A New Generalization of Power Rayleigh Distribution: Properties and Estimation Based on Type II Censoring

By

Hanaa R. Hafez

Associate Lecturer

Department of Statistics, Faculty of Commerce, AL-Azhar University Girls' Branch – Cairo

hanaareda@yahoo.com

Dr. Nahed M. Helmy

Professor of Statistics

Department of Statistics, Faculty of Commerce, AL-Azhar University Girls' Branch – Cairo

Nahedhelmy755@gmail.com

Dr. Rabab E. Abd El-Kader

Lecturer of Statistics

Department of Statistics, Faculty of Commerce, AL-Azhar University Girls' Branch – Cairo

rbb_ibrahim@yahoo.com

Dr. Neama T. AL-Sayed

Lecturer of Statistics

Department of Statistics, Faculty of Commerce, AL-Azhar University Girls' Branch – Cairo

neamaalsayed2452.el@azhar.edu.eg

Scientific Journal for Financial and Commercial Studies and Research (SJFCSR)

Faculty of Commerce – Damietta University

Vol.6, No.1, Part 1., January 2025

APA Citation:

Hafez, R. H.; **Helmy**, N. M.; **Abd El-Kader**, R. E. and **AL-Sayed**, N. T. (2025). A New Generalization of Power Rayleigh Distribution: Properties and Estimation Based on Type II Censoring, ***Scientific Journal for Financial and Commercial Studies and Research***, Faculty of Commerce, Damietta University, 6(1)1, 625-654.

Website: https://cfdj.journals.ekb.eg/

New Generalization of Power Rayleigh Distribution: Properties and Estimation Based on Type II Censoring

Hanaa R. Hafez; Dr. Nahed M. Helmy; Dr. Rabab E. Abd El-Kader and Dr. Neama T. AL-Sayed

Abstract

The aim of this study is to generalize the power Rayleigh distribution using the quadratic rank transmutation map to obtain the transmuted power Rayleigh distribution. Some important statistical and reliability properties of the transmuted power Rayleigh distribution such as hazard rate, moments, quantiles, order statistics and some sub-model distributions are obtained. The maximum likelihood method based on Type II censoring is applied to estimate the unknown parameters, reliability and hazard rate functions of the transmuted power Rayleigh distribution. In addition, a simulation study is performed to illustrate the theoretical results and the precision of the maximum likelihood estimates. Finally, two applications are applied to demonstrate the importance and flexibility of the new model.

Keywords: *Transmuted power Rayleigh distribution; Type II censored samples; maximum likelihood estimation method.*

1. Introduction

In the probability theory, various univariate standard distributions have been widely used for analyzing data which were collected from many areas of science such as engineering, bio-medicine, actuarial science, finance and economics. During various investigations it was observed that data obtained from different applied subjects such as finance, environmental science and other related areas, does not follow these distributions. Clearly, there is a need to enhance the utility of these distributions by making their extensions and generalizations so that they become suitable for such data. In this direction statisticians have made a lot of work and introduced various methods and techniques to construct new models from baseline distributions. The modified models proved more flexible than the standard distributions. One of these methods is the *quadratic rank transmutation map* (QRTM) which was proposed by Shaw and Buckley (2007) to generate a new distribution.

The transmuted family of distributions has been receiving a high attention over the past few years. A technique for adding a new parameter to an already existing distribution would provide more flexibility for this distribution.

The method of QRTM includes the parent distribution as a special case and makes it more flexible to model different types of data. The QRTM offers an avenue to many theoretical statisticians to improve on statistical methodologies for explaining life testing problems with increased accuracy.

Many authors used the transmutation procedure to achieve new generalized distributions such as: Aryal and Tsokos (2009) introduced the transmuted Gumbel distribution. They derived some of its mathematical properties and estimated its parameters by the *maximum likelihood* (ML) method. Also, Aryal and Tsokos (2011) presented the transmuted Weibull distribution as a generalized form of the standard Weibull distribution. In addition, they derived various statistical properties and estimated its parameters by the ML method.

Faton (2013) generalized the *Rayleigh distribution* (RD) using the QRTM to obtain a *transmuted Rayleigh distribution* (TRD). He derived some mathematical properties of this distribution along with its reliability behavior such as mean, variance, moments and moment generating function. However, he estimated the parameters by the method of ML. Also, he derived the information matrix. He used the likelihood ratio statistic to compare the model with its baseline model. Finally, he showed through an application of the TRD to real data that the proposed distribution can be used quite effectively to provide better fits than the RD.

Elbatal and Elgarhy (2013) used the transmuted Lindley distribution to obtain the transmuted Quasi Lindley distribution. They derived its quantile function, moments, moment generating function and order statistics. Also, they estimated its parameters by the ML, least squares and weighted least squares methods. In addition, Tian *et al.* (2014) introduced the transmuted linear exponential distribution. In addition, they derived its moments, mode, quantiles and estimated its parameters by the method of ML. Moreover, Khan and King (2015) proposed the transmuted modified inverse RD. Also, they derived various statistical properties and estimated its parameters by the method of ML.

Saboor *et al.* (2016) suggested the transmuted exponentiated Weibull geometric distribution. Also, they derived its moments, moment generating function and order statistic. Moreover, they estimated its parameters by ML method. Further, Yousof *et al.* (2017) presented a family of continuous distributions called the transmuted Topp-Leone G family. They studied some mathematical properties including probability weighted moments, moments, moment generating function, order statistics, incomplete moments, stress-strength model, moment of residual and reversed residual life. They used ML method to estimate its parameters. They used The Monte Carlo simulation for assessing the performance of the ML estimates.

Umar *et al.* (2019) introduced transmuted Lindley exponential distribution. They derived some mathematical properties including survival function, hazard function and order statistic, they also estimated its parameters using the method of ML and they used real dataset which the results give evidence that the transmuted Lindley exponential distribution is better than the Lindley exponential distribution. Also, Toumaj *et al.* (2021) presented three parameter distribution called the transmuted inverted Nadarajah-Haghighi distribution. They discussed several mathematical properties of the proposed distribution such as the quantile function, skewness, kurtosis, moments, *reversed hazard rate function* (rhrf), and order statistics. They estimated the parameters using the method of ML and presented an application to assessing the performance of the ML estimates.

Christophe *et al.* (2022) introduced the two parameter transmuted continuous Bernoulli distribution for fitting proportional data sets. They derived some mathematical properties of the proposed distribution and used the ML estimation technique to estimate the unknown parameters. They applied a Monte Carlo simulation to examine and confirm the asymptotic behavior of the obtained estimates. In order to show the applicability of the proposed distribution, they analyzed three proportional data sets and compared the results obtained with competitive distributions. Empirical findings reveal that the transmuted continuous Bernoulli distribution promises more flexibility in fitting proportional data sets than its competitors.

Joseph and Ravindran (2023) proposed a generalization of the exponentiated Kumaraswamy distribution called transmuted exponentiated Kumaraswamy distribution. They used the QRTM to obtain the proposed distribution. They provided some mathematical properties of the proposed distribution including the moments, incomplete moments, moment generating function, quantile function, entropy, mean deviation and order statistics. They performed survival analysis and estimated the distribution parameters using the ML method. Finally, they performed a simulation study to investigate the performance of the estimates.

Adetunji (2023) introduced the extension of the Ailamujia distribution using QRTM to propose a two parameter lifetime distribution. He explored shapes of the *probability density function* (pdf) and *reliability function* (rf) and presented its various mathematical properties. Comparison of the proposed distribution with other related lifetime distributions revealed that it gives better fit in most cases. In addition, Abdulhameed *et al*. (2024) proposed transmuted cosine Topp-Leone G family. They derived some of its statistical properties such as survival and hazard functions, moments, moment generating function. Moreover, they estimated the model parameters using the ML method and they performed a Monte Carlo simulation to assessing the behavior and the consistency of the estimates. Finally, they demonstrated the applicability of the family on the two lifetime datasets.

This paper aims to introduce a new generalization of the *power Rayleigh distribution* (PRD) called the *transmuted power Rayleigh distribution* (TPRD) using the QRTM.

