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PELL POLYNOMIAL SOLUTION OF THE NONLINEAR VARIABLE ORDER SPACE FRACTIONAL PDES

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Abstract. In this paper, multi-term, space fractional variable order nonlinear partial differential equations (VONPDEs) are considered. This type of the equations covers the form of all space fractional VONPDEs containing the first-order time derivative. Here fractional derivatives are defined in the Caputo sense. The presented method is a combination of the Pell collocation method and the finite difference method. Firstly, after discretization with respect to the time variable, the finite difference method is applied to the multi-term, space fractional VONPDE in time derivative. This leads to a space-fractional variable order nonlinear ordinary differential equation. Then, the approximate solution of the space fractional variable order nonlinear ordinary differential equation is expressed in the form of truncated Pell series with unknown coefficients. Finally, the Pell collocation method transform the fractional variable order nonlinear ordinary differential equation into a system of nonlinear equations. Thus, the approximate solution of the nonlinear system is obtained by using the Newton method and unknown coefficients of the truncated Pell series are computed. The error and convergence analysis of the method is studied. In addition, the accuracy of the method is also supported by numerical examples. The numerical results also confirm the convergence and computational efficiency of the presented method. All numerical results are obtained by building fast algorithms using Matlab programming.

1. Introduction

As it is known, real-world phenomena change instantaneously according to time and space. Therefore the best way to model these events is to use partial differential equations of variable order, which makes variable order partial differential equations (VOPDE) more popular among researchers [40, 4, 18, 27, 6, 13] in the recent years. However, since the order of the partial differential equation (PDE) in the VOPDE

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depends on the time and space variables, the construction of methods to obtain the solution of the problem becomes a little more difficult than integer or fractional order problems. Moreover, when VOPDEs are nonlinear, it is often impossible to obtain analytical solutions of such equations. This situation has encouraged researchers to investigate various solution methods and it has become necessary to develop numerical methods for solving such problems.

In general, time fractional VONPDEs attract more attention from researchers [12, 10, 3, 6, 37, 13, 15, 5, 21, 36, 44, 17, 32, 30, 2, 25]. An optimization method based on the generalized shifted Chebyshev polynomials, the homotopy perturbation method and spectral tau method based on the shifted Gegenbauer polynomials for solving time fractional VONPDEs have been presented in [12, 10, 3], respectively. Time fractional variable order nonlinear (VON) advection-diffusion equation has been solved by using the two-grid method and the backward substitution method in [6, 37], respectively. An optimization method related with the generalized polynomials has been applied to the time fractional VON Klein-Gordon equation in [13]. A hybrid method based on new generalized Bernoulli-Laguerre polynomials has been introduced for solving a general class of coupled nonlinear systems of variable-order fractional partial differential equations [15]. The solution of the nonlinear two-dimensional variable-order fractional optimal control problems has been obtained by using the generalized Bessel polynomials [5]. Two-dimensional Legendre wavelets method have been applied to the time fractional VON Klein-Gordon equation in [21]. The least squares scheme has been proposed for solving time fractional VON two-dimensional diffusion-wave equation in [36]. A linearized spectral Galerkin approach has been applied to the time fractional VON diffusion-reaction equations in [44]. Time fractional VON coupled sine-Gordon equations have been solved by a method based on the Chebyshev cardinal functions in [17]. A numerical scheme based on the Haar wavelets coupled with the nonstandard finite difference scheme has been applied to the VO time-fractional generalized Burgers' equation in [32]. An adaptive finite difference scheme for VO time-fractional subdiffusion equations has been studied in [30]. Spectral collocation method based on the generalized shifted Jacobi polynomials has been developed for solving multi-term VO time-fractional diffusion-wave equations in [2]. A numerical technique based on the operational matrix of differentiation with fractional-order Lagrange polynomials has been introduced to solve a class of VO time-fractional advection-diffusion equations in [25].

Some methods have been constructed for solving space-time fractional VONPDEs in [6, 22, 16, 19, 14]. The discrete finite difference implicit method and Kansa's method have been proposed for the space-time fractional VON advection-diffusion equation in [6, 22], respectively. In [22], advection-diffusion equation has nonlinear coefficients and nonlinear source term. The collocation method based on the Chebyshev cardinal functions has been constructed for the space-time fractional VON diffusion-wave equation [16]. The space-time fractional VON KdV-Burgers-Kuramoto equation has been solved by using the method based on the discrete Legendre polynomials and the collocation scheme [19]. The VO space-time fractional telegraph equation has been solved by using the approximation method based on the optimization techniques and the transcendental Bernstein series [14].

In the recent years, the Pell polynomials have been used to obtain the solution of the fractional differential equations such as the time-fractional convection equation [28], the time-fractional Black-Scholes equation [38], nonlinear time-fractional Burgers equations [39], nonlinear fractal-fractional optimal control problems [33]. However, to our knowledge there is no work based on the Pell polynomial for solving multi-term NVOPDEs. In this paper, we consider the general form of the space fractional VONPDE of the form:

$$\frac{\partial u(x,t)}{\partial t} = F(u, D_x^{\alpha_1(x,t)}u(x,t), D_x^{\alpha_2(x,t)}u(x,t), ..., D_x^{\alpha_r(x,t)}u(x,t)), \tag{1}$$
$$0 < x < R,\ \ (R \geq 1),\ \ 0 < t \leq T,$$

with the initial condition

$$u(x,0) = f(x),\ \ 0 < x < R,\ \ (R \geq 1), \tag{2}$$

and the boundary conditions for $0 < t \leq T$

$$u(b_j,t) = d_j(t),\ \ b_0 = 0 < b_1 < ... < b_{n_r-2} < b_{n_r-1} = R,\ \ j = 0,1,...,n_r-1, \tag{3}$$

where x is a space variable, t is a time variable; the continuous function $\alpha_k(x,t), k = 1,...,r$, denotes to the order of variable-order fractional derivative in the Caputo sense with respect to the space variable. $0 < \alpha_1(x,t) < ... < \alpha_r(x,t)$, $n_i - 1 < \alpha_i(x,t) \leq n_i, n_i \in \mathbb{N}$, $i = 1,2,...,r$, $f(x)$ and $d_j(t)$, $j = 0,1,...,n_r-1$ are known continuous functions. F is known and Lipschitz continuous function with respect to u, $D_x^{\alpha_1(x,t)}u(x,t)$,...,$D_x^{\alpha_r(x,t)}u(x,t)$.

