أثر الأصول غير الملموسة على استمرارية الربح الاضافي للشركة

يرى البعض (Lev and Daum 2004) أن الاصول غير الملموسة تمثل أحد العوامل المحتملة للمساهمة في التباين بين قيمة الشركة الدفترية وقيمتها السوقية. وقد أشار (Saunders 2010) إلى أن الاصول غير الملموسة هي السبب وراء قدرة الشركات على المنافسة في الاقتصاد المعاصر. وقد بدأت الحكومات في تسليط الضوء على أهمية الأصول غير الملموسة كمحركات للنمو الاقتصادي، وتشجيع الشركات لتوجيه اهتمام اكثر لهذا النوع من الأصول. وقد حظي الدور الذي تلعبه هذه الأصول كمنشأ للقيمة والنمو قبولا واسعا بين الاقتصاديين، والمستثمرين، والمديرين (Marr, et al, 2003).

يعتمد الباحث في هذا البحث على نظرية منظور الموارد للمنشأة Resource-based view (RBV) والتي تشير إلى أن الموارد غير الملموسة تمثل أحد المحركات الرئيسية لاستمرارية اختلاف الأداء بين الشركات. و توصلت Villalonga (2004) إلى أدلة عن أن الأصول غير الملموسة يمكن أن تؤدي إلى استمرارية الميزة التنافسية. حيث تلعب الأصول غير الملموسة دورا هاما في عملية خلق قيمة الشركة كي تنافس بنجاح والذي ينعكس بدوره على القيمة السوقية للشركة، كما توصلت Villalonga (2004) إلى أنه يمكن تعريف استمرارية الميزه التنافسية على أنها درجة استمرارية الربح الإضافي للمنشأة. هناك رابط بين ربحية المنشأة واستمرارية الميزة التنافسية، حيث أن بناء واستمرارية ميزة تنافسية يترتب عليه تحقيق الشركة لأرباح عالية. طبقا لـ (Hill and Jones, 2013) فإنه سوف تتحقق الميزة التنافسية عندما تكون ربحية الشركة أعلى من متوسط ربحية الشركات الأخرى في نفس الصناعة.

يهدف هذا البحث إلى دراسة العلاقة بين الأصول غير الملموسة واستمرارية الربح الإضافي للشركة. كما اعتمد الباحث على عينة تتكون من 112 شركة مصرية مدرجة خلال الفترة من 2008 إلى 2015. وتشير النتيجة الرئيسية لهذا البحث إلى وجود علاقة إيجابية بين الأصول غير الملموسة واستمرارية الربح الإضافي للشركة بعد إدخال نوع الصناعة كمتغير منظم. وبعبارة أخرى، فإن تأثير الأصول غير الملموسة على استمرار الربح الإضافي للشركة يكون أقوى في الشركات الصناعية أكثر من الشركات غير الصناعية.

Sensitivity Analysis For optimal Control Problem

Using Different Co-state Values

Ahmed Mohamed Mohamed El-Sayed

Al-Obour High Institute For Management & Informatics
Department of Basic Science
Kilo 21 Cairo-Belbies Road, P.O. Box 27 Obour City, Egypt(*)
atabl18@yahoo.com - atabl@oi.edu.eg

Abstract

This paper provides an extension to an optimal control problem of multi-item inventory model with deteriorating items using the negative logarithm of deterioration and spoilage function as an objective function that depending on the alternative quadratic exponential form. In this paper, the different co-state values are used separately they have negative values along the optimal trajectory. The effect of increasing and decreasing this value on the optimal solution is investigated. Also, the sensitivity analysis that reflects the effect of changes of the deterioration and spoilage parameters values on the optimal solution is explained. Finally, we compared the obtained results with the results that have been obtained when the co-state value is equal negative one.

***Keywords*: Co-state Variables, Sensitivity Analysis, Multi-item Inventory System, Demand Rates, Deterioration, Spoilage, Pontryagin principle.**

***Mathematics Subject Classification 2010* : 49-XX - 97M40 - 90-XX - 90Bxx - 00A69 - 49J15 - 49J20.**

1 Introduction

An optimal control problem of multi-item inventory model has a wide importance in practice. El-Sayed [1] has studied the effect of different types of demand rates on the objective function, which refers to the negative value of logarithm of deterioration and spoilage function, using the Pontryagin principle for the negative one value for the co-state variable.

In this paper we will extend this study using different values, less or more than negative one, for the co-state variable λ_0. The purpose of this study is to indicate whether the changes in the co-state value affect the optimal solution or not, especially on the optimal production rates and on the objective function values. As it is expected the optimal inventory levels may not be affected or affected slightly as we shall see later.

