$\frac{\partial v_2}{\partial t} = 0 \qquad (7)$$

$$\rho_2\,\frac{\partial u_2}{\partial t} + \rho_2 v_{o2}\frac{\partial u_2}{\partial y} = -\,\frac{\partial p_2}{\partial x} + \mu_2\,\frac{\partial^2 u_2}{\partial y^2} - \frac{\sigma_2\beta_o^2}{1+m^2}\,(u_2 + m\,w_2) \qquad (8)$$

$$\rho_2\,\frac{\partial w_2}{\partial t} + \rho_2 v_{o2}\frac{\partial w_2}{\partial y} = \mu_2\,\frac{\partial^2 w_2}{\partial y^2} - \frac{\sigma_2\beta_o^2}{1+m^2}\,(w_2 - m\,u_2) \qquad (9)$$

$$\rho_2 c_{p2}\,\frac{\partial T_2}{\partial t} + \rho_2 c_{p2} v_{o2}\frac{\partial T_2}{\partial y} = k_2\frac{\partial^2 T_2}{\partial y^2} + \mu_2\left[\left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial w_2}{\partial y}\right)^2\right] + \frac{\sigma_2\beta_o^2}{1+m^2}\,({u_2}^2 + {w_2}^2) \qquad (10)$$

In this work, the continuity of velocity, temperature, and shear stress heat flux between the fluids layers are assumed at the interface (y=0). Therefore, initial and boundary conditions for immiscible fluids on the velocity and temperature fields are respectively given by:

At $t \leq 0 \quad u_1 = u_2 = w_1 = w_2 = 0; \quad T_1 = T_2$ (11a)

At $t > 0$:

$$u_1 = 0; \quad T_1 = T_{w1} \quad at\ y = -h \tag{11b}$$

$$u_2 = U_{0;} \ T_2 = T_{w2} \quad at\ y = h \tag{11c}$$

$$u_1 = u_2; \quad T_1 = T_2; \ \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y}; \ k_1 \frac{\partial T_1}{\partial y} = k_2 \frac{\partial T_2}{\partial y}; \ at\ y = 0 \tag{11d}$$

Equations (3)-(10) will be presented with the following non-dimensional variables:

$x_i{}^* = \frac{x_i}{h}, \ y_i{}^* = \frac{y_i}{h}, \ z_i{}^* = \frac{z_i}{h}, \ t_i{}^* = \frac{t_i\,\mu_1}{\rho_1 h^2}, \ u_i{}^* = \frac{u_i \rho_1 h}{\mu_1}, \ w_i{}^* = \frac{w_i \rho_1 h}{\mu_1}, \ p_i{}^* = \frac{p_i \rho_1 h^2}{\mu_1},$

$T_i{}^* = \frac{T_i - T_{w1}}{T_{w2} - T_{w1}}, \ S_i = \frac{\rho_1 h\, v_{o1}}{\mu_1}$.

The below formats will apply to the non-dimensional conservation equations:

Region-I

$$\frac{\partial u_1^*}{\partial t^*} + S_1 \frac{\partial u_1^*}{\partial y^*} = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1^*}{\partial y^{*2}} - \frac{Ha_1^2}{(1+m^2)} (u_1^* + m\, w_1^*) \tag{12}$$

$$\frac{\partial w_1^*}{\partial t^*} + S_1 \frac{\partial w_1^*}{\partial y^*} = \frac{\partial^2 w_1^*}{\partial y^{*2}} - \frac{Ha_1^2}{(1+m^2)} (w_1^* - m\, u_1^*) \tag{13}$$

$$\frac{\partial T_1^*}{\partial t^*} + S_1 \frac{\partial T_1^*}{\partial y^*} = \frac{1}{Pr} \frac{\partial^2 T_1^*}{\partial y^{*2}} + Ec\left[\left(\frac{\partial u_1^*}{\partial y^*}\right)^2 + \left(\frac{\partial w_1^*}{\partial y^*}\right)^2\right] + \frac{Ec\, Ha^2}{(1+m^2)} \left({u_1^*}^2 + {w_1^*}^2\right) \tag{14}$$