The RD is one of the most commonly used distributions, which was first introduced by Rayleigh (1880). It plays a key role in modeling and analyzing lifetime data such as project effort loading modeling, survival statistics and clinical research. Several researchers have developed extensive extensions to the RD to give greater flexibility to this distribution, since it is used in many fields such as; reliability analysis, theory of communication, physical sciences and technology.

Let the random variable Y follows the RD with scale parameter α, then its pdf and *cumulative distribution function* (cdf) are, respectively, given by

$$f(y;\alpha) = \frac{y}{\alpha^2} e^{-\left(\frac{y^2}{2\alpha^2}\right)}, \qquad y > 0, \alpha > 0, \tag{1}$$

and

$$F(y,\alpha) = 1 - e^{-\left(\frac{y^2}{2\alpha^2}\right)}, \qquad y > 0, \alpha > 0. \tag{2}$$

The PRD formulated by Bhat and Ahmad (2020) to enhance the flexibility of RD by formulating an extended version of the model based on power transformation technique.

Let Y is a random variable follow the RD with parameter α, then the transformed variable $X = Y^{\frac{1}{\beta}}$ follow the PRD with parameters α and β.

A random variable X with parameters α (scale parameter) and β (shape parameter) is said to follow the PRD if its pdf and cdf are, respectively, given by:

$$g(x;\alpha,\beta) = \frac{\beta}{\alpha^2}\, x^{2\beta-1}\, e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}, \qquad x > 0, \alpha, \beta > 0, \tag{3}$$

and

$$G(x;\alpha,\beta) = 1 - e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}, \qquad x > 0, \alpha, \beta > 0. \tag{4}$$

Kilany *et al*. (2023) proposed the PRD for specifying the confirmed total deaths of Corona virus (Covid-19) in Egypt. They studied statistical and reliability properties of the PRD such as survival function, failure rate function, mean residual life, order statistic. Also, they used the ML method to estimate the unknown parameters and applied a simulation study. Finally, they used two sets of real life data and showed that the proposed model provided a best fit than other well-known distributions.

This paper is organized as follows: The construction of the proposed model and the graphical description of the pdf and *hazard rate function* (hrf) are introduced in Section 2. In Section 3, several statistical properties of the TPRD are derived and some sub-model distributions are obtained. The ML estimators of the parameters, rf and hrf are obtained in Section 4. In addition, the *asymptotic confidence intervals* (ACIs) of the parameters, rf and the hrf of the TPRD based on Type II censored samples are developed. A simulation study is presented to evaluate the performance of the ML estimates in Section 5. Moreover, two applications on COVID-19 data in some countries are performed to demonstrate the superiority of the proposed distribution over some known distributions. Finally, general conclusions are introduced in Section 6.

2. Transmuted Power Rayleigh Distribution

Let $g(x)$ and $G(x)$ be the pdf and cdf of the baseline distribution, respectively, of a random variable X, then the cdf and pdf of the transmuted family of distributions are, respectively, given by:

$$F(x) = (1 + \lambda)\, G(x) - \lambda\, [G(x)]^2, \qquad -1 \le \lambda \le 1,\ x \in \mathbb{R}, \tag{5}$$

and

$$f(x) = (1 + \lambda)\, g(x) - 2\lambda\, g(x)G(x), \qquad x \in \mathbb{R}. \tag{6}$$

The QRTM is used to generalize the PRD. Equations (4) and (5) are used to obtain the cdf of the TPRD as follows:

$$F(x;\alpha,\beta,\lambda) = \left[1 - e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]\left[1 + \lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right],$$

$$x > 0, \alpha, \beta > 0, -1 \le \lambda \le 1, \tag{7}$$

hence, the pdf of the TPRD with parameters α, β and λ is

$$f(x;\alpha,\beta,\lambda) = \frac{\beta}{\alpha^2}\, x^{2\beta-1}\, e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)} \left[1 - \lambda + 2\lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right],$$

$$x > 0, \alpha, \beta > 0, -1 \le \lambda \le 1, \tag{8}$$

where α is a scale parameter, β is a shape parameter and λ is a transmuting parameter of the distribution.

When $\lambda = 0$, the TPRD reduces to the baseline distribution of the random variable X.

The rf of the TPRD, based on (7), is given by

$$r(x;\alpha,\beta,\lambda) = e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\left[1 - \lambda + \lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right],$$

$$x > 0, \alpha, \beta > 0, -1 \le \lambda \le 1. \tag{9}$$

The hrf of the TPRD is given by

$$h(x;\alpha,\beta,\lambda) = \frac{\frac{\beta}{\alpha^2}\, x^{2\beta-1}\left[1-\lambda+2\lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]}{\left[1-\lambda+\lambda\, e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]},$$

$$x > 0, \alpha, \beta > 0, -1 \le \lambda \le 1. \tag{10}$$

2.1 Graphical description

The plots of the pdf and hrf of the TPRD are provided to show the flexibility of pdf and hrf of the TPRD, which allow this distribution to fit different types of lifetime data.

Figure 1 displays the plots of the pdf of the TPRD for selected values of the parameters.

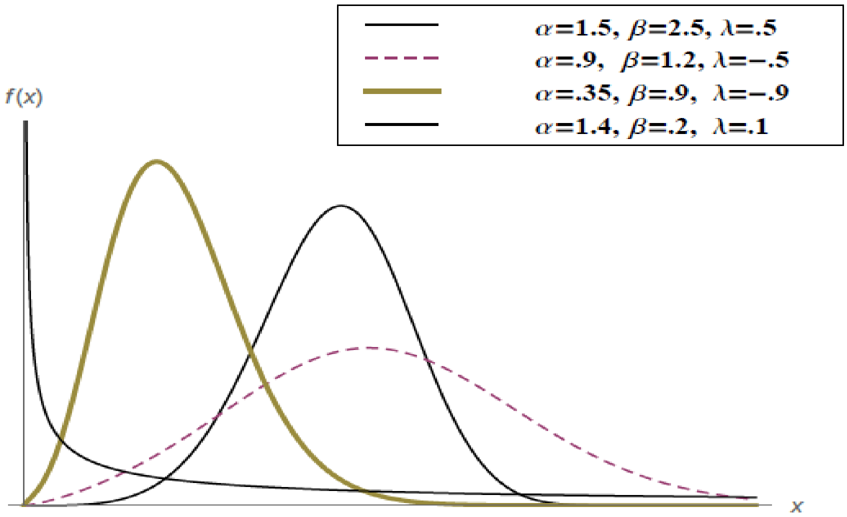


Figure 1: Plots of the probability density function of the TPRD at different values of the parameters

From Figure 1, one can observe that the pdf curves of the TPRD can take various shapes including bell shaped, approximately symmetric, monotone decreasing and then approximately constant, so the TPRD is more flexible for changing the values of the parameters.

The plots of hrf of the TPRD are provided in Figure 2 for different values of the parameters.

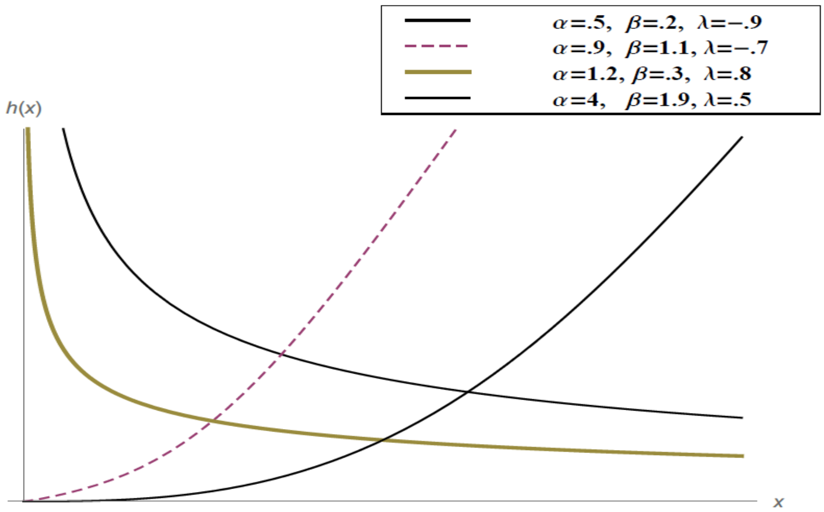


Figure 2: Plots of the hazard rate function of the TPRD at different values of parameters

From Figure 2, one can see that the hrf curves of the TPRD can take various shapes including decreasing, increasing, monotone decreasing and then approximately constant, so the TPRD is more flexible for changing the values of the parameters.