Our aim is to find the approximate solution of the problem (1)-(3) in the form of the truncated Pell polynomial series with unknown coefficients. Note that Eq. (1) is the most general form of the important physical PDE such as space fractional Kaup-Kupershmidt, Fisher's, KdV-Burgers-Kuramoto equations. Therefore, these kind of equations can be also solved by using the presented method. Note that space fractional VONPDEs have been solved by using the Laguerre collocation method and finite difference method in [43]. Here, VONPDE has only nonlinear source term. In this paper, we use the same procedure, but we consider the more general form of the VONPDEs. Furthermore, we investigate convergence and error analysis.

The paper is organized as follows: In Section 2, we introduce the fundamental concepts of the variable-order Caputo fractional derivative and Pell polynomials. In Section 3, a collocation scheme based on the finite difference method and Pell polynomial is presented. The convergence analysis of the proposed method is discussed in Section 4. Applications of the method is presented in Section 5. Finally, conclusion is given in Section 6.

2. Preliminaries and notations

2.1. The variable-order Caputo Fractional Derivative.

Definition 2.1. The variable order fractional derivative of order $\alpha(x,t)$ of the function $u(x,t)$ with respect to the variable x in the Caputo type is defined by (see, for example, [35],[7])

$$D_x^{\alpha(x,t)}u(x,t) = \frac{1}{\Gamma(n-\alpha(x,t))}\int_0^x (x-\tau)^{n-\alpha(x,t)-1}\frac{\partial^n u(\tau,t)}{\partial \tau^n}d\tau.$$

where $n-1<\alpha(x,t)<n$ and $\Gamma(.)$ is a Gamma function.
Remark 1. Based on the definition of the variable-order fractional derivative in the Caputo sense, the fractional derivative of a polynomial $x^m, m \in \mathbb{N}$, can be obtained as follows:

$$D_x^{\alpha(x,t)} x^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-\alpha(x,t)+1)} x^{m-\alpha(x,t)}, & m \geq n \\ 0, & m < n. \end{cases} \tag{4}$$

2.2. **Some properties of the Pell Polynomials.** The recurrence form of the Pell polynomials is given by the following relation [20]:

$$P_{j+1}(x) = 2xP_{j+1}(x) + P_j(x), \;\; P_0(x) = 0, \;\; P_1(x) = 1.$$

The Pell polynomials $P_j(x)$ can be defined in terms of x^k [1]

$$P_{j+1}(x) = \sum_{k=0}^{j} 2^k \eta_{j+k} \binom{\frac{j+k}{2}}{\frac{j-k}{2}} x^k, \tag{5}$$

where

$$\eta_r = \begin{cases} 1, & r \;\; even \\ 0, & r \;\; odd \end{cases}. \tag{6}$$

The analytic form of $P_j(x)$ can be written alternatively as [20]

$$P_j(x) = \sum_{m=0}^{\lfloor (j-1)/2 \rfloor} \binom{j-m-1}{m} 2^{j-2m-1} x^{j-2m-1}, \tag{7}$$

where $\lfloor j \rfloor$ is the largest integer less than or equal to j.
Remark 2. If m is a nonnegative integer, then the following inversion formula holds [1]:

$$x^m = 2^{1-m} \sum_{\substack{r=0 \\ (r+m) even}}^{m} \frac{(-1)^{(m-r)/2}(r+1)m!}{(m+r+2)(\frac{m+r}{2})!(\frac{m-r}{2})!} P_{r+1}(x). \tag{8}$$

Remark 3. The following inequality holds for Pell polynomials [1]:

$$\mid P_{k+1}(x) \mid \leq \sqrt{4R^2+2}, \;\; k \geq 0, \quad \forall x \in [0, R]. \tag{9}$$

Theorem 2.1. *If $f(x)$ is an infinitely differentiable function, then it can be expressed in the form of the Pell polynomials* [28]

$$f(x) = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^r k(2r+k-1)! u_{2r+k-1}}{2^{2r+k-1} r!(r+k)!} P_k(x) \tag{10}$$

where $u_j = f^{(j)}(0)/j!$.

Lemma 2.1. *The following equality holds for Pell polynomials for $n-1<\alpha(x,t)<n$, $x \in (0, R]$, $t \in [0, T]$*

$$D^{\alpha(x,t)} P_{j+1}(x) = x^{-\alpha(x,t)} \sum_{m=0}^{j} \Theta(j,m) P_{m+1}(x), \tag{11}$$

where

$$\Theta(j,m) = \sum_{n=m(n+m \ \ even)}^{j} b_{jn} d_{nm},$$

$$b_{jn} = \frac{2\eta_{j+n} \frac{n+j}{2}! n!}{\frac{j-n}{2}!\Gamma(n+1-\alpha(x,t))}, \ d_{nm} = \frac{(m+1)(-1)^{\frac{n-m}{2}}}{\frac{n+m}{2}!\frac{n-m}{2}!(n+m+2)}.$$

Proof. The proof can be seen from Eq.(4), Eq.(5) and Eq.(8). □

Lemma 2.2. *Assume that there exists positive constants* $M_{0,k}, ..., M_{j,k}, C_k$ *such that* $\mid \Theta(j,m) \mid \leq M_{m,k}(m = 0, 1, ..., j)$ *and* $\mid x^{-\alpha_k(x,t)} \mid \leq C_k$ *for every* $(x,t) \in (0,R] \times [0,T]$, $n_k - 1 < \alpha_k(x,t) < n_k$, $k = 1, ..., r$, *then the following inequality holds for the Pell polynomials*

$$\mid D^{\alpha_k(x,t)} P_{j+1}(x) \mid \leq \overline{M_k}(2R\sqrt{2})^{j+1}, \tag{12}$$

where $\overline{M_k} = C_k.max\{M_{0,k}, ..., M_{j,k}\}(j \geq 0)$.