(*)*Old address*: **High Institute for Specific Studies, Department of Management Information Systems, Nazlet Al-Batran, Giza, Egypt.**

Also, we need to know the sensitivity analysis for changing the deterioration and spoilage parameters, specially the effects on the optimal solution.

Finally, to complete this study we must compare the obtained results with the results when the co-state variable λ_0 equals -1.

Zhao and Prentice [4] presented the quadratic exponential form (QEF) for the two correlated variables X_1, X_2 as:

$$f(x_1, x_2) = \frac{1}{\sum_{x_1, x_2} \exp\{\theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2\}} \exp\{\theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2\}. \quad (1)$$

Elsayed [1] supposed that θ_1 represents $\psi_1 u_1$, θ_2 represents $\psi_2 u_2$ and θ_{12} represents $\psi_{12} u_1 u_2$, where the spoilage parameters ψ_1, ψ_2 and ψ_{12} depend on the control variables U_1 and U_2, $\psi's > 0$.

Since θ_1, θ_2 and θ_{12} represent the deterioration parameters, $\theta 's > 0$.

So, we can use the normalizing term, $\sum_{x_1, x_2} \exp\{\theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2\}$, in the function (1) to be rewritten in the exponential form [2] as shown below:

$$f(x_1, x_2) = \exp\{\theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2 - \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2)\}. \quad (2)$$

The negative logarithm of this function can be used as the objective function which is needed to be minimum value.

This paper can be ordered as follow: section 2 presents the mathematical form for the optimal control problem and the controlled systems, section 3 presents the numerical solution for the controlled systems with different types of demand rates, section 4 presents the sensitivity analysis for the deterioration and spoilage parameters, and finally section 5 gives some conclusions.

2. Mathematical Form

Let us define the following parameters, as used in El-sayed [1], which are used in the mathematical formulation of the optimal control model:

$X_i(t)$: The inventory levels at time t.

$U_i(t)$: The production rates at time t.

T : The length of planning period.

x_{i0} : The initial inventory levels.

a_{ii} : The deteriration coefficient due to self-contact of item x_i.

a_{ij} : The deteriration coefficient of x_i due to presence a unit of x_j, $i \neq j = 1,2$.

$D_i(x_1, x_2, t)$: The demand rates of (x_1, x_2).

ψ_1 : The spoilage rate of of x_1, $\psi_1 > 0$.

ψ_2 : The spoilage rate of of x_2, $\psi_2 > 0$.

ψ_{12} : The spoilage rate of (x_1, x_2), jointly, $\psi_{12} > 0$. θ_1 : The natural deterioration rate of x_1, $\theta_1 > 0$. θ_2 : The natural deterioration rate of x_2, $\theta_2 > 0$. θ_{12} : The natural deterioration rate of (x_1, x_2), jointly, $\theta_{12} > 0$.

As mentioned before, the negative logarithm of the function (2), which represents the deterioration and spoilage function, is used as the objective function:

$$J = -\ln f(x_1, x_2) = -\theta_1 x_1 - \theta_2 x_2 - \theta_{12} x_1 x_2 + \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2). \quad (3)$$

So, the problem can be formulated as

$$\text{Minimize} \quad \left\{J = -\theta_1 x_1 - \theta_2 x_2 - \theta_{12} x_1 x_2 + \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2)\right\}, \quad (4)$$

subject to:

$$\dot{x}_1 = -x_1\left(\theta_1 + a_{12} x_2 + a_{11} x_1\right) - D_1 + u_1, \quad (5)$$

$$\dot{x}_2 = -x_2\left(\theta_2 + a_{21} x_2 + a_{22} x_2\right) - D_2 + u_2, \quad (6)$$

and

$$x_1 = x_1(t) \geq 0, \quad x_2 = x_2(t) \geq 0, \quad u_1 = u_1(t) \geq 0, \quad u_2 = u_2(t) \geq 0, \quad (7)$$

where,

$$t \in T, \quad D = D(x_1, x_2, t) \geq 0, \quad \theta_1, \theta_2, \theta_{12} > 0, \quad \psi_1, \psi_2, \psi_{12} > 0.$$

Using the Pontryagin principle, let us define $J = \dot{x}_0$, and introduce the co-state variables λ_0, λ_1 and λ_2 corresponding to the state variables X_0, X_1 and X_2 respectively. From (4), (5) and (6), we can write the Hamiltonian function as follows:

$$H = \lambda_0 \dot{x}_0 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2, \quad (8)$$

