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Research article

Statistical Properties and Applications of a New Truncated Zubair-Generalized Family of Distributions

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Abstract: This paper proposed a truncated family of probability distributions named the doubly truncated Zubair-generalized family of truncated distributions. Certain properties of doubly truncated Zubair-Generalized family of truncated distributions are worked on. These properties are the quantile, median, the non-central moments, central moments, order statistics, entropy measures such as Rényi, Shannon, Tsallis entropies, also the mean residual life, mean past lifetime and mean time to failure. The doubly truncated Zubair-Weibull distribution is a special sub-model of the doubly truncated Zubair-generalized family. Some important statistical properties of the doubly truncated Zubair-Weibull distribution are studied, such as the reliability, hazard, reversed hazard and cumulative hazard rate functions. In addition, the moments, quantile, order statistics, entropies and some important special sub-models of the doubly truncated Zubair-Weibull distribution are given. The maximum likelihood estimation approach is applied to estimate the unknown parameters, reliability and hazard rate functions. A simulation study is conducted to evaluate the performance of the maximum likelihood estimates. Two life-time real data sets are applied to show the flexibility and applicability of the proposed model.

Keywords: Doubly truncated Zubair-generalized family; Maximum likelihood method; Markov Chain Monte Carlo method; Moments; Quantile function; doubly truncated Zubair-Weibull distribution.
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1. Introduction

Truncated distributions are conditional distributions that results restricting the domain of original distribution to a smaller one. Truncation in probability distributions may occur in many studies such as reliability and life testing. A truncated distribution occurs when there is no ability to detect or record the events above or below a set threshold or inside or outside a certain range such as the study of plant growth, which cannot be studied before the growth of the plant over the soil, so that the truncated distributions have an important role in various fields such as agriculture, medicine, engineering and physics etc. When the restriction occurs on both sides of the range, it is called doubly truncated. If occurrences are limited to values which lie above a given threshold, the upper (right) truncated distribution arises, if occurrences are limited to values which lie below a given threshold, the lower (left) truncated distribution is obtained.

The importance of truncated distributions appears when using some distributions that have negative domain to estimate the reliability function (rf) that can't be estimated by these distributions such as the logistic, normal and Cauchy distributions.

It was noticed that some studies apply their results to all items of society, while it may be necessary to apply these results to a specific group of society that has the characteristics required for the study, so it may be a need to truncate a part of the society to produce a society that fits the subject of the study where the unimportant part is truncated, especially if this part is outside the scope of the research or outside the researcher interest. In this case, it is not certain that the truncated study society will have the same distribution as the original society. Therefore, the researcher may need to find the probabilistic probability of the truncated data, including the probability density and estimation of parameters for the truncated distributions.

Truncation that occurs on both sides of the range is called doubly truncated. If X is a random variable (rv) from a population with pdf, $f\left(x;\underline{\theta}\right)$ where $x\epsilon(-\infty,\infty)$ and $\underline{\theta}$ is the vector of the parameters, then the general form of the pdf and cdf of the doubly truncated distribution can be written, respectively, as:

$$f_{DT}\left(x;\underline{\theta}\right)=\frac{f\left(x;\underline{\theta}\right)}{F\left(d;\underline{\theta}\right)-F(c;\underline{\theta})},\qquad c<x<d,\quad \underline{\theta}>\underline{0}\,, \tag{1.1}$$

and

$$F_{DT}\left(x;\underline{\theta}\right)=\frac{F\left(x;\underline{\theta}\right)-F\left(c;\underline{\theta}\right)}{F\left(d;\underline{\theta}\right)-F\left(c;\underline{\theta}\right)},\qquad c<x<d,\quad \underline{\theta}>\underline{0}\,, \tag{1.2}$$

where c and d are the points of truncation.

Various truncated distributions have been introduced by several authors such as: Balakrishnan and Aggarwala [14] studied the right truncated generalized half logistic distribution. Al-Yousef [10] studied the doubly truncated Burr distribution. AL-Hussaini et al. [5] obtained the truncated Type I generalized logistic distribution. Nadarajah [19] studied truncated inverted beta distributions. Ateya and AL-Hussaini [12] introduced truncated version of generalized Cauchy distribution suggested by Rider [26] in a special setting.

Raschke [24] proposed the estimator for the right truncation point of the truncated exponential distribution. Singh et al. [31] discussed the truncated Lindley distributions called upper, lower and doubly truncated Lindley distribution. Okasha and Al-qanoo [23] presented inference on the two-parameter doubly truncated gamma distribution. Salih and Taqi [27] discussed the two parameters truncated logistic distribution. Nurminen et al. [22] discussed the technical report gives analytical formulae for the mean and covariance matrix of a multivariate normal distribution with one component truncated from both below and above.

Kizilersu et al. [16] proposed the goodness-of-fit testing for the left-truncated two-parameter Weibull distributions with known truncation point. Najarzadegan et al. [20] studied the family of distributions as an alternative beta-G distribution with flexible hazard rate and greater reliability which is called the truncated Weibull-G distribution. Aydin [13] proposed the five-parameter doubly truncated exponentiated inverse Weibull distribution with known truncation points. Al-Omari [9] introduced the acceptance sampling plan problem based on truncated life tests for Sushila distribution.

Al-Marzouki [6] introduced the truncated Weibull power Lomax distribution. Akbarinasab et al. [4] studied the truncated log-logistic family of distributions and presented four baseline models of this family to generate special models which are exponential, Weibull, gamma and generalized exponential distributions.