Proof. From Remark 3, we can write

$$\mid P_j(x) \mid \leq (2R\sqrt{2})^{j-1}, \ j \geq 1, \ \forall x \in (0,R]. \tag{13}$$

From Lemma 2.2, we have

$$\begin{aligned} \mid D^{\alpha_k(x,t)} P_{j+1}(x) \mid &\leq C_k(M_{0,k} + M_{1,k}2R\sqrt{2} + M_2(2R\sqrt{2})^2 + ... + M_{j,k}(2R\sqrt{2})^j) \\ &\leq \overline{M_k}(1 + 2R\sqrt{2} + (2R\sqrt{2})^2 + ... + (2R\sqrt{2})^j) \\ &\leq \overline{M_k}\frac{(2R\sqrt{2})^{j+1} - 1}{2R\sqrt{2} - 1} \\ &\leq \overline{M_k}(2R\sqrt{2})^{j+1}. \end{aligned}$$

□

Let us discretize the interval $[0,T]$ such that for positive integer N, $h = \frac{T}{N}$ denotes the step size of the time variable and $t_i = ih(i = 0, 1, ..., N)$.

Theorem 2.2. *Let* $u(x,t_i)$ *be an infinitely differentiable function and* $u(x,t_i) = \sum_{k=1}^{\infty} c_k(t_i).P_k(x)$ *.If there exits positive constant* λ_i, $i \geq 0$, *such that* $|\frac{d^j u}{dx^j}(0,t_i)| \leq \lambda_i^j$ *for* $j \geq 0$, *then the series is absolutely convergent and*

$$\mid c_k(t_i) \mid \leq \frac{\lambda_i^{k-1}}{2^{k-1}(k-1)!}cosh(\lambda_i). \tag{14}$$

Proof. The proof of the inequality (14) can be seen from [28]. From (13) and (14), we can write

$$\mid \sum_{k=1}^{\infty} c_k(t_i).P_k(x) \mid \leq cosh\lambda_i \sum_{k=1}^{\infty} \frac{(\lambda_i R\sqrt{2})^{k-1}}{(k-1)!} = cosh\lambda_i e^{\lambda_i R\sqrt{2}}. \tag{15}$$

□

3. Numerical scheme

In this section, we apply the Pell collocation method to the problem (1)-(3). Firstly, applying the finite difference method to Eq.(1), Eq.(1) is reduced to a ordinary differential equation with fractional derivative. Then, the obtained fractional differential equation is solved by using the Pell collocation method.

Let the interval $[0,T]$ be partitioned by points $t_i = i.h \ (i = 0, 1, 2, ..., N)$, where $h = T/N$ is the step size. Using the forward difference formula for the derivative with respect to t in the Eq.(1), the discrete version of Eq.(1) can be written as

$$\frac{u_{i+1}(x) - u_i(x)}{h} = F(u_{i+1}(x), u_{i+1}^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^{\alpha_{r,i+1}(x)}) \tag{16}$$

or

$$u_{i+1}(x) - u_i(x) = hF(u_{i+1}(x), u_{i+1}^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^{\alpha_{r,i+1}(x)}), \tag{17}$$

where $u_i(x) = u(x, t_i)$, $u_i^{\alpha_{k,i}(x)} = D_x^{\alpha_k(x,t_i)} u(x, t_i), k = 1, 2, .., r.$

The function $u(x, t_i)$ can be expanded in terms of the Pell polynomials

$$u_{i+1}(x) = \sum_{n=1}^{\infty} c_n(t_{i+1}) P_n(x).$$

Assume that the approximate solution of the problem (1)-(3) at the point t_{i+1} can be written as follows

$$u_{i+1}^m(x) = \sum_{n=1}^{m+1} c_n(t_{i+1}) P_n(x). \tag{18}$$

Substituting Eq.(18) into Eq.(17), collocating the obtained equation at points $x_p (p = 1, ..., m - n_r + 1)$, we have the following system

$$(P.Y^{i+1} - PY^i)_p = h.F((P.Y^{i+1})_p, (B_1 Y^{i+1})_p, ..., (B_r Y^{i+1})_p), \tag{19}$$

where

$$Y^{i+1} = [c_1(t_{i+1}), c_2(t_{i+1}), ..., c_{m+1}(t_{i+1})]^*,$$

$$P = \begin{bmatrix} P_1(x_1) & P_2(x_1) & ... & P_{m+1}(x_1) \\ P_1(x_2) & P_2(x_2) & ... & P_{m+1}(x_2) \\ . & . & ... & . \\ P_1(x_{m-n_r+1}) & P_2(x_{m-n_r+1}) & ... & P_{m+1}(x_{m-n_r+1}) \end{bmatrix},$$

$$B_k = \begin{bmatrix} D^{\alpha_k(x_1,t_{i+1})} P_1(x_1) & ... & D^{\alpha_k(x_1,t_{i+1})} P_{m+1}(x_1) \\ D^{\alpha_k(x_2,t_{i+1})} P_1(x_2) & ... & D^{\alpha_k(x_2,t_{i+1})} P_{m+1}(x_2) \\ . & ... & . \\ D^{\alpha_k(x_{m-n_r+1},t_{i+1})} P_1(x_{m-n_r+1}) & ... & D^{\alpha_k(x_{m-n_r+1},t_{i+1})} P_{m+1}(x_{m-n_r+1}) \end{bmatrix},$$

$*$ denotes the sign of the transposition. $()_p$ shows the pth component of the vector. $D^{\alpha_k(x,t)} P_j(x)$ can be computed from Eq.(4) and Eq.(7) as follows

$$D^{\alpha_k(x,t)} P_j(x) = \sum_{m=0}^{\lfloor (j-1)/2 \rfloor} \frac{2^{j-2m-1}(j-m-1)!}{m!\Gamma(j-2m-\alpha(x,t))} x^{j-2m-1-\alpha_k(x,t)}, \ \ (j - 2m - 1 \geq n_k).$$

Substituting Eq.(18) into the boundary conditions (3), we have the following matrix representation

$$C.V^{i+1} = D^{i+1}, \tag{20}$$

$$\begin{aligned} D^{i+1} &= [d_0(t_{i+1}), d_1(t_{i+1}), ..., d_{n_r-1}(t_{i+1})]^*, \\ C &= \begin{bmatrix} P_1(b_0) & P_2(b_0) & ... & P_{m+1}(b_0) \\ P_1(b_1) & P_2(b_1) & ... & P_{m+1}(b_1) \\ . & . & ... & . \\ P_1(b_{n_r-1}) & P_2(b_{n_r-1}) & ... & P_{m+1}(b_{n_r-1}) \end{bmatrix}. \end{aligned}$$