3. Some Statistical Properties

In this section, some important statistical properties such as rhrf, *cumulative hazard rate function* (chrf), the non-central and central moments, mean, variance, skewness, kurtosis, quantile function, moment generating function, median, order statistics and some sub-models of the proposed distribution are obtained.

- **Reversed hazard rate function**

The rhrf which is known by the dual of the hrf; describes the probability of an immediate past failure, given that the unit has already failed at time x, as opposed to the immediate future failure. The rhrf is given by

$$rh(x;\alpha,\beta,\lambda)=\frac{\frac{\beta}{\alpha^2}\,x^{2\beta-1}\,e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]}{\left[1-e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]\left[1+\lambda e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]},$$

$$x>0,\alpha,\beta>0,-1\le\lambda\le 1. \tag{11}$$

- **Cumulative hazard rate function**

The chrf of the TPRD is defined as:

$$\begin{aligned}H(x;\alpha,\beta,\lambda)&=\int_0^x h(t)\,dt\\&=-\,ln\left\{e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\left[1-\lambda+\lambda\,e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]\right\}\\&=\frac{x^{2\beta}}{2\alpha^2}-ln\left[1-\lambda+\lambda\,e^{-\left(\frac{x^{2\beta}}{2\alpha^2}\right)}\right]=-\,ln[r(x)].\end{aligned} \tag{12}$$

The chrf must satisfy the following three conditions:

i. $H(x)$ is a non-decreasing function for all $x\ge 0$.

ii. $H(0)=0$.

iii. $\lim_{x\to\infty} H(x)=\infty$.

- **Non-central and central moments**

The r^{th} non-central moment about the origin of the TPRD (α,β,λ) is given by

$$\acute{\mu}_r = E(x^r) = \int_0^\infty x^r f(x;\alpha,\beta,\lambda)\,dx$$
$$= \alpha^{\frac{r}{\beta}}\ \Gamma\left(\frac{r}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{r}{2\beta}}+\lambda\right],$$
$$\alpha,\beta > 0,\ -1 \le \lambda \le 1,\ r = 1,2,3,\ldots \qquad (13)$$

- **The mean**

at $r = 1$, the mean of the TPRD is given by

$$\acute{\mu}_1 = \alpha^{\frac{1}{\beta}}\ \Gamma\left(\frac{1}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right],$$
$$\alpha,\beta > 0, -1 \le \lambda \le 1. \qquad (14)$$

at $r = 2$, the second non-central moment of the TPRD is as follows

$$\acute{\mu}_2 = \alpha^{\frac{2}{\beta}}\ \Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right],$$
$$\alpha,\beta > 0, -1 \le \lambda \le 1. \qquad (15)$$

The central moment of the TPRD can be obtained by using the relationship between the central and the non-central moments in (13) as follows

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \acute{\mu}_{r-j}, \qquad r = 1,2,3,\ldots \qquad (16)$$

- **Variance**

Substituting $r = 2$ in (16), then the variance of the TPRD is:

$$V(x) = \alpha^{\frac{2}{\beta}}\ \Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right] - \alpha^{\frac{2}{\beta}}\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^2\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^2. \qquad (17)$$

The standard moments of the TPRD (α,β,λ) can be obtained as follows

$$\alpha_r = \frac{\mu_r}{\sqrt{\mu_2{}^r}}\,, \qquad r = 1,2,3,\ldots \qquad (18)$$

- **Skewness and kurtosis**

The measures of skewness and kurtosis can be obtained using Equation (18) as follows

- When $r = 3$ in (18), then the skewness of the TPRD (α,β,λ) is given by

$$\alpha_3=\frac{\Gamma\left(\frac{3}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{3}{2\beta}}+\lambda\right]-3\,\Gamma\left(\frac{1}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]\Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right]+2\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^3\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^3}{\left(\Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right]-\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^2\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^2\right)^{\frac{3}{2}}}. \tag{19}$$

- When $r=4$ in (18), then the kurtosis of the TPRD (α,β,λ) is given by

$$\alpha_4=\frac{\Gamma\left(\frac{2}{\beta}+1\right)\left[(1-\lambda)2^{\frac{2}{\beta}}+\lambda\right]-4\,\Gamma\left(\frac{1}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]\Gamma\left(\frac{3}{2\beta}+1\right)\left[(1-\lambda)2^{\frac{3}{2\beta}}+\lambda\right]}{\left(\Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right]-\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^2\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^2\right)^2}$$
$$+\frac{6\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^2\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^2\Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right]-3\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^4\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^4}{\left(\Gamma\left(\frac{1}{\beta}+1\right)\left[(1-\lambda)2^{\frac{1}{\beta}}+\lambda\right]-\alpha^{\frac{2}{\beta}}\left[\Gamma\left(\frac{1}{2\beta}+1\right)\right]^2\left[(1-\lambda)2^{\frac{1}{2\beta}}+\lambda\right]^2\right)^2}. \tag{20}$$

- **Quantile function**

The quantile function $x=Q(p)=F^{-1}(p)$, for $0<p<1$, of the TPRD is obtained from Equation (7), as follows

$$x=\left[-2\,\alpha^2\,ln\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda p}}{2\lambda}\right)\right]^{\frac{1}{2\beta}}. \tag{21}$$

Special quartile may be obtained using (21). For example, if $p=0.5$, the median of the TPRD is

$$\text{Median}=F^{-1}(0.5)=\left[-2\,\alpha^2\,ln\left(1-\frac{(1+\lambda)-\sqrt{1+\lambda^2}}{2\lambda}\right)\right]^{\frac{1}{2\beta}},$$
$$x>0,\alpha,\beta>0,-1\le\lambda\le 1. \tag{22}$$

- **Moment generating function**

The moment generating function, denoted by $M_x(t)$, of a random variable X of the TPRD can be obtained as below:

$$M_x(t)=E(e^{tx})=\int_0^\infty e^{tx}\,f(x;\alpha,\beta,\lambda)dx$$
$$=\sum_{\iota_1=0}^{\infty}\frac{t^{\iota_1}}{\iota_1!}\;\alpha^{\frac{\iota_1}{\beta}}\,\Gamma\left(\frac{\iota_1}{2\beta}+1\right)\left[(1-\lambda)\,2^{\frac{\iota_1}{2\beta}}+\lambda\right]. \tag{23}$$

- **Order statistics**

Order statistics is one of the most fundamental tools in non-parametric statistics and inference. It is used in the problems of estimation and hypothesis tests in different ways. Therefore, some properties of the order statistics for the TPRD are introduced.

Suppose that $x_1, x_2, \dots, x_n$ is a random sample with the pdf, $f(x)$, and the cdf, $F(x)$. Let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ denote the corresponding order statistics, then $x_{(i)}$ is called the i^{th} order statistics, $i = 1,2, \dots n$.