Khalaf and Al-Kadim [15] presented the truncated Rayleigh Pareto distribution. Abid and Jani [1] proposed the properties of doubly truncated generalized gamma distribution and doubly truncated generalized inverse Weibull distribution. Al-Noor and Hadi [7] considered properties and applications of the truncated exponential Marshall Olkin Weibull distribution.

Al-Noor and Hilal [8] presented a truncated distribution as a sub-model with three parameters called truncated exponential Topp Leone exponential distribution. Neamah and Qasim [21] introduced the left-truncated Gumbel distribution within the period $(0, \infty)$. Abid and Khadhim [2] studied the doubly truncated exponentiated inverted gamma distribution. Shrahili and Elbatal [30] introduced the truncated Cauchy power odd Fréchet-G family of distributions. Turjoman and Neamah [34] derived three parameters of truncated distribution, called doubly truncated Weibull Pareto distribution.

Tahir and Cordeiro [32] proposed the complementary exponentiated-G Poisson family of distributions with the following probability density function (pdf):

$$f\left(x;\alpha,\lambda,\underline{\xi}\right)=\frac{\lambda\alpha g\left(x;\underline{\xi}\right)G^{\lambda-1}\left(x;\underline{\xi}\right)e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^{2}}}{e^{\alpha}-1},x\epsilon\mathbb{R},\quad \alpha,\lambda,\underline{\xi}>\underline{0}. \tag{1.3}$$

Ahmed [3] introduced the Zubair- Generalized (Z-G) family of distributions as a special case of the complementary exponentiated-G Poisson family of distributions when $\lambda = 2$ in (1.3). Thus, the pdf, cumulative distribution function (cdf) and rf of Z-G family of distributions are given, respectively, by:

$$f\left(x;\alpha,\underline{\xi}\right)=\frac{2\alpha g(x;\underline{\xi})G(x;\underline{\xi})e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^{2}}}{e^{\alpha}-1},\qquad \alpha,\underline{\xi}>\underline{0},\quad x\epsilon\mathbb{R}\,, \tag{1.4}$$

$$F\left(x;\alpha,\underline{\xi}\right)=\frac{e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^{2}}-1}{e^{\alpha}-1},\qquad \alpha,\underline{\xi}>0\quad ,x\,\epsilon\,\mathbb{R}, \tag{1.5}$$

and

$$R\left(x;\alpha,\underline{\xi}\right)=\frac{e^{\alpha}-e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^{2}}}{e^{\alpha}-1},\qquad \alpha,\underline{\xi}>0,\qquad x\epsilon\mathbb{R}, \tag{1.6}$$

where $G\left(x;\underline{\xi}\right)$ and $g\left(x;\underline{\xi}\right)$ are the cdf and pdf of the baseline model.

This paper aims to introduce a new family of truncated distributions with flexible hazard rate function (hrf) and high reliability which is named truncated Zubair– Generalized (TZ-G) family with truncated Zubair-Weibull (TZ-W) distribution as a sub-model of this family. The new family represents a wider range of behaviors in real-life data compared to existing families, particularly situations involve truncated data.

Some reasons of using TZ-G family of distributions are:

- The TZ-G family is highly flexible in modeling various types of data, including skewed, heavy-tailed, and multimodal data.
- The TZ-G family often extends or generalizes existing distributions, allowing for more robust modeling. It can have different shapes for hrfs and pdfs, making it useful in survival analysis and reliability testing.
- These distributions often provide better goodness-of-fit measures (such as AIC, BIC) compared to other models when applied to real-life data sets. These distributions are preferred for modeling complex data.
- The TZ-G family is used in various applications, including risk analysis, lifetime modeling, reliability testing, and failure time data, especially where other distributions might not provide an acceptable fit.
- Many models within the TZ-G family provide better estimation methods such as maximum likelihood (ML) and Bayesian for the unknown parameters, which can improve the performance of inferential methods and prediction.

The advantages of using the TZ-G family: (1.1) it allows for a wide range of shapes for the pdf and hrf. This flexibility makes it flexible for modeling various real-life data, particularly data with unique skewness or kurtosis properties that cannot be easily represented by traditional distributions. It can include both light-tailed and heavy-tailed distributions. (1.2) TZ-G family offers improved modeling of extreme values or outliers. This makes it useful in reliability studies and risk analysis where tail behavior (either high or low) is important. (1.3) The TZ-G family is a strong competitor for use in both real-world and simulated data sets because of its flexibility to achieve a closer fit to various types of data, through comparing measures of information criteria. (1.4) The TZ-G family includes several sub-models (e.g. TZ-W), each of which adds specific characteristics to the overall family, allowing researchers to select models that best fit their data without having to develop entirely new models. (1.5) The TZ-G family can be applied across different fields, such as reliability engineering, survival analysis and actuarial science. Its ability to model both increasing and decreasing hazard rate functions makes it flexible for analyzing distinct real-world phenomena like lifetimes of mechanical components or biological survival times.

The TZ-G family and its sub-models can effectively be utilized for both theoretical and practical applications across various fields. It can be employed for theoretical applications through statistical modeling, estimation theory and simulation studies to explore the properties of estimators, hypothesis testing and model fitting. It can be used for practical applications in economics, finance, engineering and quality control, biostatistics and environmental science.

This paper targets to fill the following gaps:

- The TZ-G family and TZ-W distribution as a sub-model perform effectively in modeling real-world data with truncation due to several key features that distinguish it from traditional distributions since truncation allows the TZ-W distribution to manage extreme values better than traditional Weibull or Zubair-generalized models.
- The TZ-G family and TZ-W distribution as a sub-model exhibit greater shape flexibility compared to its traditional distributions, allowing for a variety of hrfs including increasing, decreasing, or bathtub-shaped hrfs.
- Moreover, empirical studies show that the TZ-G family and TZ-W distribution as a sub-model provide a better fit for truncated data than traditional distributions, especially when comparing measures of information criteria. The TZ-W distribution's ability to handle truncation makes it ideal for analyzing censored or truncated datasets, commonly found in survival analysis, reliability testing, and medical studies.