Moreover, to obtain the co-state equations and the Lagrange multipliers associated with the constraints (5) and (6), we formulate the Lagrangian function as follows:

$$L = H + \mu_1(t) x_1 + \mu_2(t) x_2 + \mu_3(t) u_1 + \mu_4(t) u_2 \ , \quad (9)$$

where, $\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)$ are known as Lagrange multipliers. These Lagrange multipliers satisfy the conditions:

$$\mu_1(t) \geq 0, \mu_2(t) \geq 0, \mu_3(t) \geq 0, \mu_4(t) \geq 0, \quad \mu_1 x_1(t) = 0, \quad \mu_2 x_2(t) = 0, \quad \mu_3 u_1(t) = 0, \quad \mu_4 u_2(t) = 0. \quad (10)$$

From (9), we can easily obtain the co-state equations

$$\dot{\lambda}_i(t) = -\frac{\partial L}{\partial x_i}, \quad i = 0, 1, 2, \quad (11)$$

then,

$$\dot{\lambda}_0(t) = -\frac{\partial L}{\partial x_0} = 0, \quad \dot{\lambda}_1(t) = -\frac{\partial L}{\partial x_1}, \quad \dot{\lambda}_2(t) = -\frac{\partial L}{\partial x_2}, \quad (12)$$

The first equation of (12) shows that the co-state variable $\lambda_0(t)$ remains constant along the optimal trajectory, and the Pontryagin principle requires that this constant should be a negative value [3]. Here, we will use different values for this co-state variable $\lambda_0(t)$.

$$\lambda_0(t) = -10 \quad \text{or} \quad \lambda_0(t) = -2 \quad \text{or} \quad \lambda_0(t) = -0.001, \tag{13}$$

Substituting from (4), (5), (6),(8) and (13) in (9), we can write the Hamiltonian function, L, in the form [when $\lambda_0(t) = -10$]:

$$\begin{aligned} L = {} & 10[\theta_1 x_1 + \theta_2 x_2 + \theta_{12} x_1 x_2 - \log(1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2)] \\ & + \lambda_1\left[-x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1\right] + \lambda_2\left[-x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2\right]. \\ & + \mu_1 x_1 + \mu_2 x_2 + \mu_3 u_1 + \mu_4 u_2. \end{aligned} \tag{14}$$

From conditions (7) and (10), we get

$$\mu_1(t) = \mu_2(t) = \mu_3(t) = \mu_4(t) = 0. \tag{15}$$

Substituting from (13) and (14) into (12) we get

$$\dot{\lambda}_1 = \lambda_1\left(\frac{\partial D_1}{\partial x_1} + 10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + \lambda_2\left(\frac{\partial D_2}{\partial x_1} + a_{21}x_2\right) - \theta_1 - 10\theta_{12}x_2, \tag{16}$$

$$\dot{\lambda}_2 = \lambda_2\left(\frac{\partial D_2}{\partial x_2} + 10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + \lambda_1\left(\frac{\partial D_1}{\partial x_2} + a_{12}x_1\right) - \theta_2 - 10\theta_{12}x_1, \tag{17}$$

with boundary conditions

$$\lambda_i(T) \neq 0, \quad i = 1,2. \tag{18}$$

Where T is the length of planning period which can be suggested.

To obtain the optimal production rates (control variables) U_i, $i = 1,2$, we differentiate the Lagrange function (14) with respect to u_1, u_2 respectively and putting it equal to zero, we get

$$\frac{\partial L}{\partial u_1} = -\frac{10(\psi_1 + \psi_{12}u_2)}{1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2} + \lambda_1 = 0.$$

$$\frac{\partial L}{\partial u_2} = -\frac{10(\psi_2 + \psi_{12}u_1)}{1 + \psi_1 u_1 + \psi_2 u_2 + \psi_{12} u_1 u_2} + \lambda_2 = 0.$$

Then,

$$u_1^*(t) = \frac{1}{\lambda_1} - \frac{1 + \psi_2 \hat{u}_2}{10(\psi_1 + \psi_{12}\hat{u}_2)}, \quad \lambda_1 \neq 0, \tag{19}$$

$$u_2^*(t) = \frac{1}{\lambda_2} - \frac{1 + \psi_1 \hat{u}_1}{10(\psi_2 + \psi_{12}\hat{u}_1)}, \quad \lambda_2 \neq 0 \tag{20}$$

Since, $\hat{u}_1$ and $\hat{u}_2$ are goal levels of production rates at the end of the planning period, T. Then using the equations (5), (6), (16) and (17) we get the controlled system of non-linear ordinary differential equations:

$$
\left.
\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\frac{\partial D_1}{\partial x_1} + 10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + \lambda_2\left(\frac{\partial D_2}{\partial x_1} + a_{21}x_2\right) - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\frac{\partial D_2}{\partial x_2} + 10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + \lambda_1\left(\frac{\partial D_1}{\partial x_2} + a_{12}x_1\right) - \theta_2 - 10\theta_{12}x_1
\end{aligned}
\right\}, \qquad (21)
$$