Combining Eq. (19) and Eq. (20), we have a system of non-linear equations as follows

$$\mathbf{U}(Y^{i+1}, Y^i) = \mathbf{0}_{m+1}, \ \ i = 0, 1, ...N-1, \tag{21}$$

where

$$\begin{aligned} \mathbf{U}(Y^{i+1}, Y^i) &= \begin{bmatrix} P \\ C \end{bmatrix}.Y^{i+1} - \begin{bmatrix} P \\ \mathbf{0}_{n_r} \end{bmatrix}.Y^i \\ &- \begin{bmatrix} h.F((P.Y^{i+1})_1, (B_1V^{i+1})_1, ..., (B_rV^{i+1})_1) \\ \vdots \\ h.F((P.Y^{i+1})_{m-n_r+1}, (B_1V^{i+1})_{m-n_r+1}, ..., (B_rV^{i+1})_{m-n_r+1}) \\ D^{i+1} \end{bmatrix}, \end{aligned}$$

$\mathbf{0}_k$ is the zero vector with k components.

Let us consider the initial condition (2). Substituting Eq.(18) into the initial condition (3), and collocating the resulting equation at $m+1$ points we obtain the vector $Y^0 = [c_1(0), c_2(0), ..., c_{m+1}(0)]^*$.

For the non-linear system (21) with unknown Y^j ($j = 1, ..., N$), the following iteration formula can be written by using Newton iteration method

$$\begin{aligned} Y^{j,k+1} &= Y^{q,k} - J^{-1}(Y^{q,k}).\mathbf{U}(Y^{j,k}, Y^{j-1}), \\ Y^{j,0} &= Y^{j-1}, \ \ j = 1, ..., N, \ \ k = 0, 1, ... \end{aligned} \tag{22}$$

where $J^{-1}(Y^{j,k})$ is the inverse of the Jacobian matrix, $Y^{j,k}$ is approximate solution of Y^j. Using the iteration formula (22), approximate solution of the problem (1)-(3) is obtained.

4. Error Analysis

In this section, we investigate the convergence of the presented method.

From the Taylor series expansion for a continuously differentiable function $f(x)$, we have

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h). \tag{23}$$

From Eq.(23), Eq.(17) can be written as

$$u_{i+1}(x) - u_i(x) = hF(u_{i+1}(x), u_{i+1}^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^{\alpha_{r,i+1}(x)}) + O(h^2). \tag{24}$$

Note that if the exact solution of Eq.(1) is $u(x,t)$, then $u(x, t_{i+1})$ is the exact solution of Eq.(24). Let $u_{i+1}^m(x)$ be the approximate solution of Eq.(24). Then $u_{i+1}^m(x)$ satisfy the following equation

$$u_{i+1}^m(x) - u_i^m(x) = hF(u_{i+1}^m(x), (u_{i+1}^m)^{\alpha_{1,i+1}(x)}, ..., (u_{i+1}^m)^{\alpha_{r,i+1}(x)}) + O(h^2) + R_m(x),$$

where $R_m(x)$ is the residual function. Subtracting Eq.(17) from the last relation, we have the following relation

$$\begin{aligned} R_m(x) &= e_{i+1}^m(x) - e_i^m(x) - h\Big[F(u_{i+1}^m(x), u_{i+1}^m(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^m(x)^{\alpha_{r,i+1}(x)}) \\ &- F(u_{i+1}(x), u_{i+1}(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}(x)^{\alpha_{r,i+1}(x)})\Big] + O(h^2). \end{aligned}$$

where $e_i^m(x) = u_i^m(x) - u_i(x)$. Then we have

$$\begin{aligned} \mid R_m(x) \mid &\leq \mid e_i^m(x) \mid + \mid e_{i+1}^m(x) \mid + h \mid F(u_{i+1}^m(x), u_{i+1}^m(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^m(x)^{\alpha_{r,i+1}(x)}) \\ &- F(u_{i+1}(x), u_{i+1}(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}(x)^{\alpha_{r,i+1}(x)}) \mid + O(h^2). \end{aligned} \tag{25}$$

From (13) and (14), we can obtain the following inequality

$$\begin{aligned} \mid e_i^m(x) \mid &\leq \sum_{k=m+2}^{\infty} \mid c_k(t_i).P_k(x) \mid \leq cosh\lambda_i \sum_{k=m+2}^{\infty} \frac{(\lambda_i R\sqrt{2})^{k-1}}{(k-1)!} \\ &\leq cosh\lambda_i [e^{\lambda_i R\sqrt{2}} - \sum_{k=1}^{m+1} \frac{(\lambda_i R\sqrt{2})^{k-1}}{(k-1)!}] \\ &\leq cosh\lambda_i e^{\lambda_i R\sqrt{2}} \frac{\Gamma(m+1) - \Gamma(m+1, \lambda_i R\sqrt{2})}{\Gamma(m+1)} \\ &\leq cosh\lambda_i e^{\lambda_i R\sqrt{2}} \frac{(\lambda_i R\sqrt{2})^{m+1}}{(m+1)!}, \end{aligned} \tag{26}$$

where $\Gamma(.,.)$ is the incomplete gamma function [28].

Similarly for $k = 1, 2, ..., r$ using Lemma 2.3 we have

$$\begin{aligned} \mid e_i^m(x)^{\alpha_{k,i}(x)} \mid &\leq \sum_{j=m+2}^{\infty} \mid c_j(t_i).D^{\alpha_{k,i}(x)} P_j(x) \mid \\ &\leq cosh\lambda_i.2L\sqrt{2}\overline{M_k} \sum_{j=m+2}^{\infty} \frac{(\lambda_i R\sqrt{2})^{j-1}}{(j-1)!} \\ &\leq cosh\lambda_i.2R\sqrt{2}\overline{M_k} e^{\lambda_i R\sqrt{2}} \frac{(\lambda_i R\sqrt{2})^{m+1}}{(m+1)!}. \end{aligned} \tag{27}$$