This paper is arranged as follows: The doubly TZ-G (DTZ-G) family of distributions is presented in Section 2. In Section 3, some statistical properties of DTZ-G family of distributions are studied. The ML method is used to estimate the unknown parameters of DTZ-G family of distributions in Section 4. In Section 5, the DTZ-Weibull (DTZ-W) distribution and several sub models are proposed. In Section 6, some statistical properties of the DTZ-W distribution are derived. In Section 7, the ML estimator of the unknown parameters, rf and hrf of the DTZ-W distribution based on a complete sample are obtained. A simulation study is conducted to evaluate the performance of the ML estimates and concluding remarks in Section 8. Two life-time real data sets are presented in Section 9.

2. The Doubly Truncated Zubair-G Family of Distributions

Let X is a rv having the DTZ-G family of distributions with parameter vector $\underline{\psi} = (\alpha,\ \underline{\xi}, c, d)$, and taking values in the interval [c, d]. Substituting (1.4), (1.5) in (1.1), then the pdf of the DTZ-G family is

$$f_{TZG}\left(x;\underline{\psi}\right) = \frac{2\alpha g(x;\underline{\xi})G(x;\underline{\xi})e^{\alpha\left[G(x;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}},\quad c < x < d,\ \underline{\psi} > \underline{0}\,. \tag{2.1}$$

The cdf, rf, hrf, reversed hrf (rhrf) and cumulative hrf (chrf) of the DTZ-G family of distributions are given, respectively, by:

$$F_{TZG}\left(x;\underline{\psi}\right) = \frac{e^{\alpha\left[G(x;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}},\qquad c < x < d,\ \ \underline{\psi} > \underline{0}, \tag{2.2}$$

$$R_{TZG}\left(x;\underline{\psi}\right) = 1 - F_{TZG}\left(x;\underline{\psi}\right) = \frac{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}},\quad c < x < d,\ \ \underline{\psi} > \underline{0}, \tag{2.3}$$

$$h_{TZG}\left(x;\underline{\psi}\right) = \frac{f_{TZG}\left(x;\underline{\psi}\right)}{R_{TZG}\left(x;\underline{\psi}\right)} = \frac{2\alpha g(x;\underline{\xi})G(x;\underline{\xi})e^{\alpha\left[G(x;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(x;\underline{\xi})\right]^2}},\qquad c < x < d,\ \ \underline{\psi} > \underline{0}\,, \tag{2.4}$$

$$r_{TZG}\left(x;\underline{\psi}\right)=\frac{f_{TZG}\left(x;\underline{\psi}\right)}{F_{TZG}\left(x;\underline{\psi}\right)}=\frac{2\alpha g(x;\underline{\xi})G(x;\underline{\xi})e^{\alpha[G(x;\underline{\xi})]^2}}{e^{\alpha[G(x;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}},\quad c<x<d,\ \ \underline{\psi}>\underline{0}, \tag{2.5}$$

and

$$H_{TZG}\left(x;\underline{\psi}\right)=-\ln R_{TZG}\left(x;\underline{\psi}\right)=-\ln\left(\frac{e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(x;\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}}\right),\quad c<x<d,\ \ \underline{\psi}>\underline{0}. \tag{2.6}$$

3. Some Statistical Properties of Doubly Truncated Zubair-G Family of Distributions

This section presented some statistical properties of the DTZ-G family of truncated distributions such as: the quantile function, median, central and non-central moments, order statistics, entropy measures, mean residual life (MRL), mean past lifetime (MPL) and mean time to failure (MTTF).

3.1. Quantile and median

The u^{th} quantile function, say $x_u = Q(u)$ for a continuous rv X has cdf, $F(x)$, is defined as:

$$x_{u_{TZG}}=F^{-1}_{TZG}(u)=G^{-1}_{TZG}\left(\frac{\ln\{u\left[e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}\right]+e^{\alpha[G(c;\underline{\xi})]^2}\}}{\alpha}\right)^{\frac{1}{2}}. \tag{3.1}$$

The median can be obtained by substituting $u = 0.5$ in (3.1) as follows:

$$x_{0.5_{TZG}}=G^{-1}_{TZG}\left(\frac{\ln\{0.5\left[e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}\right]+e^{\alpha[G(c;\underline{\xi})]^2}\}}{\alpha}\right)^{\frac{1}{2}}. \tag{3.2}$$

3.2. Moments

3.2.1. Non-central moments

The r^{th} non-central moment of the DTZ-G family of distributions is given by:

$$\begin{aligned}\mu'_{r_{TZG}}&=E\left(X^r\right)=\int_x x^r f_{TZG}\left(x;\underline{\psi}\right)dx\\&=\frac{2\alpha}{e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}}\int_x x^r g\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha[G(x;\underline{\xi})]^2}dx,\qquad r=1,2,3,\ldots.\end{aligned} \tag{3.3}$$

The mean of the DTZ-G family of distributions is given by:

$$\mu'_{1_{TZG}}\equiv\mu=\frac{2\alpha}{e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}}\int_x xg\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha[G(x;\underline{\xi})]^2}dx. \tag{3.4}$$

The second non-central moment of the DTZ-G family of distributions is given by:

$$\mu'_{2_{TZG}}=\frac{2\alpha}{e^{\alpha[G(d;\underline{\xi})]^2}-e^{\alpha[G(c;\underline{\xi})]^2}}\int_x x^2 g\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha[G(x;\underline{\xi})]^2}dx. \tag{3.5}$$