Region-II

$$\frac{\partial u_2^*}{\partial t^*} + S_2 \frac{\partial u_2^*}{\partial y^*} = -\frac{1}{\varphi} \frac{\partial p_2}{\partial x} + \frac{\alpha}{\varphi} \frac{\partial^2 u_2^*}{\partial y^{*2}} - \frac{Ha_2^2 \alpha}{\varphi\,(1+m^2)} (u_2^* + m\, w_2^*) \tag{15}$$

$$\frac{\partial w_2^*}{\partial t^*} + S_2 \frac{\partial w_2^*}{\partial y^*} = \frac{\alpha}{\varphi} \frac{\partial^2 w_2^*}{\partial y^{*2}} - \frac{Ha_2^2 \alpha}{\varphi\,(1+m^2)} (w_2^* - m\, u_2^*) \tag{16}$$

$$\frac{\partial T_2^*}{\partial t^*} + S_2 \frac{\partial T_2^*}{\partial y^*} = \frac{k_R}{c_{pR}\, \varphi Pr} \frac{\partial^2 T_2^*}{\partial y^{*2}} + \frac{Ec\, \alpha}{c_{pR}\, \varphi}\left[\left(\frac{\partial u_2^*}{\partial y^*}\right)^2 + \left(\frac{\partial w_2^*}{\partial y^*}\right)^2\right] + \frac{Ec\, Ha_2^2 \alpha}{c_{pR}\, \varphi\ (1+m^2)} \left({u_2^*}^2 + {w_2^*}^2\right) \tag{17}$$

Where, the Hartmann number squared $\mathrm{Ha}_i^2 = \frac{\sigma_i B_0^2 h^2}{\mu_i}$, the Eckert number $\mathrm{Ec} = \frac{\mu_1^2}{\rho_1^2 h^2 C_{p1}\,(T_{w2} - T_{w1})}$, the Prandtle number $Pr = \frac{\mu_1 C_{p1}}{k_1}$, α viscosities ratio, k_R thermal conductivities Ratio, c_{pR} specific heat ratio, and φ densities ratio.

The following are the dimensionless beginning and boundary conditions for the heat and immiscible fluids flow issues, respectively:

At $t \leq 0$: $\ u_1 = u_2 = w_1 = w_2 = 0; \quad T_1 = T_2 = 0$ (18a)

At $t > 0$:

$$u_1 = 0; \quad T_1 = 0 \quad at\ y = -1 \tag{18b}$$

$$u_2 = 1;\ T_2 = 1 \quad at\ y = 1 \tag{18c}$$

$$u_1 = u_2;\ \ T_1 = T_2;\ \ \mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y};\ k_1 \frac{\partial T_1}{\partial y} = k_2 \frac{\partial T_2}{\partial y};\ \ at\ y = 0 \tag{18d}$$

2. Numerical solution

The coupled, unsteady non-linear partial differential equations (12)–(17) under the initial and boundary conditions (18a)–(18d) can be solved numerically with an implicit finite differences approach using Crank-Nicolson technique. The computational domain is discretized with uniform grid of dimension Δy and Δt respectively. The procedure we have adopted involves dividing the solutions into grid points and approximating the differential equation by the finite difference equations. The finite difference equations are obtained by writing the equations at the mid-point of the computational cell and then replacing the differential terms by their second order central difference approximation in the t direction at $(j - 1)$ and $(j + 1)$ [55-59].

3. Results and discussion

The results of unsteady MHD Hartman flow for two immiscible fluids between two horizontal infinite parallel porous plates are presents and discuss with considering the heat transfer and Hall Effect for various parametric conditions.