Since the function F satisfies the Lipschitz condition with respect to $u, u^{\alpha_{1,t_{i+1}}}, ..., u^{\alpha_{r,t_{i+1}}}$, thus we have

$$\begin{aligned} &\mid F(u_{i+1}^m(x), u_{i+1}^m(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}^m(x)^{\alpha_{r,i+1}(x)}) \\ - &F(u_{i+1}(x), u_{i+1}(x)^{\alpha_{1,i+1}(x)}, ..., u_{i+1}(x)^{\alpha_{r,i+1}(x)}) \mid \leq L_0 \mid e_{i+1}^m(x) \mid + L_1 \mid e_{i+1}^m(x)^{\alpha_{1,i+1}(x)} \mid \\ &+ ... + L_r \mid e_{i+1}^m(x)^{\alpha_{r,i+1}(x)} \mid \\ \leq\ &cosh\lambda_{i+1}.e^{\lambda_{i+1} R\sqrt{2}} \frac{(\lambda_{i+1} R\sqrt{2})^{m+1}}{(m+1)!} \overline{C}, \end{aligned} \tag{28}$$

where $L_0, ..., L_r$ are positive constants and $\overline{C} = L_0 + L_1 R 2\sqrt{2M_1} + L_r L 2\sqrt{2M_r}$.

Substituting inequalities (26), (27) and (28) into (25), we have

$$\begin{aligned} &\mid R_m(x) \mid \leq \mid cosh\lambda_{i+1}.e^{\lambda_{i+1} R\sqrt{2}} \frac{(\lambda_{i+1} R\sqrt{2})^{m+1}}{(m+1)!} (1 + \overline{C}) \\ + &cosh\lambda_i.e^{\lambda_i R\sqrt{2}} \frac{(\lambda_i R\sqrt{2})^{m+1}}{(m+1)!} + O(h^2). \end{aligned}$$

5. Applications

Example 1. Consider the following space fractional VONPDE with initial and boundary conditions

$$\begin{aligned} u_t(x,t) &= x^{2\alpha(x,t)}(D_x^{\alpha(x,t)}u(x,t))^2u(x,t) - x^{3\theta(x,t)}(D_x^{\theta(x,t)}u(x,t))^3(u(x,t))^2 \\ &- xtu^2(x,t) + t^5x^3e^{2x}(E_{1,2-\theta})^2 - t^3x^4e^x(E_{1,3-\alpha})^2 + e^x + xt^3e^{2x} \\ &, \quad 0 < x < 1, \ 0 < t \leq 1, \\ u(x,0) &= 0, \ 0 < x < 1, \ u(0,t) = t, \ u(1,t) = et, \ 0 < t \leq 1, \end{aligned}$$

where $(E_{1,3-\alpha})$ and $(E_{1,2-\theta})$ are the Mittag-Leffler functions (see, for example, [31]), $\alpha(x,t) = \frac{5+\sin(x+t)}{4}$, $\beta(x,t) = \frac{1+\sin(x+t)}{4}$. The exact solution is $u_e(x,t) = e^x t$.

By using the iteration formula (22), the absolute errors for the approximate solution $u_m(x,t), m = 4,6,7,9,11$, are computed. Table 1 shows the absolute errors for $m = 4,6,7,9,11; N = 10^4$ at $t = 0.2; 0.8$. From Table 1, it is observed that as the values m increase, the approximate solution approaches to the exact solution.

Example 2.Let us consider

$$\begin{aligned} & u_t(x,t) = -x^{2\alpha(x,t)}(\Gamma(6-\alpha(x,t)))^2(D_x^{\alpha(x,t)}u(x,t))^2 + (D_x^{\theta(x,t)}u(x,t))^3u(x,t) \\ + \ & u(x,t) + (x^5 - x^2 + 1)(3t - t^2 - 1) - (x^5 - x^2 + 1)(t^2 - 4)^4\Big(\frac{120x^{5-\theta(x,t)}}{\Gamma(6-\theta(x,t))} \\ - \ & \frac{2x^{2-\theta(x,t)}}{\Gamma(3-\theta(x,t))}\Big) + 120^2(t^2 - t)^2x^{10}, \ 0 < x < 2, \ 0 < t \leq 1, \\ & u(x,0) = 0, \ 0 < x < 2, \\ & u(0,t) = t^2 - t, \ u(1,t) = t^2 - t, \ u(2,t) = 29(t^2 - t), \ 0 < t \leq 1, \end{aligned}$$

where $\alpha(x,t) = \frac{8+x+t}{4}$, $\theta(x,t) = \frac{4+x+t}{4}$. The exact solution is $u_e(x,t) = (t^2 - t)(x^5 - x^2 + 1)$.

Table 2 shows the absolute errors for the approximate solution $u_5(x, 0.5)$ for $N = 10^4; 10^5; 10^6; 10^7$. It can be concluded from Table 2 that the values of the absolute errors go to zero as the value N increases.

Example 3. Consider

$$\begin{aligned} u_t(x,t) &= \Gamma(5-\alpha(x,t))x^{\alpha(x,t)}D_x^{\alpha(x,t)}u(x,t) - (\Gamma(5-\theta(x,t)))^2x^{2\theta(x,t)} \\ &. \ (D_x^{\theta(x,t)}u(x,t))^2u(x,t) + \cos(t)(x^4+1)24\sin(t)x^4 + 576\sin(t)x^8 \\ &. \ (x^4+1), \ 0 < x < 3, \ 0 < t < 1, \\ u(x,0) &= 0, \ 0 < x < 3, \ u(0,t) = \sin(t), \ u(3,t) = 82\sin(t), \ 0 < t \leq 1, \end{aligned}$$

where $\alpha(x,t) = \frac{5+x+t}{5}$, $\theta(x,t) = \frac{1+x+t}{5}$. Note that the exact solution to this problem is $u_e(x,t) = (x^4+1)\sin(t)$.

Table 3 displays the absolute errors for the approximate solution $u_4(x, 0.5)$ for $N = 10^4; 10^5; 10^6; 10^7$. From the table it is seen that better results are obtained when the value of N increases.