The variance of the DTZ-G family of distributions can be defined as

$$V_{TZG}(X) = \frac{2\alpha}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \int_x x^2 g(x;\underline{\xi}) G(x;\underline{\xi}) e^{\alpha[G(x;\underline{\xi})]^2} dx - \left[\frac{2\alpha}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \int_x x g(x;\underline{\xi}) G(x;\underline{\xi}) e^{\alpha[G(x;\underline{\xi})]^2} dx \right]^2. \quad (3.6)$$

3.2.2. Central moments

The central moments of the DTZ-G family of distributions can be defined as:

$$\mu_{r_{TZG}} = E(x-\mu)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j (\mu)^j \grave{\mu}_{r-j}, \qquad r = 1, 2, 3, \ldots . \quad (3.7)$$

3.3. Order statistics

Let $x_1,\ x_2, \ldots,\ x_n$ be a random sample of size n obtained from the DTZ-G family of distributions with $f_{TZG}(x;\underline{\psi})$ and $F_{TZG}(x;\underline{\psi})$. Suppose $x_{(1)} \le\ x_{(2)} \le \ldots\ \le x_{(n)}$ then, the pdf of the r^{th} order statistics is given by

$$f_{r:n_{TZG}}(x;\underline{\psi}) = \frac{n!}{(r-1)!\,(n-r)!} f_{TZG}(x;\underline{\psi}) \left[F_{TZG}(x;\underline{\psi})\right]^{r-1} \left[1 - F_{TZG}(x;\underline{\psi})\right]^{n-r}, \quad c < x_{(r)} < d, \quad (3.8)$$

where $f_{TZG}(x;\underline{\psi})$ and $F_{TZG}(x;\underline{\psi})$ are the pdf and cdf of the DTZ-G family of distributions, then the pdf of the r^{th} order statistics of the DTZ-G family of distributions is

$$f_{r:n_{TZG}}(x;\underline{\psi}) = \frac{n!}{(r-1)!\,(n-r)!} \frac{2\alpha g(x_{(r)};\underline{\xi}) G(x_{(r)};\underline{\xi}) e^{\alpha[G(x_{(r)};\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \times \left(\frac{e^{\alpha[G(x_{(r)};\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right)^{r-1} \left[\frac{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(x_{(r)};\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right]^{n-r}, \quad c < x_{(r)} < d. \quad (3.9)$$

Special cases

When $r = 1$, then the pdf of the smallest order statistics can be written as follows:

$$f_{1:n_{TZG}}(x;\underline{\psi}) = n f_{TZG}(x;\underline{\psi}) \left[1 - F_{TZG}(x;\underline{\psi})\right]^{n-1} = n \left(\frac{2\alpha g(x_{(1)};\underline{\xi}) G(x_{(1)};\underline{\xi}) e^{\alpha[G(x_{(1)};\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right) \left[\frac{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(x_{(1)};\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right]^{n-1}, \quad c < x_{(1)} < d. \quad (3.10)$$

When $r = n$, then the pdf of the largest order statistics can be written as follows:

$$f_{n:n_{TZG}}(x;\underline{\psi}) = n f_{TZG}(x;\underline{\psi}) \left[F_{TZG}(x;\underline{\psi})\right]^{n-1} = n \left(\frac{2\alpha g(x_{(n)};\underline{\xi}) G(x_{(n)};\underline{\xi}) e^{\alpha[G(x;\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right) \left[\frac{e^{\alpha[G(x_{(n)};\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}}{e^{\alpha[G(d;\underline{\xi})]^2} - e^{\alpha[G(c;\underline{\xi})]^2}} \right]^{n-1}, \quad c < x_{(n)} < d. \quad (3.11)$$

3.4. Entropy measures

Rényi entropy

Entropy usually measures variation or uncertainty of a rv X. Also, it measures the randomness of rv. Entropy is extensively applied in physics and molecular imaging of tumors. The Rényi entropy of order δ, where $\delta > 0$ and $\delta \neq 1$ [see Rényi [25]]. It can be defined as

$$En_{\delta_{TZG}}(x) = \frac{1}{(1-\delta)} ln\left(\int_0^\infty \left[f_{TZG}(x;\underline{\psi})\right]^\delta dx\right), \qquad \delta > 0 \text{ and } \delta \neq 1. \tag{3.12}$$

When $X \sim$ DTZ-G$(\underline{\psi})$ family of distributions, Rényi entropy can be expressed as

$$En_{\delta_{TZG}}(x) = \frac{1}{(1-\delta)} ln\left(\left[\int_c^d \left(\frac{2\alpha g(x;\underline{\xi})G(x;\underline{\xi})e^{\alpha\left[G(x;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}}\right)^\delta\right] dx\right), \quad \delta > 0 \text{ and } \delta \neq 1. \tag{3.13}$$

Shannon entropy

Shannon entropy is a concept introduced by Shannon [29]. The Shannon entropy is a special case of the Rényi entropy when $\delta \to 1$. Mathematically, Shannon entropy, H is defined as an expectation of $E[-ln(f_{TZG}\left(x;\underline{\psi}\right))]$, which is equivalent to,

$$H_{TZG} = -\int_0^\infty f_{TZG}\left(x;\underline{\psi}\right) ln f_{TZG}\left(x;\underline{\psi}\right)\ dx. \tag{3.14}$$

If X~ DTZ-G$(\underline{\psi})$ family of truncated distributions, then the Shannon entropy is

$$H_{TZG} = -\int_c^d \left[\left(\frac{2\alpha g\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}}\right) ln\left(\frac{2\alpha g\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}}\right)\right] dx. \tag{3.15}$$