Figure 2 presents the velocity components u^* and w^* profiles and temperature distribution T^* for different values α in the two regions (Region-I and Region-II) and for φ=1, Ha=1, m=1, S_1= S_2=1, Pr = 0.7, $c_{pR} = 1$, $k_R = 3$, and Ec=1. The figure illustrates how the temperature profiles T* and velocity components u* and w* reduce as the α increases. As α increases, the fluid in the two regions grows thicker, which results in a reduction in the flow velocity components and a reduction in the temperature distribution also.

Figure 3 indicates the effect of m on the profiles of u^*, $w^*$and T^* in the two regions and for α=0.333, φ=1, *Ha*=1, S_1= S_2=0.5, P_r=0.7, $c_{pR} = 1$, $k_R = 3$, and E_c=1. The figure 3(a) indicates that, u^* is rises as the m. This is due to the fact that, an increase in m decreases the effective conductivity $\frac{\mathrm{Ha}^2}{(1+\mathrm{m}^2)}$ and hence the magnetic damping. On the other hand the w^* decreasing with increasing m as shown in figure 3(b), this is for reducing the origin expression w^* and lengthening its damping duration expression $(-\frac{Ha_2^2}{(1+m^2)}(\mathrm{w} - \mathrm{m\,u}))$. Also, it is seen in figure 3(c) the temperatures of the fluid T^* increasing with increasing m, this is as a result of the Joule dissipation term's increased contribution.

The impact of different Ha on the u*, w* and T^* and for α=0.333, φ=1, m=1, S1= S2=1, Pr=0.7, $c_{pR} = 1$, $k_R = 3$, and Ec=1 are illustrated in Figure 4. Figure 4(a) shows that, u^* decreases with increasing the Ha. It is clear that, w^* is increases with increasing the Ha at any point in two regions as shown in Figure 4(b). Also, Figure 4(c) indicate, the temperature decreases as increasing the Ha. Figure 5 illustrated the effect of S on the the u*, w* and T^* profile (α=0.333, φ=1, Ha=2, m=1, $c_{pR} = 1$, $k_R = 3$, Pr=0.7, and Ec=1). It is noticed that the u*, w* and T^* decrease with increases the S at any point in two regions.

The effects of k_R, Pr, and Ec (φ=1, Ha=2, m=1, S1= S2=1, $c_{pR} = 1$, α = 0.333, Pr=0.7, and Ec=1) on the T^* in the two regions is displayed in figures 6-8. It is predicted that T^* decrease as k_R increases. This means the tendency to cool down the thermal state in the fluid for two regions, as shown in figure 6. It seen that, when the Pr or Ec increases the $T^*$is increases at any point in the two regions, this is becouse increase the strength of the heat sources in the temperature equations, as shown in figures 7 – 8.

4. Conclusions

Unsteady flow and heat transfer of two immiscible fluids between two horizontal parallel plates have been investigated. Both fluids were assumed to be Newtonian, electrically conducting, and a uniform magnetic field was subjected to the fluids and perpendicular to the plates. The following points can be concluded:

- The velocity and temperature profiles drop as the viscosity ratio rises.
- The paramter m is direct proportional to $u^*$and T^*while it is inverse proportional to w^*.
- With increased Ha the u* and $T^*$are decreases, while w* is increases at any point in two regions.
- The influence of the k_R, Pr, and Ec, on T^* has been achieved. The distribution T^* decreases with the increase in the k_R, while it increases with the increase in the Pr, and Ec at any position in the two regions.

Future research might focus on exploring more complicated boundary conditions and geometries, which are frequently seen in real-world applications to improve fluid flow system innovation and optimization in engineering and biological applications with more study and improvement.