The pdf of the i^{th} order statistics $x_{(i)}$ is given by

$$g(x_{(i)}) = \frac{n!}{(i-1)!\,(n-i)!}\, f(x_{(i)})[F(x_{(i)})]^{i-1}\,[1-F(x_{(i)})]^{n-i},$$

$$-\infty < x_{(i)} < \infty. \tag{24}$$

The pdf of the i^{th} order statistics $x_{(i)}$ of the TPRD can be obtained by substituting (7) and (8) in (24), then

$$g(x_{(i)}) = \frac{n!}{(i-1)!\,(n-i)!}\,\frac{\beta}{\alpha^2}\, x_{(i)}^{2\beta-1}\, e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]$$

$$\times\left\{\left[1-e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]\left[1+\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]\right\}^{i-1}\left\{1-\left[1-e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]\left[1+\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]\right\}^{n-i}$$

$$= \frac{n!}{(i-1)!(n-i)!}\,\frac{\beta}{\alpha^2}\, x_{(i)}^{2\beta-1}\sum_{\iota_2=0}^{n-i}\sum_{\iota_3=0}^{\iota_2+i-1}\sum_{\iota_4=0}^{\iota_2+i-1}(-1)^{\iota_2+\iota_3}\binom{n-i}{\iota_2}\binom{\iota_2+i-1}{\iota_3}\binom{\iota_2+i-1}{\iota_4}$$

$$\times\lambda^{\iota_4} e^{-(\iota_3+\iota_4+1)\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right],$$

$$x_{(i)} > 0, \alpha, \beta > 0,\ -1 \le \lambda \le 1, i = 1,2,3, \dots n, \tag{25}$$

Special cases:

- If $i = 1$, in (25), the pdf of the smallest order statistics can be obtained as follows:

$$g(x_{(1)}) = \frac{n\beta}{\alpha^2}\, x_{(1)}^{2\beta-1}\sum_{\iota_2=0}^{n-1}\sum_{\iota_3=0}^{\iota_2}\sum_{\iota_4=0}^{\iota_2}(-1)^{\iota_2+\iota_3}\binom{n-1}{\iota_2}\binom{\iota_2}{\iota_3}\binom{\iota_2}{\iota_4}$$

$$\times \lambda^{\iota_4} e^{-(\iota_3+\iota_4+1)\left(\frac{x_{(1)}^{2\beta}}{2\alpha^2}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x_{(1)}^{2\beta}}{2\alpha^2}\right)}\right],$$

$$x_{(1)} > 0, \alpha, \beta > 0, -1 \le \lambda \le 1. \tag{26}$$

- If $i = n$, in (25), the pdf of the largest order statistics can be obtained as follows:

$$g(x_{(n)}) = \frac{n\beta}{\alpha^2} x_{(n)}^{2\beta-1} \sum_{\iota_3=0}^{n-1} \sum_{\iota_4=0}^{n-1} (-1)^{\iota_3} \binom{n-1}{\iota_3} \binom{n-1}{\iota_4} \lambda^{\iota_4} e^{-(\iota_3+\iota_4+1)\left(\frac{x_{(n)}^{2\beta}}{2\alpha^2}\right)}$$

$$\times\left[1-\lambda+2\lambda e^{-\left(\frac{x_{(n)}^{2\beta}}{2\alpha^2}\right)}\right], \quad x_{(n)} > 0, \alpha, \beta > 0, -1 \le \lambda \le 1. \tag{27}$$

3.1. Some sub-model distributions

The TPRD (α, β, λ) contains some special well-known distributions, which are discussed in lifetime literature.

- **Rayleigh distribution**

One parameter RD is a special case from the TPRD (α, β, λ) when the transmuted parameter $\lambda = 0$ and shape parameter $\beta = 1$ in Equation (8).

- **Power Rayleigh distribution**

The pdf of the TPRD (α, β, λ) reduces to the pdf of the PRD with two parameters when transmuted parameter $\lambda = 0$ in Equation (8).

- **Transmuted Rayleigh distribution**

The pdf of the TPRD (α, β, λ) reduces to the pdf of the TRD with two parameters when shape parameter $\beta = 1$ in Equation (8) as follows:

$$f(x; \alpha, \lambda) = \frac{x}{\alpha^2} e^{-\left(\frac{x^2}{2\alpha^2}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x^2}{2\alpha^2}\right)}\right],$$

$$x > 0, \alpha > 0, -1 \le \lambda \le 1. \tag{28}$$

4. Maximum Likelihood Estimation

In this section, the ML estimators of the parameters, rf and hrf of the TPRD are derived. Additionally, ACIs of the parameters, rf and hrf of the TPRD are obtained.

Let $x_1, x_2, \dots, x_n$ be a random sample from TPRD (α, β, λ). The *likelihood function* (LF) based on Type II censoring sample is given by

$$L(\underline{\theta}; \underline{x}) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f\left(x_{(i)}; \underline{\theta}\right)\right]\left[1 - F\left(x_{(r)}; \underline{\theta}\right)\right]^{n-r}, \tag{29}$$

where $\underline{\theta} = (\alpha, \beta, \lambda)$, $f\left(x_{(i)}; \underline{\theta}\right)$ and $F\left(x_{(r)}; \underline{\theta}\right)$ are given by (7) and (8). Then substituting (7) and (8) in (29) yields

$$L(\underline{\theta}; \underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left(\frac{\beta}{\alpha^2}\right)x_{(i)}^{2\beta-1}e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\left[1 - \lambda + 2\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right]$$

$$\times\left\{e^{-\left(\frac{x_{(r)}^{2\beta}}{2\alpha^2}\right)}\left[1 - \lambda + \lambda e^{-\left(\frac{x_{(r)}^{2\beta}}{2\alpha^2}\right)}\right]\right\}^{n-r}. \tag{30}$$

The natural logarithm of the LF is given by

$$\ell \propto r\, ln(\beta) - 2r\, ln(\alpha) - \sum_{i=1}^{r}\frac{x_{(i)}^{2\beta}}{2\alpha^2} + (2\beta - 1)\sum_{i=1}^{r} ln\left[x_{(i)}\right]$$

$$+\sum_{i=1}^{r} ln\left[1 - \lambda + 2\lambda e^{-\left(\frac{x_{(i)}^{2\beta}}{2\alpha^2}\right)}\right] - (n-r)\left(\frac{x_{(r)}^{2\beta}}{2\alpha^2}\right) + (n-r)\, ln\left[1 - \lambda + \lambda e^{-\left(\frac{x_{(r)}^{2\beta}}{2\alpha^2}\right)}\right]. \tag{31}$$

4.1 Estimation of the parameters

The ML estimators of the parameters $\underline{\theta} = (\alpha, \beta, \lambda)$ can be obtained by differentiating (31) with respect to α, β and λ and then equating to zero. Hence

$$\frac{\partial \ell}{\partial \alpha} = \frac{-2r}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3}\sum_{i=1}^{r} x_{(i)}^{2\hat{\beta}} + \frac{2\hat{\lambda}}{\hat{\alpha}^3}\sum_{i=1}^{r} \frac{x_{(i)}^{2\hat{\beta}}\, e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}}{\left[1-\hat{\lambda}+2\hat{\lambda}e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}\right]}$$

$$+\frac{(n-r)}{\hat{\alpha}^3}\left[x_{(r)}^{2\hat{\beta}}\right] + \left\{\frac{(n-r)\hat{\lambda}\, x_{(r)}^{2\hat{\beta}}\, e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}}{\hat{\alpha}^3\left[1-\hat{\lambda}+\hat{\lambda}e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}\right]}\right\} = 0, \qquad (32)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{r}{\hat{\beta}} - \frac{1}{\hat{\alpha}^2}\sum_{i=1}^{r} x_{(i)}^{2\hat{\beta}}\, ln\left[x_{(i)}\right] + 2\sum_{i=1}^{r} ln\left[x_{(i)}\right] - \frac{2\hat{\lambda}}{\hat{\alpha}^2}\sum_{i=1}^{r}\frac{x_{(i)}^{2\hat{\beta}}\ \ ln\left[x_{(i)}\right] e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}}{\left[1-\hat{\lambda}+2\hat{\lambda}e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\hat{\alpha}^2}\right)}\right]}$$

$$-\frac{(n-r)\, x_{(r)}^{2\hat{\beta}}\ \ ln\left[x_{(r)}\right]}{\hat{\alpha}^2} - \frac{(n-r)\,\hat{\lambda} x_{(r)}^{2\hat{\beta}}\ \ ln\left[x_{(r)}\right] e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}}{\hat{\alpha}^2\left[1-\hat{\lambda}+\hat{\lambda}e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]} = 0, \qquad (33)$$

and

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{r}\frac{\left[-1+2e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]}{\left[1-\hat{\lambda}+2\hat{\lambda}e^{-\left(\frac{x_{(i)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]} + (n-r)\left\{\frac{\left[-1+e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]}{\left[1-\hat{\lambda}+\hat{\lambda}e^{-\left(\frac{x_{(r)}^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]}\right\} = 0. \qquad (34)$$

The system of non-linear Equations (32)-(34) can`t be solved in closed form but can be solved numerically using Newton-Raphson method to obtain the ML estimates of the parameters α, β and λ.