Tsallis entropy

The Tsallis entropy was introduced by Tsallis [33]. Tsallis entropy is a generalization of the standard Boltzmann-Gibbs entropy. The Tsallis entropy is defined by

$$T_{\rho_{TZG}}(x) = \left(\frac{1}{\rho-1}\right)\left(1 - \int_0^\infty \left[f_{TZG}(x;\underline{\psi})\right]^\rho\right) dx, \qquad \rho > 0 \text{ and } \rho \neq 1. \tag{3.16}$$

If X~ DTZ-G$(\underline{\psi})$ family of truncated distributions, then the Tsallis entropy is given by

$$T_{\rho_{TZG}}(x) = \left(\frac{1}{\rho-1}\right)\left[1 - \int_c^d \left(\frac{2\alpha g\left(x;\underline{\xi}\right)G\left(x;\underline{\xi}\right)e^{\alpha\left[G\left(x;\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}}\right) dx\right]. \tag{3.17}$$

3.5. Mean residual life, Mean past lifetime and Mean time to failure

The MRL is the expected remaining life, $X - t$, given that the item has survived to time t. Let X be a non-negative rv (usually representing the lifetime of some engineering or biological component) with rf; $R(t) = P(x \geq t)$.

If $E(X) < \infty$, the MRL of X is defined by

$$
\begin{aligned}
m_{TZG}(t) = E(X - t|\ X \geq t) &= \frac{1}{R_{TZG}(t;\underline{\psi})} \int_t^d R_{TZG}\left(x;\underline{\psi}\right) dx \\
&= \frac{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(t;\underline{\xi})\right]^2}} \int_t^d \left(\frac{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(x;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}} \right) dx.
\end{aligned}
\tag{3.18}
$$

The MPL in a real-life situation, where systems often are not monitored continuously, one might be interested in getting inference more about the history of the system, e.g. when the individual components have failed. Assume that a component with lifetime X has failed at or some time before t, $t > 0$. Consider the conditional rv, $t - X|X \leq t$. This conditional rv shows, in fact, the time elapsed from the failure of the component given that its lifetime is less than or equal to t. Hence, the MPL of the component denoted by $m^*(t)$, is defined by

$$
\begin{aligned}
m^*_{TZG}(t) = E(t - X|\ X \leq t) &= \frac{\int_c^t F_{TZG}(x;\underline{\psi}) dx}{F_{TZG}(t;\underline{\psi})} \\
&= \frac{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}}{e^{\alpha\left[G(t;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}} \int_c^t \frac{e^{\alpha\left[G(x;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}} dx,
\end{aligned}
\tag{3.19}
$$

[see Asadi [11]].

MTTF is the expected (average) time that the system is likely to operate successfully before a failure occurs. [see Shafiq et al. [28]]. The MTTF of the DTZ-G family of distributions is as follows:

$$
MTTF_{TZG} = \int_c^d R_{TZG}\left(t;\underline{\psi}\right) dt = \int_c^d \frac{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(t;\underline{\xi})\right]^2}}{e^{\alpha\left[G(d;\underline{\xi})\right]^2} - e^{\alpha\left[G(c;\underline{\xi})\right]^2}} dt. \tag{3.20}
$$

4. Maximum likelihood estimation for the doubly truncated Zubair-G family

Let $x_1, \ x_2, \ \ldots, \ x_n$ be a random sample from the DTZ-G $\left(\underline{\psi}\right)$ with pdf $f_{TZG}\left(x;\underline{\psi}\right)$. The likelihood function of the DTZ-G $\left(\underline{\psi}\right)$ is

$$
L_{TZG}\left(\underline{\psi};\underline{x}\right) \propto \prod_{i=1}^{n} f_{TZG}\left(x_{(i)},\underline{\psi}\right) = \prod_{i=1}^{n} \frac{2\alpha g\left(x_{(i)};\underline{\xi}\right) G\left(x_{(i)};\underline{\xi}\right) e^{\alpha\left[G\left(x_{(i)};\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}},
$$

where $\underline{\psi} = \alpha, \underline{\xi}, c$ and d.

The log likelihood function can be rewritten as

$$\begin{aligned} l_{TZG} \equiv lnL_{TZG}\left(\underline{\psi};\underline{x}\right) &= ln\prod_{i=1}^{n}\frac{2\alpha g\left(x_{(i)};\underline{\xi}\right)G\left(x_{(i)};\underline{\xi}\right)e^{\alpha\left[G\left(x_{(i)};\underline{\xi}\right)\right]^2}}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}-e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}} \\ &= \sum_{i=1}^{n} ln\left[2\alpha g\left(x_{(i)};\underline{\xi}\right)G\left(x_{(i)};\underline{\xi}\right)e^{\alpha\left[G\left(x_{(i)};\underline{\xi}\right)\right]^2}\right] - nln\left[[e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}-e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}\right] \\ &= \sum_{i=1}^{n}\left\{ln\left(2\right) + ln(\alpha) + ln\left[g\left(x_{(i)};\underline{\xi}\right)\right] + ln\left[G\left(x_{(i)};\underline{\xi}\right)\right] + \alpha\left[G\left(x_{(i)};\underline{\xi}\right)\right]^2\right\} \\ &\qquad - nln\left[e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}-e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}\right]. \end{aligned} \tag{4.1}$$