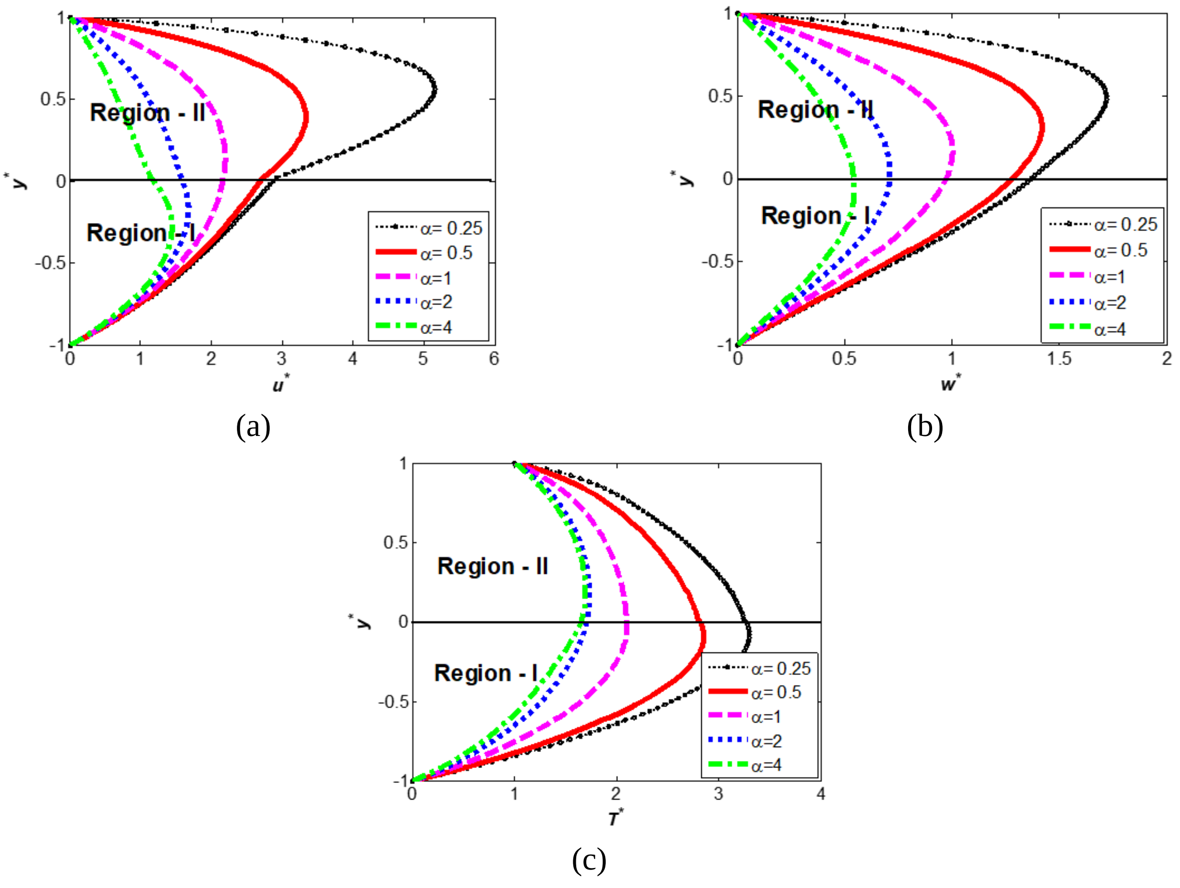
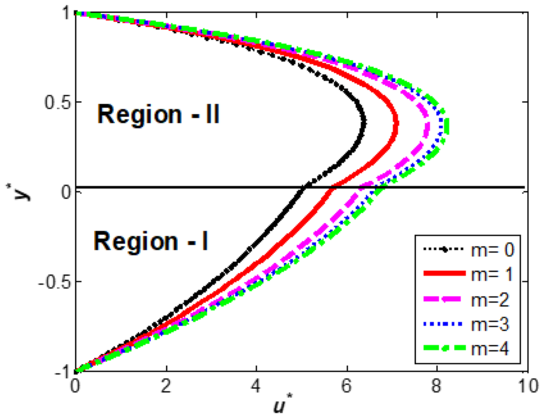
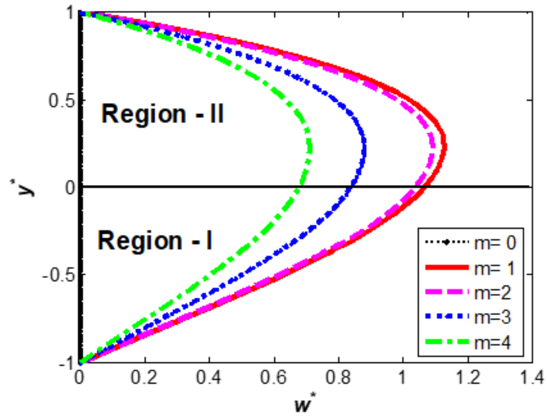