4.2 Estimation of the reliability and hazard rate functions

The invariance property of the ML estimators enables us to obtain the ML estimators of the rf and hrf by replacing the parameters α, β and λ by their ML estimators in (9) and (10), respectively, as follows:

$$\hat{r}(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda}) = e^{-\left(\frac{x^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\left[1 - \hat{\lambda} + \hat{\lambda} e^{-\left(\frac{x^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right],$$

$$x > 0,\ \hat{\alpha}, \hat{\beta} > 0, -1 \leq \hat{\lambda} \leq 1, \qquad (35)$$

and

$$\hat{h}(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda}) = \frac{\frac{\hat{\beta}}{\hat{\alpha}^2}\, x^{2\hat{\beta}-1}\left[1-\hat{\lambda}+2\hat{\lambda} e^{-\left(\frac{x^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]}{\left[1-\hat{\lambda}+\hat{\lambda} e^{-\left(\frac{x^{2\hat{\beta}}}{2\,\hat{\alpha}^2}\right)}\right]},$$

$$x > 0,\ \hat{\alpha}, \hat{\beta} > 0\,, -1 \leq \hat{\lambda} \leq 1, \qquad (36)$$

where $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ are the ML estimators of α, β and λ.

4.3 Asymptotic confidence intervals

The asymptotic Fisher information matrix $\hat{I}_{ij}(\underline{\theta})$ for the ML estimators of the parameters $\underline{\theta} = (\alpha, \beta, \lambda)$ is the 3 × 3 symmetric matrix of the negative second partial derivatives of ln L, given by (31) with respect to the parameters $\underline{\theta} = (\alpha, \beta, \lambda)$. That is

$$\hat{I}_{ij}(\underline{\theta}) \approx -\left[\frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j}\right]_{|\underline{\theta}=\underline{\hat{\theta}}}, \qquad i, j = 1,2,3, \qquad (37)$$

where $\theta_1 = \alpha, \theta_2 = \beta$ and $\theta_3 = \lambda$.

The asymptotic variance-covariance matrix $\hat{V}$ for the ML estimators is the inverse of the asymptotic Fisher information matrix.

The asymptotic Fisher information matrix enables us to construct ACIs for the parameters based on the limiting normal distribution, where $V(\hat{\theta}_i) = \sigma_{ii}$, the i^{th} diagonal elements of the matrix $\hat{V}$.

ACIs for the parameters can be obtained by using the asymptotic normality of the ML estimators. Hence, a two sided approximate $100(1-\tau)\%$ ACIs for θ_i are given by

$$L_{\theta_i} = \hat{\theta}_i - Z_{(1-\tau/2)}\sqrt{\widehat{Var}(\hat{\theta}_i)} \text{ and } U_{\theta_i} = \hat{\theta}_i + Z_{(1-\tau/2)}\sqrt{\widehat{Var}(\hat{\theta}_i)}, \quad i = 1,2,3, \tag{38}$$

where $\theta_1 = \alpha, \theta_2 = \beta$ and $\theta_3 = \lambda$, L_{θ_i} and U_{θ_i} are the *lower limit* (LL) and *upper limit* (UL) and $(1-\tau/2)$ is the percentile of the standard normal distribution.

5. Numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation on the basis of simulated and real data.

5.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates on the basis of generated data from the TPRD. The ML estimates of the parameters, rf and hrf based on Type II censoring sample are computed. Moreover, ACIs of the parameters, rf and hrf are calculated. Simulation study is performed using Mathematica 9 to illustrate the obtained results.

The following steps are used to compute the ML estimates of the parameters, rf and hrf based on Type II censoring sample from the TPRD (α, β, λ).

a. Conducting a simulation for different sample sizes generated from the TPRD using different parameter values.

b. For each sample size n sort the $x_i's$, such that $x_1 \leq x_2 \leq \cdots \leq x_n$.

c. Choose the number of failures r to be less than or equal to the sample size n.

d. Repeat all the previous steps *number of replications* (NR) times where NR represents a fixed number of simulated samples.

For the number of the units r and the population parameter values, the ML estimates and the ACIs of the parameters are obtained. Also, the rf, hrf and their ACIs are calculated using the ML estimates of the parameters.

The performance of the estimates is considered through some measurements of accuracy. In order to study the precision and variation of the estimates, it is convenient to use the

$$mean\ square\ error\ \text{(MSE)} = variance(estimate) + bias^2(estimate)$$

Simulation results of ML estimates are displayed in Tables 1 and 2, where NR = 2000 and (n=30, 60, 100, 200 and 500), are the sample sizes. For each sample size, the censoring sizes are 20%, 50% and 100%.

Table 1 displays the ML estimates, MSEs and 95% ACIs of the unknown parameters based on Type II censoring for all percentages under different sample sizes.

Table 1: ML estimates, MSEs and 95% ACIs of the parameters based on Type II censoring of the TPRD for different samples of size n (NR =2000, $\alpha = 0.5,\ \beta = 1.4$ and $\lambda = 0.2$)

n	r	parameters	ML estimates	MSEs	ACIs		
					UL	LL	Length
30	6	α	0.5694	0.0262	0.8559	0.2829	0.5730
		β	1.4168	0.0971	2.0268	0.8069	1.2199
		λ	0.5584	0.5896	1.8894	0.0000	1.8894
	15	α	0.5040	0.0154	0.7475	0.2605	0.4870
		β	1.4504	0.0960	2.0497	0.8510	1.1987
		λ	0.3124	0.5055	1.6885	0.0000	1.6885
	30	α	0.5259	0.0110	0.7255	0.3263	0.3992
		β	1.4572	0.0481	1.8722	1.0421	0.8301
		λ	0.4309	0.4167	1.6124	0.0000	1.6124
60	12	α	0.5713	0.0251	0.8485	0.2941	0.5544
		β	1.4020	0.0603	1.8832	0.9209	0.9623
		λ	0.5401	0.5632	1.8513	0.0000	1.8513
	30	α	0.5268	0.0128	0.7419	0.3117	0.4302
		β	1.4297	0.0587	1.9008	0.9586	0.9422
		λ	0.3971	0.4086	1.5890	0.0000	1.5890
	60	α	0.4987	0.0055	0.6444	0.3529	0.2915
		β	1.4282	0.0255	1.7364	1.1199	0.6165
		λ	0.3694	0.2657	1.3236	0.0000	1.3236
100	20	α	0.5569	0.0224	0.8285	0.2853	0.5432
		β	1.4207	0.0366	1.7935	1.0479	0.7456
		λ	0.5022	0.3949	1.5821	0.0000	1.5821
	50	α	0.5245	0.0082	0.6953	0.3536	0.3417
		β	1.3897	0.0348	1.7550	1.0244	0.7306
		λ	0.4585	0.3864	1.5665	0.0000	1.5665
	100	α	0.5125	0.0026	0.6085	0.4165	0.1920
		β	1.4191	0.0127	1.6371	1.2011	0.4360
		λ	0.2435	0.2269	1.1734	0.0000	1.1734

200	40	α β λ	0.5654 1.3941 0.4857	0.0217 0.0225 0.3487	0.8244 1.6880 1.4986	0.3063 1.1002 0.0000	0.5181 0.5878 1.4986
	100	α β λ	0.5166 1.4145 0.3167	0.0042 0.0190 0.1465	0.6387 1.6830 1.0311	0.3946 1.1460 0.0000	0.2441 0.5370 1.0311
	200	α β λ	0.5006 1.4106 0.2845	0.0010 0.0080 0.0902	0.5633 1.5848 0.8495	0.4379 12365 0.0000	0.1254 0.3483 0.8495
500	100	α β λ	0.5560 1.3893 0.4346	0.0194 0.0133 0.2952	0.8056 1.6145 1.3952	0.3064 1.1641 0.0000	0.4992 0.4504 1.3952
	250	α β λ	0.5123 1.4097 0.2824	0.0026 0.0075 0.0942	0.6101 1.5784 0.8618	0.4146 1.2409 0.0000	0.1955 0.3375 0.8618
	500	α β λ	0.5012 1.4020 0.2122	0.0007 0.0032 0.0274	0.5531 1.5122 0.5356	0.4494 .1.2917 0.0000	0.1037 0.2205 0.5356

From Table 1, it is observed that the ML estimates for the parameters are very close to the population parameter values as the sample size increases. The MSEs of the parameters are decreasing when the sample sizes are increasing. This is indicative of the fact that the estimates are consistent and approach the population parameter values as the sample size increases and the lengths of the ACIs of the parameters become narrower as the sample size increases.