The ML estimators of the truncation points c and d are $\hat{c}_{ML} = \arg\max L_{TZG}\left(\underline{\psi};\underline{x}\right) = x_{(1)}$, and $\hat{d}_{ML} = \arg\max L_{TZG}\left(\underline{\psi};\underline{x}\right) = x_{(n)}$. The ML estimators of the parameters α and $\underline{\xi}$ can be obtained by solving the following log-likelihood equations:

$$\frac{\partial l_{TZG}}{\partial\alpha} = \sum_{i=1}^{n}\left\{\frac{1}{\alpha}+\left[G\left(x_{(i)};\underline{\xi}\right)\right]^2\right\} - \frac{n}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}-e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}}\left\{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}\left[G\left(d;\underline{\xi}\right)\right]^2 - e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}\left[G\left(c;\underline{\xi}\right)\right]^2\right\}. \tag{4.2}$$

$$\begin{aligned} \frac{\partial l_{TZG}}{\partial\underline{\xi}} &= \sum_{i=1}^{n}\left\{\frac{g`\left(x_{(i)};\underline{\xi}\right)}{g\left(x_{(i)};\underline{\xi}\right)}+\frac{G`\left(x_{(i)};\underline{\xi}\right)}{G\left(x_{(i)};\underline{\xi}\right)}+2\alpha\left[G\left(x_{(i)};\underline{\xi}\right)\right]\left[G`\left(x_{(i)};\underline{\xi}\right)\right]\right\} \\ &- \frac{n}{e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2}-e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}}\left\{2\alpha\left[G\left(d;\underline{\xi}\right)\right]\left[G`\left(d;\underline{\xi}\right)\right]e^{\alpha\left[G\left(d;\underline{\xi}\right)\right]^2} - 2\alpha\left[G\left(c;\underline{\xi}\right)\right]\left[G`\left(c;\underline{\xi}\right)\right]e^{\alpha\left[G\left(c;\underline{\xi}\right)\right]^2}\right\} \end{aligned} \tag{4.3}$$

By equating these partial derivatives in (4.2) and (4.3) with zeros and solving numerically, hence the ML estimators of α and $\underline{\xi}$ can be obtained. One can apply the invariance property of the ML estimators to obtain the ML estimators of the rf and hrf by replacing the parameters $\underline{\psi} = (\alpha, \underline{\xi})$ in (2.3) and (2.4) with their ML estimators. Then the ML estimators of $R_{TZG}\left(x;\underline{\psi}\right)$ and $h_{TZG}\left(x;\underline{\psi}\right)$ are given, respectively, by:

$$\hat{R}_{TZG}\left(x;\underline{\widehat{\psi}}\right) = \frac{e^{\widehat{\alpha}\left[G\left(\hat{d};\underline{\widehat{\xi}}\right)\right]^2}-e^{\widehat{\alpha}\left[G\left(x;\underline{\widehat{\xi}}\right)\right]^2}}{e^{\widehat{\alpha}\left[G\left(\hat{d};\underline{\widehat{\xi}}\right)\right]^2}-e^{\widehat{\alpha}\left[G\left(\hat{c};\underline{\widehat{\xi}}\right)\right]^2}}, \quad \hat{c} < x < \hat{d}, \tag{4.4}$$

and

$$\hat{h}_{TZG}\left(x;\underline{\widehat{\psi}}\right) = \frac{2\widehat{\alpha}g(x;\underline{\widehat{\xi}})G(x;\underline{\widehat{\xi}})e^{\widehat{\alpha}\left[G\left(x;\underline{\widehat{\xi}}\right)\right]^2}}{e^{\widehat{\alpha}\left[G\left(\hat{d};\underline{\widehat{\xi}}\right)\right]^2}-e^{\widehat{\alpha}\left[G\left(x;\underline{\widehat{\xi}}\right)\right]^2}}, \qquad \hat{c} < x < \hat{d}. \tag{4.5}$$

5. Doubly Truncated Zubair-Weibull Distribution

A special sub-model of the DTZ-G family of truncated distributions, called DTZ-Weibull (DTZ-W) distribution will be considered and studied. The cdf and pdf of the Weibull distribution are given by

$$G\left(x;\ \underline{\xi}\right) = 1 - e^{-\gamma x^{\theta}}, \qquad x > 0,\ \ \underline{\xi} > \underline{0}\,, \tag{5.1}$$

and

$$g\left(x;\ \underline{\xi}\right) = \theta\gamma x^{\theta-1}\ e^{-\gamma x^{\theta}}, \quad x > 0,\ \ \underline{\xi} > \underline{0}, \tag{5.2}$$

where $\underline{\xi} = (\theta, \gamma)$.

Then, the pdf and cdf of the DTZ-W distribution can be expressed as given below. The following functions are the main characteristic functions of the DTZ-W distribution.

$$f_{TZW}\left(x;\underline{\psi}\right) = \frac{2\alpha\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}(1-e^{-\gamma x^{\theta}})e^{\alpha(1-e^{-\gamma x^{\theta}})^2}}{e^{\alpha(1-e^{-\gamma d^{\theta}})^2}-e^{\alpha(1-e^{-\gamma c^{\theta}})^2}}, \quad 0 < c < x < d < \infty,\ \ \underline{\psi} > \underline{0}, \tag{5.3}$$

and

$$F_{TZW}\left(x;\underline{\psi}\right) = \frac{e^{\alpha(1-e^{-\gamma x^{\theta}})^2}-e^{\alpha(1-e^{-\gamma c^{\theta}})^2}}{e^{\alpha(1-e^{-\gamma d^{\theta}})^2}-e^{\alpha(1-e^{-\gamma c^{\theta}})^2}}, \quad 0 < c < x < d < \infty,\ \underline{\psi} > \underline{0}. \tag{5.4}$$

where $\underline{\psi} = \left(\ \alpha,\ \underline{\xi}, c, d\right)$ *and* $\underline{\xi} = (\theta, \gamma)$.