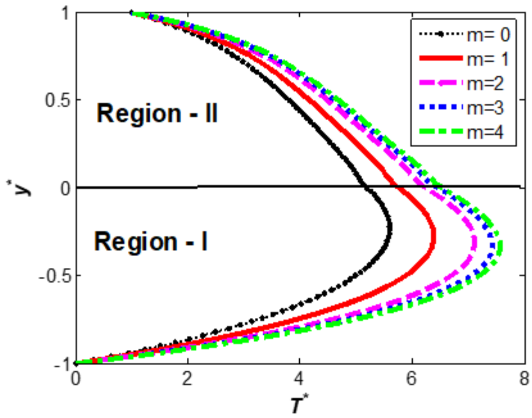
Fig. 2: Effect of α on u^*, w^* and T^* (a) u^* Profile; (b) w^* Profile; (c) T^* Profile.



(a)

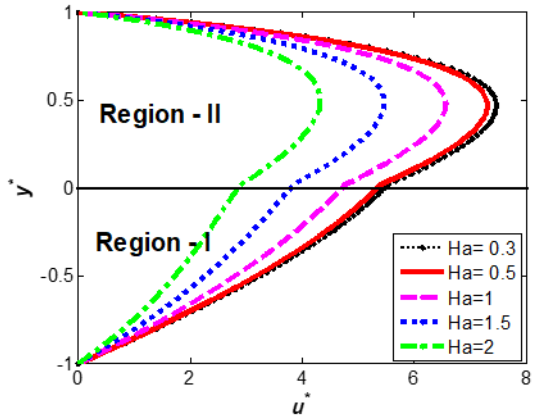


(b)

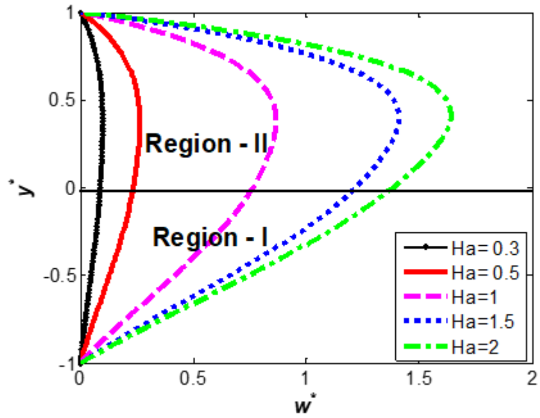


(c)

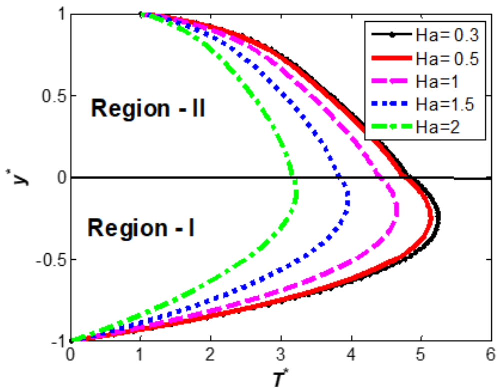
Fig. 3: Effect of m on u^*, w^* and T^* (a) u^* Profile; (b) w^* Profile; (c) T^* Profile



(a)



(b)



(c)

Fig. 4: Effect of Ha u^*, w^* and T^* (a) u^* Profile; (b) w^* Profile; (c) T^* Profile.