Table 2 presents the ML estimates, MSEs and 95% ACIs of the rf and hrf based on Type II censoring for all percentages under different sample sizes.

Table 2: ML estimates, MSEs and 95% ACIs of the reliability and hazard rate functions based on Type II censoring of the TPRD for different samples of size n
(NR =2000, $x_0 = 0.1,\ \alpha = 0.5,\ \beta = 1.4$ and $\lambda = 0.2$)

n	r	rf and hrf	ML estimates	MSEs	ACIs		
					UL	LL	Length
30	6	$r(x_0)$ $h(x_0)$	0.8354 1.2804	0.0073 0.0878	1.0026 1.8607	0.6681 0.7002	0.3345 1.1605
	15	$r(x_0)$ $h(x_0)$	0.8330 1.2656	0.0029 0.0676	0.9387 1.7752	0.7273 0.7560	0.2114 1.0192
	30	$r(x_0)$ $h(x_0)$	0.8346 1.2554	0.0023 0.0577	0.9294 1.7257	0.7398 0.7851	0.1896 0.9406
60	12	$r(x_0)$ $h(x_0)$	0.8281 1.2765	0.0035 0.0444	0.9432 1.6888	0.7130 0.8642	0.2302 0.8246
	30	$r(x_0)$ $h(x_0)$	0.8312 1.2675	0.0018 0.0432	0.9145 1.6747	0.7480 0.8602	0.1665 0.8145

	60	$r(x_0)$	0.8330	0.0017	0.9134	0.7526	0.1608
		$h(x_0)$	1.2491	0.0386	1.6326	0.8656	0.7670
100	20	$r(x_0)$	0.8313	0.0011	0.8960	0.7665	0.1295
		$h(x_0)$	1.2676	0.0276	1.5933	0.9419	0.6514
	50	$r(x_0)$	0.8334	0.0010	0.8968	0.7699	0.1269
		$h(x_0)$	1.2699	0.0243	1.5754	0.9643	0.6111
	100	$r(x_0)$	0.8341	0.0007	0.8850	0.7832	0.1018
		$h(x_0)$	1.2548	0.0240	1.5579	0.9518	0.6061
200	40	$r(x_0)$	0.8335	0.0005	0.8780	0.7890	0.0890
		$h(x_0)$	1.2695	0.0146	1.5065	1.0325	0.4740
	100	$r(x_0)$	0.8344	0.0005	0.8774	0.7913	0.0861
		$h(x_0)$	1.2454	0.0143	1.4758	1.0150	0.4608
	200	$r(x_0)$	0.8341	0.0004	0.8709	0.7973	0.0736
		$h(x_0)$	1.2634	0.0124	1.4816	1.0452	0.4364
500	100	$r(x_0)$	0.8338	0.0002	0.8617	0.8059	0.0558
		$h(x_0)$	1.2489	0.0076	1.4160	1.0817	0.3343
	250	$r(x_0)$	0.8334	0.0002	0.8603	0.8065	0.0538
		$h(x_0)$	1.2656	0.0069	1.4285	1.1027	0.3258
	500	$r(x_0)$	0.8342	0.0002	0.8596	0.8089	0.0507
		$h(x_0)$	1.2681	0.0066	1.4276	1.1087	0.3189

From Table 2, one can observe that the MSEs of the rf and hrf are decreasing when the sample sizes are increasing and the lengths of the ACIs of the rf and hrf become narrower as the sample size increases.

5.2 Applications

This subsection is devoted to exhibit the applicability and flexibility of the TPRD for data modeling. Two applications are used to demonstrate the superiority of the TPRD over some existing distributions namely, PRD, TRD and RD. The ML estimates of the parameters based on three levels of Type II censoring (20%, 50% and 100%) and their *standard errors* (SEs) are obtained.

The measures of goodness of fit are *Kolmogorov-Smirnov* (K-S) goodness of fit test through the R programming language and its corresponding p-value, the -2*log likelihood statistic* (-2LL), *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC) and *Akaike information criterion corrected* (AICC) are used to compare the new distribution with some existing distributions, where

$$\mathrm{AIC} = 2\mathrm{K} - 2\mathrm{LL},$$

$$\mathrm{BIC} = \mathrm{K}\log(\mathrm{n}) - 2\mathrm{LL}$$

and

$$\mathrm{AICC} = \mathrm{AIC} + \frac{2\mathrm{K}(\mathrm{K}+1)}{(\mathrm{n}-\mathrm{K}-1)},$$

where K is the number of the parameters in the statistical model, n is the sample size and L is the natural logarithm of the value of the LF evaluated at the ML estimates.

The best distribution corresponding to the lowest values of -2LL, AIC, AICC and BIC and the highest p-value of the K-S tests.

Application 1

The first application is given by Mubarak and Almetwally (2021) and represents COVID-19 data which belong to the United Kingdom of 76 days, from 15 April to 30 June 2020. The data are formed of drought mortality rates. The data are: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019 and 11.4584.

Table 3 presents some basic descriptive statistics for the first application.

Table 3: Descriptive Statistics for Application 1

Mean	Median	Std. Deviation	Variance	Skewness	Kurtosis
2.437	1.248	2.936	8.620	1.737	2.319

From Table 3, one can notice that the value of the skewness is positive which implies that the data is right-skewed and the value of the kurtosis is less than 3 which implies that the data is Platykurtic.

Figure 3 shows the plots of the fitted pdf, histogram, PP-plot and QQ-plot for the data.

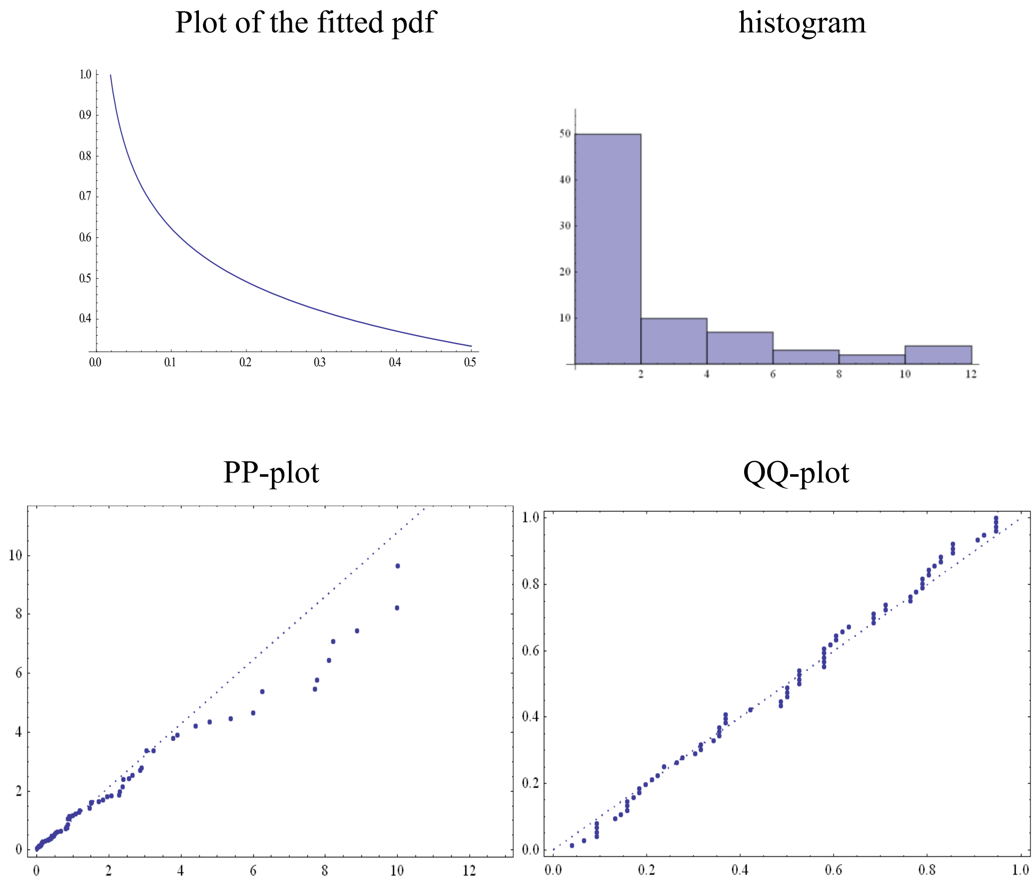


Figure 3: The fitted pdf, histogram, PP-plot and QQ-plot for Application 1.