The rf, hrf, rhrf and chrf of the DTZ-W distribution are given, respectively, by:

$$R_{TZW}\left(x;\underline{\psi}\right) = \frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}, \quad 0 < c < x < d < \infty,\ \underline{\psi} > \underline{0}, \tag{5.5}$$

$$h_{TZW}\left(x;\underline{\psi}\right) = \frac{2\alpha\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}(1-e^{-\gamma x^{\theta}})e^{\alpha(1-e^{-\gamma x^{\theta}})^2}}{e^{\alpha(1-e^{-\gamma d^{\theta}})^2}-e^{\alpha(1-e^{-\gamma x^{\theta}})^2}}, \quad 0 < c < x < d < \infty, \underline{\psi} > \underline{0}, \tag{5.6}$$

$$r_{TZW}\left(x;\underline{\psi}\right) = \frac{2\alpha\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}(1-e^{-\gamma x^{\theta}})e^{\alpha(1-e^{-\gamma x^{\theta}})^2}}{e^{\alpha(1-e^{-\gamma x^{\theta}})^2}-e^{\alpha(1-e^{-\gamma c^{\theta}})^2}}, \quad 0 < c < x < d < \infty,\ \underline{\psi} > \underline{0}, \tag{5.7}$$

and

$$H_{TZW}\left(x;\underline{\psi}\right) = -ln\left[\frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right], \quad 0 < c < x < d < \infty, \underline{\psi} > \underline{0}. \tag{5.8}$$

Some important special sub-models of DTZ-W distribution are given in Table 1.

Table 1. Special sub-models of the DTZ-W distribution

Parameter	Model	pdf
$\gamma = 1$	Doubly truncated Zubair-standard Weibull	$f_1(x;\alpha,\theta) = \frac{2\alpha\theta x^{\theta-1}e^{-x^{\theta}}\left(1-e^{-x^{\theta}}\right)e^{\alpha\left(1-e^{-x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-c^{\theta}}\right)^2}},\quad c<x<d,\ \alpha,\theta>0.$
$\theta = 1$	Doubly truncated Zubair-exponential	$f_2(x;\alpha,\gamma) = \frac{2\alpha\gamma e^{-\gamma x}\left(1-e^{-\gamma x}\right)e^{\alpha(1-e^{-\gamma x})^2}}{e^{\alpha\left(1-e^{-\gamma d}\right)^2}-e^{\alpha(1-e^{-\gamma c})^2}},\quad c<x<d,\ \alpha,\gamma,>0.$
$\theta = 2$	Doubly truncated Zubair-Rayleigh	$f_3(x;\alpha,\gamma) = \frac{4\alpha\gamma x e^{-\gamma x^2}\left(1-e^{-\gamma x^2}\right)e^{\alpha\left(1-e^{-\gamma x^2}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^2}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^2}\right)^2}},\quad c<x<d,\ \alpha,\gamma>0.$
$\theta = 1$ and $\gamma = 1$	Doubly truncated Zubair-standard exponential	$f_4(x;\alpha) = \frac{2\alpha e^{-x}(1-e^{-x})e^{\alpha(1-e^{-x})^2}}{e^{\alpha(1-e^{-d})^2}-e^{\alpha(1-e^{-c})^2}},\quad c<x<d,\ \alpha>0.$
$\theta = 2$ and $\gamma = 1$	Doubly truncated Zubair-standard Rayleigh	$f_5(x;\alpha) = \frac{4\alpha x e^{-x^2}(1-e^{-x^2})e^{\alpha(1-e^{-x^2})^2}}{e^{\alpha(1-e^{-d^2})^2}-e^{\alpha(1-e^{-c^2})^2}},\quad c<x<d,\ \alpha>0.$
$c > 0$ and $d \to \infty$	Left truncated Zubair-Weibull	$f_6\left(x;\underline{\psi}\right) = \frac{2\alpha\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}\left(1-e^{-\gamma x^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}},\quad c<x<\infty,\ \alpha,\xi>0.$
$c > 0$, $d \to \infty$ and $\gamma = 1$	Left truncated Zubair-Standard Weibull	$f_7(x;\alpha,\theta) = \frac{2\alpha\theta x^{\theta-1}e^{-x^{\theta}}\left(1-e^{-x^{\theta}}\right)e^{\alpha\left(1-e^{-x^{\theta}}\right)^2}}{e^{\alpha}-e^{\alpha\left(1-e^{-c^{\theta}}\right)^2}},\quad c<x<\infty,\ \alpha,\theta>0.$
$c > 0$, $d \to \infty$ and $\theta = 1$	Left truncated Zubair-exponential	$f_8(x;\alpha,\gamma) = \frac{2\alpha\gamma e^{-\gamma x}\left(1-e^{-\gamma x}\right)e^{\alpha(1-e^{-\gamma x})^2}}{e^{\alpha}-e^{\alpha(1-e^{-\gamma c})^2}},\quad c<x<\infty,\ \alpha,\gamma>0.$
$c > 0$, $d \to \infty$ and $\theta = 2$	Left truncated Zubair-Rayleigh	$f_9(x;\alpha,\gamma) = \frac{4\alpha\gamma x e^{-\gamma x^2}\left(1-e^{-\gamma x^2}\right)e^{\alpha\left(1-e^{-\gamma x^2}\right)^2}}{e^{\alpha}-e^{\alpha\left(1-e^{-\gamma c^2}\right)^2}},\quad c<x<\infty,\ \alpha,\gamma>0.$
$c > 0$, $d \to \infty$, $\theta = 1$ and $\gamma = 1$	Left truncated Zubair-Standard exponential	$f_{10}(x;\alpha) = \frac{2\alpha e^{-x}(1-e^{-x})e^{\alpha(1-e^{-x})^2}}{e^{\alpha}-e^{\alpha(1-e^{-c})^2}},\quad c<x<\infty,\ \alpha>0.$