Example 4. Let us consider the following problem

$$
\begin{aligned}
& u_t(x,t) = e^{3t}x^{\alpha(x,t)+\theta(x,t)}\Gamma(4-\alpha(x,t))\Gamma(4-\theta(x,t))\Gamma(2-\theta(x,t)) \\
. \quad & D_x^{\alpha(x,t)}u(x,t)D_x^{\theta(x,t)}u(x,t).u(x,t) - e^{5t}(\Gamma(4-\beta(x,t)))^2x^{2\beta}(D_x^{\beta(x,t)}u(x,t))^2 \\
. \quad & u^3(x,t) - u(x,t) + 36(x^3-x)^3x^6 - 6(x^3-x)x^3(6x^3\Gamma(2-\theta(x,t) \\
- \quad & x\Gamma(4-\theta(x,t)),\ 0<x<1,\ 0<t\le 1, \\
& u(x,0) = x^3-x,\ 0<x<1,\ u(0,t)=0,\ u(1,t)=0,\ 0<t\le 1,
\end{aligned}
$$

where $\alpha(x,t) = \frac{5+x+t}{4}$, $\beta(x,t) = \frac{6+\sin(x+t)}{4}$, $\theta(x,t) = \frac{1+x+t}{4}$. The exact solution is $u_e(x,t) = (x^3-x)e^{-t}$.

Taking $m = 3$ and $Y^0 = (0,-3/4,0,1/8)^*$, the absolute errors for $u_3(x,t)$ is computed. In Table 4, the absolute errors for $u_3(x,t)$ for $N = 10^4; 10^5; 10^6; 10^7$ and $t = 0.5; 0.8$ are presented. Table 4 indicates the influence of the value N on the absolute error. Fig.1 shows 3D plot of the approximate solution $u_3(x,t)$ for $N = 10^4$.

Example 5. Consider the following problem

$$
\begin{aligned}
& u_t(x,t) = x^{\alpha(x,t)}D_x^{\alpha(x,t)}u(x,t).u(x,t) + x^{2\theta}(D_x^{\theta(x,t)}u(x,t))^2.u(x,t) \\
+ \quad & \cos(t)\cos(x) + x^2E_{2,3-\alpha(x,t)}(-x^2).\cos(x)\sin^2(t) - x^4(E_{2,3-\theta(x,t)}(-x^2))^2 \\
. \quad & \cos(x)\sin^3(t),\ 0<x<\frac{\pi}{2},\ 0<t\le 1, \\
& u(x,0)=0,\ 0<x<\frac{\pi}{2},\ u(0,t)=\sin(t),\ u(\frac{\pi}{2},t)=0,\ 0<t\le 1,
\end{aligned}
$$

where $\alpha(x,t) = \frac{5+x+t}{4}$, $\theta(x,t) = \frac{1+x+t}{4}$. The exact solution is $u_e(x,t) = \cos(x)\sin(t)$.

Fig.2 shows 3D plot of the approximate solution $u_5(x,t)$ for $N = 10^4$. Fig.3 indicates behavior of the absolute error of the $u_5(x,t)$ for $N = 10^4$.

Example 6. Let us consider nonlinear space fractional Fisher's equation with initial and boundary conditions

$$
\begin{aligned}
& u_t(x,t) = D_x^{1.5}u(x,t) + u(x,t) - u(x,t)^2 + x^2,\ 0.0125<x<1.0125, \\
& 0<t\le 1,\ u(x,0)=x,\ 0.0125<x<1.0125, \\
& u(0.0125,t) = 0.0125(1+t) + 0.00609375t^2 - 0.082176t^3 - 0.0210541^4 \\
- \quad & 7.16634.10^{-6}t^5,\ u(1.0125,t) = 1.0125(1+t) - 0.518906t^2 \\
- \quad & 0.921366t^3 + 0.310529t^4 + 0.0845434t^5,\ 0<t\le 1.
\end{aligned}
$$

In Table 5, the approximate solution $u_9(x,t)$ is computed for $N = 10^6$ at $t = 0.1$. Table 5 illustrates the comparison between the present method and GDTM (generalized differential transform method) [29], VIM (variational iteration method)[29], RBFM (radial basis functions method)[41], QPSM (quadratic polynomial spline-based method)[9], CSCM (cubic spline Collocation Method)[11]. We conclude that the Pell collocation method is closer to the generalized differential transformation method than other methods.

6. Conclusion

In this paper, a collocation method based on the Pell polynomials with together finite difference method is applied to the general form of the space fractional VON-PDEs with initial and boundary conditions. Furthermore, convergence analysis of the method is given. In section 5, numerical results shows that the method has high accuracy and can be apply to all space fractional VONPDEs containing first order time derivative without any effort. In Example 1, as the terms of the truncated series are increased, the obtained solution excellently approximates to the exact solution. In Examples 2-5, the numerical results show that the algorithm converges to the exact solution as the number of N is increased. In Example 6, the space fractional VON Fisher'equation is solved, the analytical solution of which is unknown. The obtained numerical solutions are compared with the results of other methods in the literature and it has been seen that the results are close to each other.In section 5, all numerical results are obtained by building fast algorithms using Matlab programming.

As it is known, there are many separately written articles in the literature investigating the solution of equations that are important in the physics and engineering [4, 19, 23, 34, 42, 8, 24, 26]. One of the advantages of the presented method is also that since it can be applied to all space fractional VONPDEs containing first-order time derivative, the physical equations such as space fractional KdV, KdV-Burgers-Kuramoto, Burgers-Huxley, Benjamin-Ono, Cahn-Hilliard, Chafee Infante, Harry Dym, Rosenau-Hyman can be solved using the presented method.

Competing interests

The author has no financial or proprietary interests in any material discussed in this article.