From Figure 3, one can notice that these data are right-skewed. PP-plot, QQ-plot and the plot of the fitted pdf of the TPRD indicate that the TPRD provides a good fit to these data.

Table 4 displays the K-S statistic and its corresponding p-value, -2LL, AIC, BIC and AICC for Application 1.

Table 4: K-S statistics, P-values, -2LL, AIC, BIC and AICC of the fitted models for Application 1

Model	K–S	p-value	-2LL	AIC	BIC	AICC
TPRD	0.0658	0.9970	283.087	289.087	296.079	289.420
PRD	0.1053	0.7973	295.132	299.132	303.793	299.296
TRD	0.1974	0.1036	473.340	477.340	482.002	477.505
RD	0.1184	0.6609	569.740	571.740	574.071	571.794

The values in Table 4 indicate that the TPRD has the lowest K-S value and the highest p-value. Thus, it provides the best fit for these data compared to the other competitors of distributions and the TPRD has the smallest values of the -2LL, AIC, BIC and AICC, which implies that the proposed model is the best among the other competitors of distributions (PRD, TRD and RD).

Table 5 presents the ML estimates of the parameters and SEs under 20%, 50% and 100% levels of Type II censoring for COVID-19 data of Application 1.

Table 5: ML estimates and SEs of the parameters for Application 1

r	parameters	ML estimates	SEs
15	α	0.8649	0.2793
	β	0.5825	0.5330
	λ	0.2001	0.5999
38	α	1.2204	0.0170
	β	0.4931	0.2977
	λ	0.7702	0.5404
76	α	1.2082	0.0068
	β	0.4353	0.1455
	λ	0.5982	0.1964

From Table 5, the ML estimates of the parameters of the TPRD have smaller SEs for the case of the complete samples (100% level of censoring) comparing to the case of censored samples (20% and 50% levels of censoring). This returns to the amount of lost information through the censoring.

Application 2

The second application is given by Liu *et al.* (2021). In this application the survival times of patients suffering from the COVID-19 epidemic in China is considered. The data represent the survival times of patients from the time admitted to the hospital until death. Among them, a group of 53 COVID-19 patients were found in critical condition in hospital from January to February 2020. Among them, 37 patients (70%) were men and 16 women (30%). 40 patients (75%) were diagnosed with chronic diseases, especially including high blood pressure and diabetes. 47 patients (88%) had common clinical symptoms of the flu, 42 patients (81%) were coughing, 37 (69%) were short of breath, and 28 patients (53%) had fatigue. 50 (95%) patients had bilateral pneumonia showed by the chest computed tomographic scans.

These data are: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Table 6 presents some basic descriptive statistics for the second application.

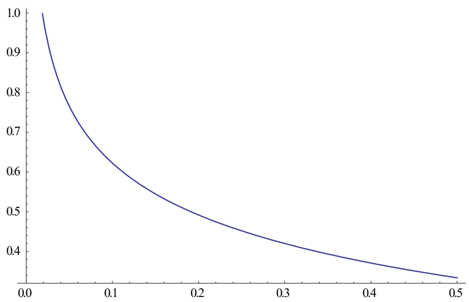
Table 6: Descriptive Statistics for Application 2

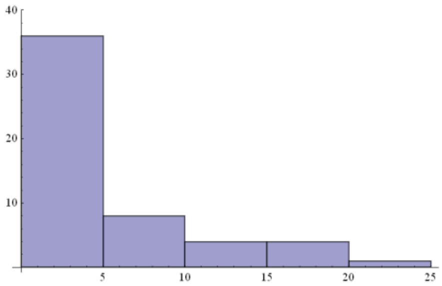
Mean	Median	Std. Deviation	Variance	Skewness	Kurtosis
4.787	3.079	5.495	30.198	1.420	1.062

From Table 6, one can notice that the value of the skewness is positive which implies that the data is right-skewed and the value of the kurtosis is less than 3 which implies that the data is Platykurtic.

Figure 4 shows the plots of the fitted pdf, histogram, PP-plot and QQ-plot for the data.

Plot of the fitted pdf histogram





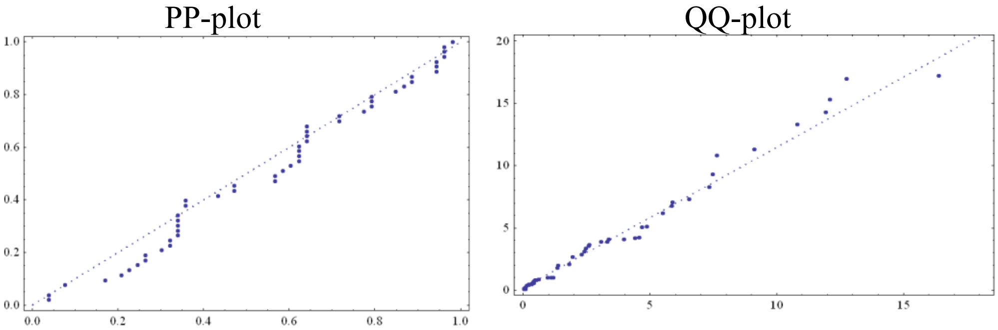


Figure 4: The fitted pdf, histogram, PP-plot and QQ-plot for Application 2.

From Figure 4, one can notice that these data are right-skewed. PP-plot, QQ-plot and the fitted pdf plot of the TPRD indicate that the TPRD provides a good fit to these data.

Table 7 displays the K-S statistic and its corresponding p-value, -2LL, AIC, BIC and AICC.

Table 7: K-S statistics, P-values, -2LL, AIC, BIC and AICC of the fitted models for Application 2

Model	K–S	p-value	-2LL	AIC	BIC	AICC
TPRD	0.1132	0.8901	267.511	273.511	279.422	274.001
PRD	0.1509	0.5844	272.386	276.386	280.327	276.626
TRD	0.2453	0.0824	367.546	371.546	375.486	371.786
RD	0.1698	0.4294	572.814	574.814	576.784	574.892

The values in Table 7 indicate that the TPRD has the lowest K-S value and the highest p-value. Thus, it provides the best fit for these data compared to the other competitors of distributions and the TPRD has the smallest values of the -2LL, AIC, BIC and AICC, which implies that the proposed model is the best among the other competitors of distributions (PRD, TRD and RD).

Table 8 presents the ML estimates of the parameters and SEs under 20%, 50% and 100% levels of Type II censoring for COVID-19 data of Application 2.

Table 8: ML estimates and SEs of the parameters for Application 2

r	parameters	ML estimates	SEs
11	α	2.2913	0.5276
	β	0.4919	0.4055
	λ	2.4993	3.9986
27	α	1.3898	0.0734
	β	0.3708	0.0595
	λ	0.3085	0.3829
53	α	1.5477	0.0318
	β	0.4118	0.1765
	λ	0.6197	0.2395

From Table 8, the ML estimates of the parameters of the TPRD have smaller SEs for the case of the complete samples (100% level of censoring) comparing to the case of censored samples (20% and 50% levels of censoring). This returns to the amount of lost information through the censoring.