We can construct this system when $\lambda_0(t) = -2$ and $\lambda_0(t) = -0.001$ respectively

$$
\left.
\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\frac{\partial D_1}{\partial x_1} + 2\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + \lambda_2\left(\frac{\partial D_2}{\partial x_1} + a_{21}x_2\right) - \theta_1 - 2\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\frac{\partial D_2}{\partial x_2} + 2\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + \lambda_1\left(\frac{\partial D_1}{\partial x_2} + a_{12}x_1\right) - \theta_2 - 2\theta_{12}x_1
\end{aligned}
\right\}, \qquad (22)
$$

and

$$
\left.
\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - D_1 + u_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - D_2 + u_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\frac{\partial D_1}{\partial x_1} + 0.001\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + \lambda_2\left(\frac{\partial D_2}{\partial x_1} + a_{21}x_2\right) - \theta_1 - 0.001\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\frac{\partial D_2}{\partial x_2} + 0.001\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + \lambda_1\left(\frac{\partial D_1}{\partial x_2} + a_{12}x_1\right) - \theta_2 - 0.001\theta_{12}x_1
\end{aligned}
\right\}, \qquad (23)
$$

The optimal control variables can be constructed when $\lambda_0(t) = -2$ and $\lambda_0(t) = -0.001$ respectively as:

$$u_1^*(t) = \frac{1}{\lambda_1} - \frac{1 + \psi_2\hat{u}_2}{2(\psi_1 + \psi_{12}\hat{u}_2)}, \quad \lambda_1 \neq 0, \qquad (24)$$

$$u_2^*(t) = \frac{1}{\lambda_2} - \frac{1 \pm \psi\hat{u}_1}{2(\psi_2 + \psi_{12}\hat{u}_1)}, \quad \lambda_2 \neq 0, \qquad (25)$$

and

$$u_1^*(t) = \frac{1}{\lambda_1} - \frac{1+\psi_2 \hat{u}_2}{0.001(\psi_1 + \psi_{12}\hat{u}_2)}, \quad \lambda_1 \neq 0, \tag{26}$$

$$u_2^*(t) = \frac{1}{\lambda_2} - \frac{1+\psi_1 \hat{u}_1}{0.001(\psi_2 + \psi_{12}\hat{u}_1)}, \quad \lambda_2 \neq 0. \tag{27}$$

This system can be used to describe the time evolution of inventory levels and production rates. The analytical solution of this system is very difficult and then we can solve it numerically.

3 Numerical Solution

The solution of optimal control problem of this model will be carried out using Pontryagin principle. The numerical solution is to be necessary when the analytical solution is absence for the non-linear systems (21, 22 and 23).

In this solution we solve the non-linear ordinary differential equations using Runge-Kutta method, using the initial and boundary values for $x_1(t), x_2(t), \lambda_1(t)$ and $\lambda_2(t)$. The numerical solution can be explained by different types of demand as:

1. The demand rates are constant:
   $D(x_1, x_2, t) = \gamma_i$.
2. The demand rates are linear functions of inventory levels and time:
   $D(x_1, x_2, t) = \gamma_i + w_i x_i$.
3. The demand rates are logistic functions of inventory levels and time:
   $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$.
4. The demand rates are periodic functions of time:
   $D(x_1, x_2, t) = 1 - b_i \cos(t)$.

where γ_i, w_i, κ_i and $b_i (i = 1,2)$ are positive constants.

Table 1 presents the values of system parameters and the initial states which are used in the numerical examples for four cases of demand rate functions as follows:

Table 1. Values and initial states of system parameters

$\hat{u}_1$	$\hat{u}_2$	θ_1	θ_2	θ_{12}	a_{12}	a_{21}	a_{11}	a_{22}	γ_1
20	20	0.05	0.07	0.06	0.8	0.9	0.05	0.04	0.6
x_{10}	x_{20}	w_1	w_2	κ_1	κ_2	b_1	b_2	T	γ_2
5	5	0.9	0.8	0.4	0.5	0.7	0.8	5	0.7
ψ_1	ψ_2	ψ_{12}	$\lambda_1(T)$	$\lambda_2(T)$					
0.05	0.07	0.06	1	1					

Hint: In the logistic function demand rates, we will use the next values

$\hat{u}_1$	$\hat{u}_2$	T	$\lambda_1(0)$	$\lambda_2(0)$
200	200	1	1	1

The reset of values remain without changes as the other three cases of the demand rates (constant, linear and periodic).

The next subsections explain the controlled system for each case of the demand rates functions with different co-state value $\lambda_0(t)$ as shown below:

3.1 As $\lambda_0(t) = -10$

In this subsection we will use different demand rates with $\lambda_0(t) = -10$:

1. Constant Rates

We will present the model with demand function as constant rates, $D(x_1, x_2, t) = \gamma_i$. Substituting in the controlled system (21) by the constant demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (28)$$

Solving the controlled system (28) numerically, we get some results as displayed in Table 2.