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TABLE 1. The absolute errors for $N = 10^4$ at $t = 0.2; 0.8$ in Example 1.

x	$m=4$		$m=6$		$m=7$		$m=9$		$m=11$	
	$t=0.2$	$t=0.7$	$t=0.2$	$t=0.7$	$t=0.2$	$t=0.7$	$t=0.2$	$t=0.7$	$t=0.2$	$t=0.7$
0.1	3.10^{-6}	10^{-5}	6.10^{-9}	2.10^{-8}	3.10^{-10}	9.10^{-10}	7.10^{-14}	3.10^{-13}	8.10^{-15}	10^{-14}
0.2	4.10^{-6}	2.10^{-5}	8.10^{-9}	2.10^{-8}	7.10^{-11}	2.10^{-10}	2.10^{-13}	5.10^{-13}	7.10^{-15}	10^{-14}
0.3	10^{-4}	4.10^{-5}	8.10^{-9}	2.10^{-8}	4.10^{-10}	6.10^{-10}	2.10^{-13}	10^{-12}	3.10^{-15}	2.10^{-14}
0.4	7.10^{-6}	3.10^{-5}	10^{-8}	5.10^{-9}	8.10^{-11}	2.10^{-9}	10^{-14}	6.10^{-13}	2.10^{-15}	2.10^{-14}
0.5	2.10^{-8}	5.10^{-6}	7.10^{-10}	7.10^{-8}	4.10^{-10}	3.10^{-9}	3.10^{-13}	4.10^{-13}	2.10^{-15}	5.10^{-14}
0.6	7.10^{-6}	10^{-5}	10^{-8}	10^{-7}	6.10^{-11}	10^{-9}	9.10^{-14}	3.10^{-12}	2.10^{-15}	6.10^{-14}
0.7	8.10^{-6}	2.10^{-5}	7.10^{-9}	7.10^{-8}	3.10^{-10}	2.10^{-11}	3.10^{-13}	2.10^{-12}	10^{-14}	5.10^{-14}
0.8	2.10^{-6}	9.10^{-6}	7.10^{-9}	5.10^{-9}	10^{-10}	2.10^{-9}	5.10^{-13}	3.10^{-13}	3.10^{-14}	4.10^{-14}
0.9	6.10^{-6}	4.10^{-5}	10^{-9}	2.10^{-8}	3.10^{-10}	2.10^{-9}	10^{-12}	2.10^{-12}	2.10^{-15}	3.10^{-14}

TABLE 2. The absolute errors for $u_5(x, 0.5)$ in Example 2.

x	$N=10^4$	$N=10^5$	$N=10^6$	$N=10^7$
0.25	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0.5	2.10^{-6}	$2.^{-7}$	2.10^{-8}	2.10^{-9}
0.75	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1.25	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1.5	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1.75	2.10^{-7}	2.10^{-8}	2.10^{-9}	2.10^{-10}

TABLE 3. The absolute errors for $u_4(x, 0.5)$ in Example 3.

x	$N = 10^4$	$N = 10^5$	$N = 10^6$	$N = 10^7$
0.25	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0.5	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0.75	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1	7.10^{-7}	$7.^{-8}$	7.10^{-9}	7.10^{-10}
1.25	3.10^{-7}	$3.7.^{-8}$	3.10^{-9}	3.10^{-10}
1.5	2.10^{-8}	2.10^{-9}	2.10^{-10}	2.10^{-11}
1.75	10^{-7}	10^{-8}	10^{-9}	10^{-10}
2	4.10^{-8}	4.10^{-9}	4.10^{-10}	4.10^{-11}
2.25	10^{-7}	10^{-8}	10^{-9}	10^{-10}
2.5	3.10^{-7}	3.10^{-8}	3.10^{-9}	3.10^{-10}
2.75	3.10^{-7}	3.10^{-8}	3.10^{-9}	3.10^{-10}

TABLE 4. The absolute errors for $m = 3$ at $t = 0.5; 0.8$ in Example 4.

x	$N = 10^3$		$N = 10^4$		$N = 10^5$		$N = 10^6$		$N = 10^7$	
	$t = 0.5$	$t = 0.8$	$t = 0.5$	$t = 0.8$	$t = 0.5$	$t = 0.8$	$t = 0.5$	$t = 0.8$	$t = 0.5$	$t = 0.8$
0.1	10^{-5}	2.10^{-5}	2.10^{-6}	2.10^{-6}	2.10^{-7}	2.10^{-7}	2.10^{-8}	2.10^{-8}	2.10^{-9}	2.10^{-9}
0.2	3.10^{-5}	3.10^{-5}	3.10^{-6}	3.10^{-6}	3.10^{-7}	3.10^{-7}	3.10^{-8}	3.10^{-8}	3.10^{-9}	3.10^{-9}
0.3	3.10^{-5}	3.10^{-5}	3.10^{-6}	3.10^{-6}	3.10^{-7}	3.10^{-7}	3.10^{-8}	3.10^{-8}	3.10^{-9}	3.10^{-9}
0.4	3.10^{-5}	4.10^{-5}	3.10^{-6}	4.10^{-6}	3.10^{-7}	4.10^{-7}	3.10^{-8}	4.10^{-8}	3.10^{-9}	4.10^{-9}
0.5	3.10^{-5}	4.10^{-5}	3.10^{-6}	4.10^{-6}	3.10^{-7}	4.10^{-7}	3.10^{-8}	4.10^{-8}	3.10^{-9}	4.10^{-9}
0.6	3.10^{-5}	3.10^{-5}	3.10^{-6}	3.10^{-6}	3.10^{-7}	3.10^{-7}	3.10^{-8}	3.10^{-8}	3.10^{-9}	3.10^{-9}
0.7	2.10^{-5}	3.10^{-5}	2.10^{-6}	3.10^{-6}	2.10^{-7}	3.10^{-7}	2.10^{-8}	3.10^{-8}	2.10^{-9}	3.10^{-9}
0.8	2.10^{-5}	2.10^{-5}	2.10^{-6}	2.10^{-6}	2.10^{-7}	2.10^{-7}	2.10^{-8}	2.10^{-8}	2.10^{-9}	2.10^{-9}
0.9	8.10^{-6}	10^{-5}	8.10^{-7}	10^{-6}	8.10^{-8}	10^{-7}	8.10^{-9}	10^{-8}	8.10^{-10}	10^{-9}

TABLE 5. The approximate solution for $u_9(x, 0.1)$ in Example 6.

x	Presented Method	VIM	GDTM	RBFM	QPSM	CSCM
0.2	0.220285	0.220589	0.220348	0.219761	0.210917	0.220600
0.4	0.439894	0.440329	0.439957	0.438920	0.425837	0.440452
0.6	0.658735	0.659214	0.658707	0.657719	0.645538	0.659483
0.8	0.876818	0.877185	0.876585	0.875592	0.869671	0.877676
1	1.094120	1.094096	1.093587	1.092409	1.093920	1.094273