Concluding remarks:

- It is observed from Table 1 that the ML estimates for the parameters are very close to the population parameter values as the sample size increases.
- The MSEs of the parameters are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approach the population parameter values as the sample size increases.
- From Table 2, one can observe that the MSEs of the rf and hrf are decreasing when the sample size is increasing.
- The lengths of the ACIs of the parameters, rf and hrf become narrower as the sample size increases.
- Values in Tables 4 and 7 indicate that the TPRD has the lowest K-S value and the highest p-value for the two applications. Thus, it provides the best fit for these data compared to the other competitors of distributions.
- Moreover, the TPRD has the smallest values of the -2LL, AIC, BIC and AICC which imply that the proposed model is the best among the other competitors of distributions (PRD, TRD and RD).
- The ML estimates of the parameters of the TPRD have smaller SEs for the case of the complete samples (100% level of censoring) comparing to the case of censored samples (20% and 50% levels of censoring). This returns to the amount of lost information through the censoring.

6. General Conclusions

In this paper, distribution called the TPRD (α, β, λ) is introduced. From the graphical description of the TPRD, one can observe that it has several curves such as decreasing, increasing, monotone decreasing and then approximately constant, bell shaped, approximately symmetric, therefore the TPRD is a very flexible reliability model. Several statistical properties of this distribution such as rhrf, chrf, the central and non-central moments, mean, variance, skewness, kurtosis, quantile function, median, moment generating function and order statistics are derived. Also, some sub-model distributions are obtained. In addition, the ML estimators for the parameters, rf and hrf of the TPRD based on Type II censored data are obtained. Simulation study is applied to investigate the precision of the ML estimates and an application using two real data sets are used to demonstrate how the results can be used in practice. In general, the ML estimates are very close to the population parameter values as the sample size increases. Also, MSEs are decreasing when the sample sizes are increasing. This is indicative of the fact that the estimates are consistent and approach the population parameter values as the sample size increases. The lengths of the ACIs of the parameters, rf and hrf become narrower as the sample size increases. The TPRD is the best fitting among many known distributions for two real data sets.

References

Abdulhameed, A. O., Sani, I. D., Abubakar, Y., Yahaya, Z. and Abubakar, U. (2024). Transmuted cosine Topp-Leone G family of distributions: properties and applications. *Journal of the Nigerian Society of Physical Sciences*. Vol. 6, No. 4, pp. 1-12.

Adetunji, A. A. (2023). Transmuted Ailamujia distribution with aplications to lifetime observations. *Asian Journal of Probability and Statistics*. Vol. 21, No. 1, pp. 1-11. https://doi:10.9734/AJPAS/2023/v21i1452.

Aryal, G. R. and Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory Methods and Application*. Vol. 71, No. 12, pp. 1401–1407. https://doi.org/10.1016/j.na.2009.01.168.

Aryal, G. R. and Tsokos, C. P. (2011). Transmuted Weibull distribution: a generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics.* Vol. 4, No. 2, pp. 89-102.

Bhat, A. A. and Ahmad, S. P. (2020). A new generalization of Rayleigh distribution: properties and applications. *Pakistan Journal of Statistics.* Vol. 36, No. 3, pp. 225-250.

Christophe, C., Festus, C. O. and Ngozi, O. U. (2022). Theory and applications of the transmuted continuous Bernoulli distribution. *Earthline Journal of Mathematical Sciences*. Vol. 10, No. 2, pp. 385-407.

https://doi.org/10.34198/ejms.10222.385407.

Elbatal, I. and Elgarhy, M. (2013). Transmuted quasi Lindley distribution: a generalization of the quasi Lindley distribution. *International Journal of Pure and Applied Sciences and Technology.* Vol. 18, No. 2, pp. 59–70.

Faton, M. (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistics.* Vol. 42, No.1, pp. 21–31.

Joseph, J. and Ravindran, M. (2023). Transmuted exponentiated Kumaraswamy distribution. *Reliability Theory and Applications Journal*. Vol. 18, No. 72, pp. 539-552.

Khan, M. S. and King, R. (2015). Transmuted modified inverse Rayleigh distribution. *Austrian Journal of Statistics*. Vol. 44, pp. 17–29.

Kilany, N. M., Mohamed, A. W. M. and Lobna, H. R. (2023). Power Rayleigh distribution for fitting total deaths of Covid-19 in Egypt. *Journal of Statistics Applications and Probability*. Vol. 12, No. 3, pp.1073-1085. http://dx.doi.org/10.18576/jsap/120316.

Liu, X., Ahmed, Z., Gemeay, A. M., Abdulrahman, A. T., Hafez, E. H. and Khalil, N. (2021). Modeling the survival times of the COVID-19 patients with a new statistical model: a case study from China. *PloS one.* Vol. 16, No. 7, pp. 1-31. https://doi.org/10.1371/journal.pone.0254999.

Mubarak, A. E. and Almetwally, E. M. (2021). A new extension exponential distribution with applications of COVID-19 data. *Journal of Financial and Business Research.* Vol. 22, No. 1, pp. 444-460.

Rayleigh, J. W. S. (1880). On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science - 5th Series*. Vol. 10, No. 60, pp. 73-78. https://doi.org/10.1080/14786448008626893.

Saboor, A., Elbatal, I. and Cordeiro, G. M. (2016). The transmuted exponentiated Weibull-geometric distribution: Theory and applications. *Hacettepe Journal of Mathematics and Statistics.* Vol. 45, No. 3, pp. 973 - 987.

Shaw, W. T. and Buckley, I. R. C. (2007). The Alchemy of probability distributions: Beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. https://www.researchgate.net/publication/228732685.

Tian, Y., Tian, M. and Zhu, Q. (2014). Transmuted linear exponential distribution: a new generalization of the linear exponential distribution. *Communications in Statistics Simulation and Computation*. Vol. 43, No. 10, pp. 2661–2677.

Toumaj, A., Mostafaee, S. M. T. K. and Hamedani, G. G. (2021). The transmuted inverted Nadarajah- Haghighi distribution with an application to lifetime data. *Pakistan Journal Statistics Operation Research.* Vol. 17, No. 2, pp. 451-466.

Umar, A. A., Eraikhuemen, I. B., Koleoso, P. O., Joel, J. and Ieren, T. G. (2019). On the properties and pplications of a transmuted Lindley-exponential distribution. *Asian Journal of Probability and Statistics.* Vol. 5, No. 3, pp. 1-13.

Yousof, H. M., Alizadeh, M., Jahanshahi, S. M. A., Ramires, T. G., Ghosh, I. and Hamedani, G.G. (2017). The transmuted Topp-Leone G family of distributions: Theory, characterizations and applications. *Journal of Data Science.* Vol. 15, No. 4, pp. 723-740.

المستخلص

تلعب توزيعات الحياة دوارًا مهماً في نمذجة وتحليل العديد من ظواهر الحياة. ونتيجة لذلك ، حاول العديد من الباحثين تقديم توزيعات تنافسية جديدة من أجل نمذجة البيانات الحقيقية التي تنشأ في العديد من المجالات مثل الطب والهندسة والاقتصاد حيث لا يمكن استخدام التوزيعات القديمة. غالبًا ما يكون تعميم التوزيع عن طريق إضافة معلمة واحدة أو أكثر إلى التوزيع الأساسي. يمكن أن تجعل المعلمات الجديدة التوزيع المعمم أكثر مرونة بمعنى أنه يمكنه نمذجة مجموعات البيانات بشكل أكثر ملاءمة.

توجد طرق عديدة لتعميم التوزيعات من بين هذه الطرق تم اقتراح نهج يسمى خريطة تحويل الرتبة التربيعية. حظيت عائلة التوزيعات المحولة باهتمام كبير خلال السنوات القليلة الماضية. تقنية إضافة معلمة جديدة إلى توزيع موجود بالفعل من شأنه أن يوفر مزيدًا من المرونة لهذا التوزيع.

لذلك تم في هذا البحث اقتراح توزيع تحت مسمى توزيع باور رايلي المحول وتم فيه دراسة خصائص هذا التوزيع المقترح بالإضافة إلي تقدير معالم هذا التوزيع ودالتي الصلاحية والفشل إعتماداً علي النوع الثاني من عينات المراقبة Type II censored samples باستخدام طريقة الإمكان الأعظم وتم عمل أسلوب محاكاة لحل المعادلات الناتجة من تقدير معالم دالتي الصلاحية والفشل والتأكد من صحة ما تم الوصول إليه نظرياً ثم التطبيق علي بعض البيانات الفعلية لإبراز أهمية التوزيع المقترح تطبيقياً.