2. Linear Rates

Also, we will present the model with demand function as linear rates, $D(x_1, x_2, t) = \gamma_i + w_i x_i$. Substituting in the controlled system (21) by the linear demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\omega_1 + \theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\omega_2 + \theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\omega_1 + 10\theta_1 + a_{12}x_2 + 2a_{11}x\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\omega_2 + 10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (29)$$

Solving the controlled system (29) numerically, we get some results as displayed in Table 2.

3. Logistic Rates

We present the model with demand function as logistic rates, $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$. Substituting in the controlled system (21) by the logistic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\big(2(\kappa_1 - x_1) + \theta_1 + a_{12}x_2 + a_{11}x_1\big) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\big(2(\kappa_2 - x_2) + \theta_2 + a_{21}x_2 + a_{22}x_2\big) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\big(2(\kappa_1 - 2x_1 + a_{11}x_1) + 10\theta_1 + a_{12}x_2\big) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\big(2(\kappa_2 - 2x_2 + a_{22}x_2) + 10\theta_2 + a_{21}x_1\big) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \qquad (30)$$

Solving the controlled system (30) numerically, we get some results as displayed in Table 2.

4. Periodic Rates

Finally, we will present the model with demand function as periodic rates, $D(x_1, x_2, t) = 1 - b_i\cos(t)$. Substituting in the controlled system (21) by the periodic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\big(\theta_1 + a_{12}x_2 + a_{11}x_1\big) - 1 + b_1\cos(t) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\big(\theta_2 + a_{21}x_2 + a_{22}x_2\big) - 1 + b_2\cos(t) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\big(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\big) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\big(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\big) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \qquad (31)$$

Solving the controlled system (31) numerically, we get some results as displayed in Table 2.

Table 2. The optimal solution when $\lambda_0(t) = -10$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.79	3.07	5.43	2.06	74.87
Linear	4.70	3.68	3.91	2.89	88.35
Logistic	143.21	31.83	-12.07	-2.07	-23.9
Periodic	6.66	3.10	5.23	2.09	76.78

As we see from Table 2, the otimal value of the objective function when $\lambda_0(t) = -10$ is achieved in the constant rate (74.87).

The logistic rate is unacceptable because the production rates in this case have negative values which is not realistic. This is reflexed on the value of an objective function (-23.9).

3.1 As $\lambda_0(t) = -2$

Also, in this subsection we use the previous demand rates with $\lambda_0(t) = -2$:

1. Constant Rates

We will present the model with demand function as constant rates, $D(x_1, x_2, t) = \gamma_i$. Substituting in the controlled system (22) by the constant demand rates, we have the controlled system:

$$\left.\begin{aligned} \dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\ \dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\ \dot{\lambda}_1 &= \lambda_1\left(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\ \dot{\lambda}_2 &= \lambda_2\left(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1 \end{aligned}\right\}, \tag{32}$$

Solving the controlled system (32) numerically, we get some results which are displayed in Table 3.

2. Linear Rates

Also, we will present the model with demand function as linear rates, $D(x_1, x_2, t) = \gamma_i + w_i x_i$. Substituting in the controlled system (22) by the linear demand rates, we have the controlled system:

$$\left.\begin{aligned} \dot{x}_1 &= -x_1\left(\omega_1 + \theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\ \dot{x}_2 &= -x_2\left(\omega_2 + \theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\ \dot{\lambda}_1 &= \lambda_1\left(\omega_1 + 10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\ \dot{\lambda}_2 &= \lambda_2\left(\omega_2 + 10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1 \end{aligned}\right\}, \tag{33}$$

Solving the controlled system (33) numerically, we get the results displayed in Table 3.

3. Logistic Rates

We present the model with demand function as logistic rates, $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$.

Substituting in the controlled system (22) by the logistic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(2(\kappa_1 - x_1) + \theta_1 + a_{12}x_2 + a_{11}x_1\right) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(2(\kappa_2 - x_2) + \theta_2 + a_{21}x_2 + a_{22}x_2\right) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(2(\kappa_1 - 2x_1 + a_{11}x_1) + 10\theta_1 + a_{12}x_2\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(2(\kappa_2 - 2x_2 + a_{22}x_2) + 10\theta_2 + a_{21}x_1\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (34)$$

Solving the controlled system (30) numerically, we get the results are displayed in Table 3.

4. Periodic Rates

Finally, we will present the model with demand function as periodic rates, $D(x_1, x_2, t) = 1 - b_i\cos(t)$. Substituting in the controlled system (22) by the periodic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - 1 + b_1\cos(t) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - 1 + b_2\cos(t) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (35)$$

Solving the controlled system (35) numerically, we get the results can be displayed in Table 3.