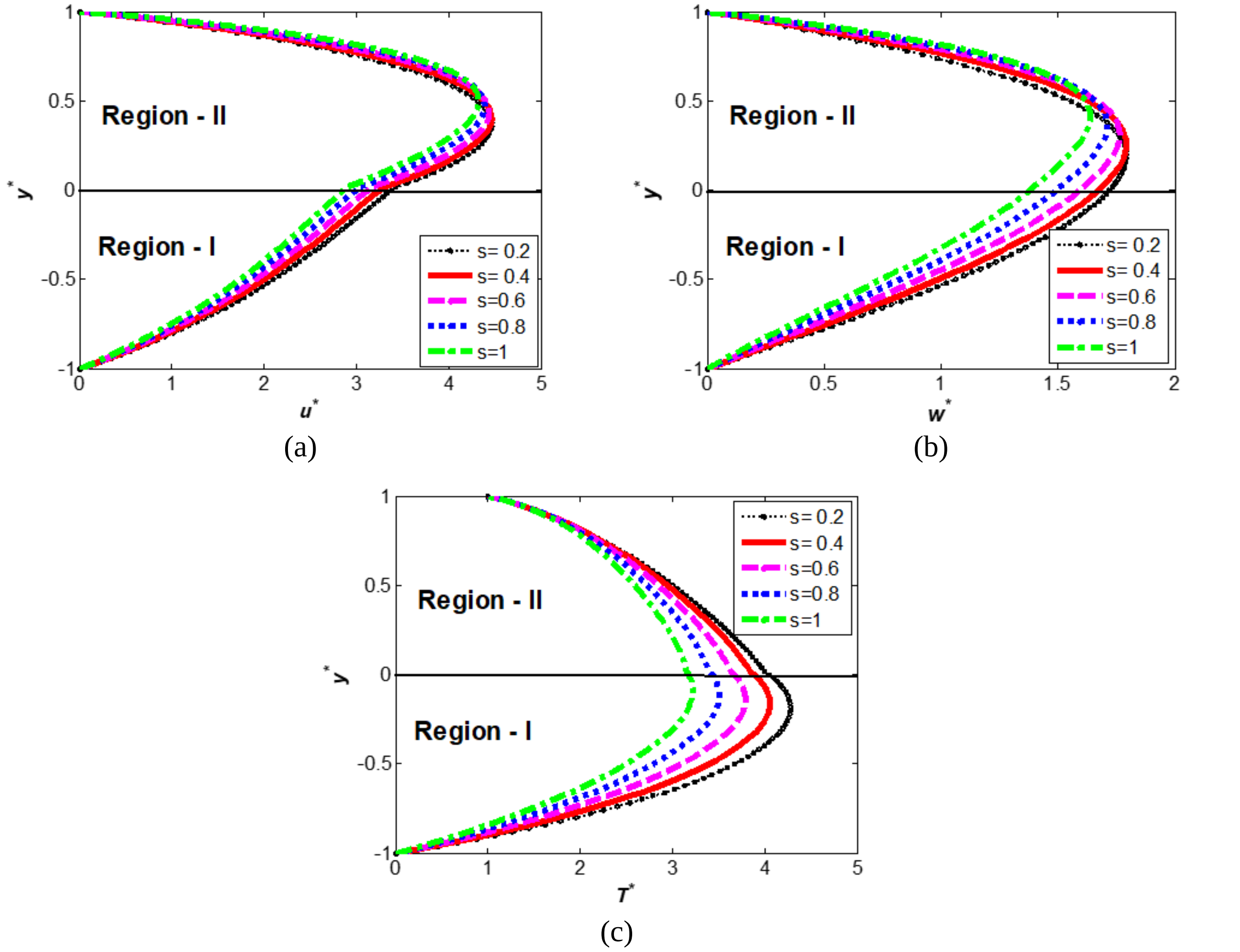


Fig. 5: Effect of S on u^*, w^* and T^* (a) u^* Profile; (b) w^* Profile; (c) T^* Profileز