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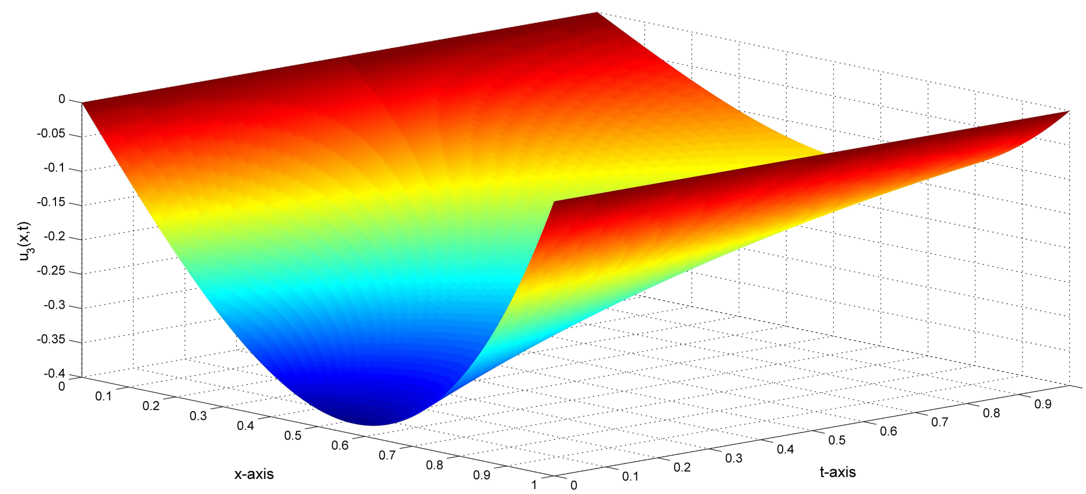

FIGURE 1. The behavior of the approximate solution $u_3(x,t)$ for $N = 10^4$, $0 < x < 1$ and $0 < t < 1$ in Example 4.

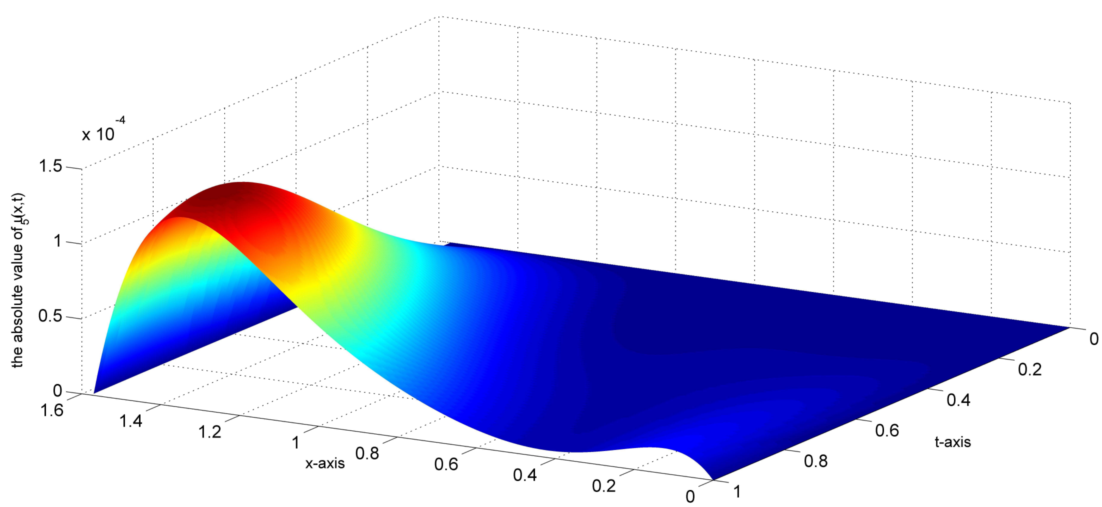

FIGURE 2. The behavior of the absolute error for $u_5(x,t)$ for $N = 10^4$, $0 < x < \frac{\pi}{2}$ and $0 < t < 1$ in Example 5.

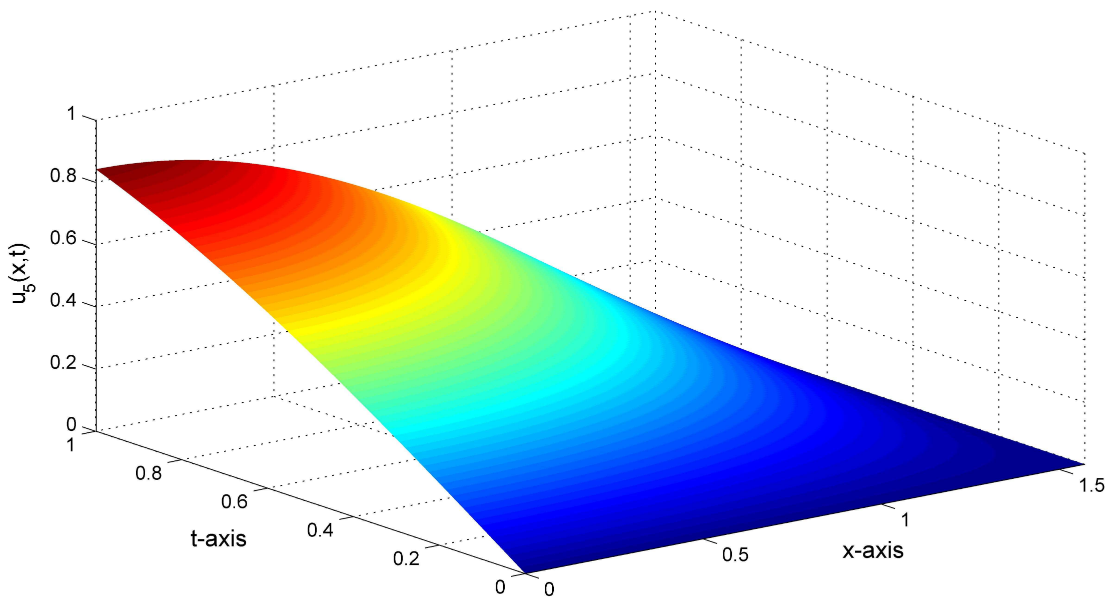

FIGURE 3. The approximate solution $u_5(x,t)$ for $N = 10^4$, $0 < x < \frac{\pi}{2}$ and $0 < t < 1$ in Example 5.