Table 3. The optimal solution when $\lambda_0(t) = -2$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.79	3.07	16.06	10.86	14.97
Linear	4.7	3.68	18.89	11.51	17.67
Logistic	157.13	31.65	-207.26	-48.15	-4.71
Periodic	6.66	3.10	15.75	10.99	15.36

As we see from Table 3, the otimal value of the objective function when $\lambda_0(t) = -2$ is achieved in the constant rate (14.97).

Also. The logistic rate is unacceptable because the production rates in this case are negative values and this unacceptable, this is reflexed on the value of an objective function (-4.71).

3.3 As $\lambda_0(t) = -0.001$

Finally, we use the previous demand rates with $\lambda_0(t) = -0.001$:

1. Constant Rates

We will present the model with demand function as constant rates, $D(x_1, x_2, t) = \gamma_i$. Substituting in the controlled system (23) by the constant demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \tag{36}$$

Solving the controlled system (36) numerically, we get the results displayed in Table 4.

2. Linear Rates

Also, we will present the model with demand function as linear rates, $D(x_1, x_2, t) = \gamma_i + w_i x_i$. Substituting in the controlled system (23) by the linear demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\left(\omega_1 + \theta_1 + a_{12}x_2 + a_{11}x_1\right) - \gamma_1 + \hat{u}_1 \\
\dot{x}_2 &= -x_2\left(\omega_2 + \theta_2 + a_{21}x_2 + a_{22}x_2\right) - \gamma_2 + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\left(\omega_1 + 10\theta_1 + a_{12}x_2 + 2a_{11}x_1\right) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\left(\omega_2 + 10\theta_2 + a_{21}x_1 + 2a_{22}x_2\right) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \tag{37}$$

Solving the controlled system (37) numerically, we get the results displayed in Table 4.

3. Logistic Rates

We present the model with demand function as logistic rates, $D(x_1, x_2, t) = 2x_i(\kappa_i - x_i)$. Substituting in the controlled system (23) by the logistic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\big(2(\kappa_1 - x_1) + \theta_1 + a_{12}x_2 + a_{11}x_1\big) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\big(2(\kappa_2 - x_2) + \theta_2 + a_{21}x_2 + a_{22}x_2\big) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\big(2(\kappa_1 - 2x_1 + a_{11}x_1) + 10\theta_1 + a_{12}x_2\big) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\big(2(\kappa_2 - 2x_2 + a_{22}x_2) + 10\theta_2 + a_{21}x_1\big) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (38)$$

Solving the controlled system (38) numerically, we get the results displayed in Table 4.

4. Periodic Rates

Finally, we will present the model with demand function as periodic rates, $D(x_1, x_2, t) = 1 - b_i \cos(t)$. Substituting in the controlled system (23) by the periodic demand rates, we have the controlled system:

$$\left.\begin{aligned}
\dot{x}_1 &= -x_1\big(\theta_1 + a_{12}x_2 + a_{11}x_1\big) - 1 + b_1\cos(t) + \hat{u}_1 \\
\dot{x}_2 &= -x_2\big(\theta_2 + a_{21}x_2 + a_{22}x_2\big) - 1 + b_2\cos(t) + \hat{u}_2 \\
\dot{\lambda}_1 &= \lambda_1\big(10\theta_1 + a_{12}x_2 + 2a_{11}x_1\big) + a_{21}\lambda_2 x_2 - \theta_1 - 10\theta_{12}x_2 \\
\dot{\lambda}_2 &= \lambda_2\big(10\theta_2 + a_{21}x_1 + 2a_{22}x_2\big) + a_{12}\lambda_1 x_1 - \theta_2 - 10\theta_{12}x_1
\end{aligned}\right\}, \quad (39)$$

Solving the controlled system (39) numerically, we get the results displayed in Table 4.

Table 4. The optimal solution when $\lambda_0(t) = -0.001$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.79	3.07	-1899.14	-1606.32	0.007
Linear	4.70	3.68	-1635.02	-1481.15	0.009
Logistic	162.41	31.83	-1244.57	-1244.58	- 0.0
Periodic	6.66	3.1	-1899.17	-1606.44	0.008

As we see from Table 4, the optimal value of the objective function when $\lambda_0(t) = -0.001$ is achieved in the constant rate (0.007), but all rates are unacceptable because the production rates in all cases are negative values and this unacceptable. So, we can conclude that the co-state value $\lambda_0(t) = -0.001$ is more effective on the production rates and then on the objective function. So, it is better for co-state value $\lambda_0(t)$ to be less than or equal negative one.