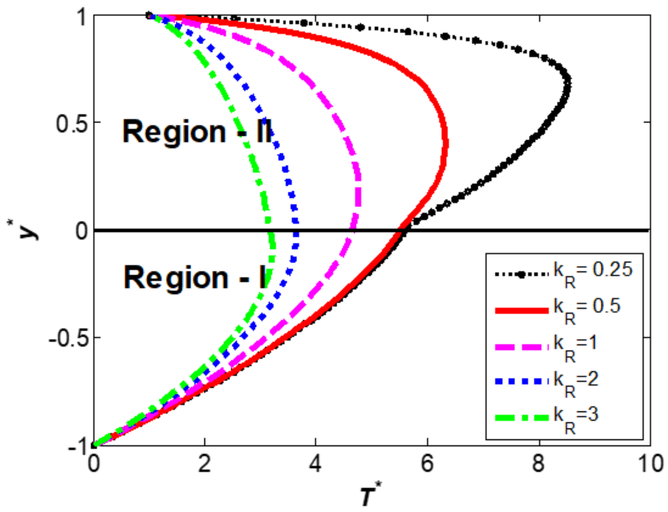


Fig. 6: Effects of k_R on T^*

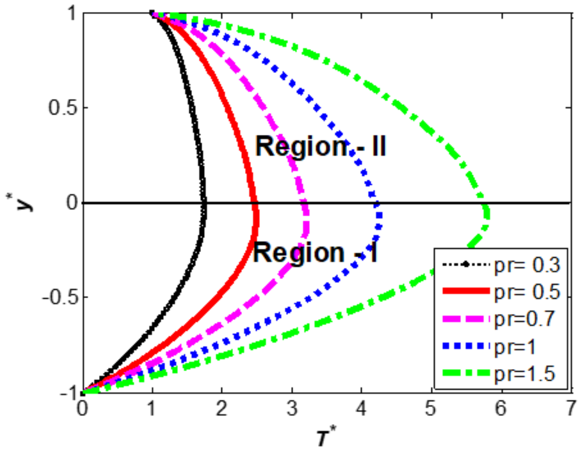


Fig. 7: Effects of Pr on T^*

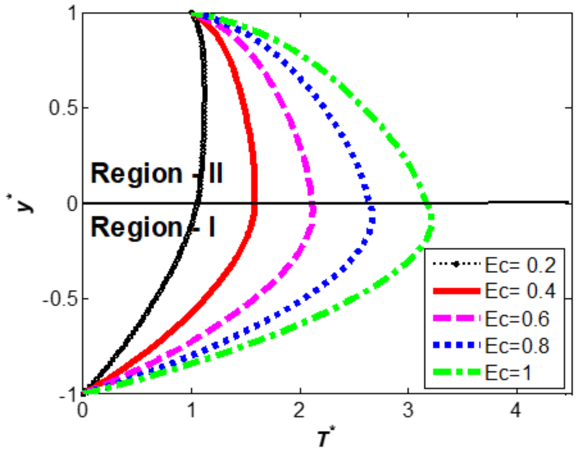


Fig. 8: Effects of Ec on T^*

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Nomenclature

B	induced magnetic field
B_0	magnetic field density
c_P	specific heat capacity
E	electric field intensity
Ec	Eckert number
Ha	Hartmann number
J	electric current density
k	thermal conductivity
m	Hall parameter
Pr	Prandtl number
S	suction parameter
t	time
t^*	dimensionless time
T	temperature
T^*	dimensionless temperature
u, w	velocity components along x, and z axes, respectively
u^*, w^*	dimensionless velocity components along x, and z axes, respectively
$\vec{v}$	Velocity vector
Greek symbols	
β	Hall factor
ρ	density
ρ^*	dimensionless density
σ	electric conductivity
μ	viscosity