4 Sensitivity Analysis

In this section we study the effect of increasing and decreasing the deterioration and spoilage parameters on the optimal solution. So, we can suggest some values in each case for the co-state value $\lambda_0(t)$, and the reset parameters remain without changes.

4.1 The increasing case

In this subsection, we will use the values in Table 1 without changes, but we change the values of deterioration and spoilage parameters as shown in Table 5.

Table 5. The values of deterioration and spoilage parameters (Increasing Case)

θ_1	θ_2	θ_{12}	ψ_1	ψ_2	ψ_{12}
0.10	0.12	0.11	0:10	0.12	0.11

Solving the controlled system (21) using the values in Tables 1 and 5, when the co-state value is $\lambda_0(t) = -10$, we obtain the results in Table 6.

Table 6. The optimal solution when $\lambda_0(t) = -10$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.59	3.13	2.81	1.19	29.35
Linear	4.65	3.68	2.17	0.14	54.49
Logistic	155.4	31.3	-6.09	-1.53	-47.12
Periodic	6.46	3.16	2.72	1.21	32.88

As we see from Table 6, the otimal value of the objective function when $\lambda_0(t) = -10$ is achieved in the constant rate (29.35). The logistic rate is unacceptable because the production rates in this case are negative values, this is reflexed on the value of an objective function (-47.12).

Also, solving the controlled system (22) using the values in Tables 1 and 5, when the co-state value is $\lambda_0(t) = -2$, we obtain the results in Table 7.

Table 7. The optimal solution when $\lambda_0(t) = -2$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.59	3.13	8.97	5.64	5.87
Linear	4.65	3.68	9.43	6.45	10.9
Logistic	157.25	32.11	-47.57	-8.76	-9.35
Periodic	6.46	3.16	8.85	5.68	6.58

As we see from Table 8, the otimal value of the objective function when $\lambda_0(t) = -2$ is achieved in the constant rate (5.87). The logistic rate is unacceptable because the production rates in this case are negative values, this is reflexed on the value of an objective function (-9.35).

Also, solving the controlled system (23) using the values in Tables 1 and 5, when the co-state value is $\lambda_0(t) = -0.001$, we obtain the results in Table 8.

Table 8. The optimal solution when $\lambda_0(t) = -0.001$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.59	3.13	-1465.9	-1312.64	0.003
Linear	4.65	3.68	-1400.11	-1220.5	0.005
Logistic	155.4	31.75	-1130.97	-1130.98	-0.0
Periodic	6.46	3.16	-1465.72	-1313.18	0.003

As we see from Table 8, the otimal value of the objective function when $\lambda_0(t) = -0.001$ is achieved in the constant and periodic rates (0.003) but all rates are unacceptable because the production rates in all cases are negative values.

From Tables 6,7 and 8 we can conclude that the effect of increasing the deterioation and spoilage parameters have little effect on the inventory levels, but high effect on the production rates in all cases. Also the value of the objective function is decreasing when the co-state value $\lambda_0(t)$ is increasing.

4.2 The decreasing case

In this subsection we will use the values in Table 1 without changes, but we change the deterioration and spoilage parameters values as shown in Table 9.

Table 9. The values of deterioration and spoilage parameters (Decreasing Case)

θ_1	θ_2	θ_{12}	ψ_1	ψ_2	ψ_{12}
0.01	0.03	0.02	0.01	0.03	0.02

Solving the controlled system (21) using the values in Tables 1 and 9, when the co-state value is $\lambda_0(t) = -10$, we obtain the results in Table 10.

Table 10. The optimal solution when $\lambda_0(t) = -10$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.96	3.03	18.77	5.7	92.79
Linear	4.74	3.68	12.26	8.07	88.5
Logistic	163.22	31.8	-53.74	-5.7	-5.12
Periodic	6.83	3.04	16.5	6	85.18

As we see from Table 11, the otimal value of the objective function when $\lambda_0(t) = -10$ is achieved in the periodic rate (85.18). Also. The logistic rate is unacceptable because the production rates in this case are negative values, this is reflexed on the value of an objective function (-5.12).

Solving the controlled system (22) using the values in Tables 1 and 9, when the co-state value is $\lambda_0(t) = -2$, we obtain the results in Table 11.

Table 11. The optimal solution when $\lambda_0(t) = -2$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.97	3.01	40.15	38.4	16.92
Linear	4.74	3.68	84.43	29.49	17.7
Logistic	160.81	31.8	118.09	- 14.75	-1.01
Periodic	6.83	3.04	38.17	40.13	17.04

As we see from Table 11, the otimal value of the objective function when $\lambda_0(t) = -2$ is achieved in the constant rate (16.92).

Also. The logistic rate is unacceptable because the production rates in this case are negative values, this is reflexed on the value of an objective function (-1.01).

Solving the controlled system (23) using the values in Tables 1 and 9. when the co-state value is $\lambda_0(t) = -0.001$, we obtain the results in Table 12.

Table 12. The optimal solution when $\lambda_0(t) = -0.001$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.97	3.01	-3852.42	[illegible]	0.008
Linear	4.74	3.68	-4146.81	[illegible]	0.009
Logistic	159.85	31.35	-1745.4	-744.01	-0.0
Periodic	6.83	3.04	-3854.57	-2856.7	0.008

As we see from Table 12, the otimal value of the objective function when $\lambda_0(t) = -0.001$ is achieved in the constant and periodic rates (0.008) but all rates are unacceptable because the production rates in all cases are negative values.

From Tables 10,11 and 12, we can conclude that the effect of decreasing the deterioation and spoilage parameters also have little effect on the inventory levels, but have high effect on the production rates in all cases. Also the value of the objective function is decreasing when the co-state value $\lambda_0(t)$ is increasing, but the value of objective function is increasing in the decreasing case.

Comparing the results that are obtained from using different values for $\lambda_0(t)$ (-10,-2 and 0.001) with these results that obtained when $\lambda_0(t) = -1$, that arise in Table 13 as shown below:

Table 13. The optimal solution when $\lambda_0(t) = -1$

Demand Rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J^*(T)$
Constant	6.79	3.07	17.48	[illegible]	7.49
Linear	4.70	3.69	35.44	[illegible]	8.83
Logistic	161.50	34.80	196.14	[illegible]	-2.38
Periodic	6.66	3.10	17.28	[illegible]	7.68

We can conclude that as co-state value $\lambda_0(t)$ is better when it is less than or equal negative one. The effect is high on the production rates and then the objective function (cause increasing) and little effect on the inventory levels.

5 Conclusions

In this study, we discussed the optimal control problem using the deterioration and spoilage function depending on the alternative quadratic exponential form. We used different values for the co-state variable $\lambda_0(t)$ which has a negative value along the optimal trajectory, and study the effect of increasing and decreasing this value on the optimal solution. Also we explained the sensitivity analysis to study the effect of changing the values of deterioration and spoilage parameters on the optimal solution. Also, we compared the obtained results with the results that obtained when $\lambda_0(t) = -1$ and conduct that the co-state value is more effective when it is less than or equal negative one. Finally, the logistic rate in all cases is unacceptable, because of the production rates are negative values and then the objective function is affected.

References

1. El-Sayed, A., (2017), Alternative quadratic exponential form to o[illegible]nal control problem, Journal of Statistical Science, ISSR, Cairo University, 61,1.

2. El-Sayed, A., Islam, M. and Alzaid, A. (2013). Estimation and te[illegible] measures of association for correlated binary data. Bulletin of the Malaysian Mathematical Scie[illegible] Society 2, 36, 4, 985-1008.

3. Sethi, S. P. and Thompson, G. L. (2000). Optimal control theor[illegible]pplications to Management Science and Economics, Kluwer Academic Publishers, Boston/ D[illegible]t/ London.

4. Zhao, L.P. and Prentice, R. L. (1990). Correlated binary regressio[illegible]sing a quadratic exponential model, Biometrika 77 (3), 642–648.

ملخص بالعربي

دراسة تحليل الحساسية لبعض قيم الحالات المساعدة

توفر هذه الورقة امتدادا لمسألة التحكم الأمثل لنموذج مخزون متعدد الوحدات مع وجود حالات متدهورة باستخدام اللوغاريتم السالب للتدهور ودالة التلف كدالة هدف تعتمد على النموذج الأسي التربيعي البديل.

في هذه الورقة ، يتم استخدام قيم مختلفة للحالات المساعدة بشكل منفصل ، والتي لها قيم سالبة على طول المسار الأمثل ، و يتم دراسة مدى تأثير زيادة أوخفض هذه القيمة على الحل الأمثل ، كما يتم شرح تحليل الحساسية الذي يعكس تأثير التغيرات في قيم معلمات التلف والفساد على الحل الأمثل ، وأخيرا، قارنا النتائج التي تم الحصول عليها مع النتائج التي تم الحصول عليها عندما كانت قيمة الحالة تساوي سالب واحد.

يمكن أن ينظم هذا البحث على النحو التالي: القسم 2 يعرض الشكل الرياضي لمشألة التحكم الأمثل والنظم الخاضعة للرقابة ، القسم 3 يعرض الحل العددي للأنظمة الخاضعة للرقابة مع أنواع مختلفة من معدلات الطلب، القسم 4 يعرض تحليل الحساسية للتدهور ومعلمات التلف، وأخيرا القسم 5 يعرض بعض الاستنتاجات .