$c > 0$, $d \to \infty$, $\theta = 2$ and $\gamma = 1$	Left truncated Zubair-Standard Rayleigh	$f_{11}(x;\alpha) = \frac{4\alpha x e^{-x^2}(1-e^{-x^2})e^{\alpha(1-e^{-x^2})^2}}{e^{\alpha}-e^{\alpha(1-e^{-c^2})^2}}$,	$c < x < \infty$, $\alpha > 0$.
$c = 0$ and $d < \infty$	Right truncated Zubair-Weibull	$f_{12}(x;\underline{\psi}) = \frac{2\alpha\theta\gamma x^{\theta-1}e^{-\gamma x^{\theta}}\left(1-e^{-\gamma x^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-1}$,	$0 < x < d$, $\alpha, \xi > 0$.
$c = 0$, $d < \infty$ and $\gamma = 1$	Right truncated Zubair-Standard Weibull	$f_{13}(x;\alpha,\theta) = \frac{2\alpha\theta x^{\theta-1}e^{-x^{\theta}}\left(1-e^{-x^{\theta}}\right)e^{\alpha\left(1-e^{-x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-d^{\theta}}\right)^2}-1}$,	$0 < x < d$, $\alpha, \theta > 0$.
$c = 0$, $d < \infty$ and $\theta = 1$	Right truncated Zubair-expontial	$f_{14}(x;\alpha,\gamma) = \frac{2\alpha\gamma e^{-\gamma x}\left(1-e^{-\gamma x}\right)e^{\alpha(1-e^{-\gamma x})^2}}{e^{\alpha\left(1-e^{-\gamma d}\right)^2}-1}$,	$0 < x < d$, $\alpha, \gamma > 0$.
$c = 0$, $d < \infty$ and $\theta = 2$	Right truncated Zubair-Rayleigh	$f_{15}(x;\alpha,\gamma) = \frac{4\alpha\gamma x e^{-\gamma x^2}\left(1-e^{-\gamma x^2}\right)e^{\alpha\left(1-e^{-\gamma x^2}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^2}\right)^2}-1}$,	$0 < x < d$, $\alpha, \gamma > 0$.
$c = 0, d < \infty$, $\theta = 1$ and $\gamma = 1$	Right truncated Zubair-Standard exponential	$f_{16}(x;\alpha) = \frac{2\alpha e^{-x}(1-e^{-x})e^{\alpha(1-e^{-x})^2}}{e^{\alpha(1-e^{-d})^2}-1}$,	$0 < x < d,\ \alpha > 0$.
$c = 0, d < \infty$, $\theta = 2$ and $\gamma = 1$	Right truncated Zubair-Standard Rayleigh	$f_{17}(x;\alpha) = \frac{4\alpha x e^{-x^2}(1-e^{-x^2})e^{\alpha(1-e^{-x^2})^2}}{e^{\alpha(1-e^{-d^2})^2}-1}$,	$0 < x < d,\ \alpha > 0$.

5.1. *Graphical description*

The plots of the pdf and hrf of the DTZ-W distribution for different values of the parameters are given in Figures 1 and 2, respectively. Figure 1 shows plots of the pdf for various values of the parameter. The pdf can take different shapes such as unimodal, increasing and decreasing. Figure 2 displays plots of the hrf for some values of the parameters. The hrf represents major shapes such as increasing, bathtub and unimodal.

6. Some Properties of the Doubly Truncated Zubair-Weibull Distribution

6.1. *Quantile and median*

The u^{th} quantile function x_u of the DTZ-W $(\underline{\psi})$ distribution is given by

$$x_{u_{TZW}} = \left\{ \frac{-1}{\gamma} \ln \left[1 - \left(\frac{\ln \left\{ u \left[e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2} \right] + e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2} \right\}}{\alpha} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{\theta}}, 0 < u < 1. \tag{6.1}$$

The median can be obtained by substituting $u = 0.5$ in (6.1) as follows:

$$x_{0.5_{TZW}} = \left\{ \frac{-1}{\gamma} \ln \left[1 - \left(\frac{\ln \left\{ 0.5 \left[e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2} \right] + e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2} \right\}}{\alpha} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{\theta}}, \tag{6.2}$$

6.2. Moments

a. Non-central moments

The r^{th} non-central moment of the DTZ-W ($\underline{\psi}$) distribution is given by

$$\mu'_{r_{TZW}} = \frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2}} \left[\frac{1}{\gamma(1+j)} \right]^{\frac{r+\theta}{\theta}} \times \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} \left\{ \Gamma\left(\frac{r+\theta}{\theta}, \gamma c^\theta (1+j) \right) - \Gamma\left(\frac{r+\theta}{\theta}, \gamma d^\theta (1+j) \right) \right\}, \tag{6.3}$$

then the first four non-central moments of the DTZ-W distribution are given, respectively, by

$$\mu'_{1_{TZW}} = \frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2}} \left[\frac{1}{\gamma(1+j)} \right]^{\frac{1+\theta}{\theta}} \times \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} \left\{ \Gamma\left(\frac{1+\theta}{\theta}, \gamma c^\theta (1+j) \right) - \Gamma\left(\frac{1+\theta}{\theta}, \gamma d^\theta (1+j) \right) \right\}, \tag{6.4}$$

$$\mu'_{2_{TZW}} = \frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2}} \left[\frac{1}{\gamma(1+j)} \right]^{\frac{2+\theta}{\theta}} \times \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} \left\{ \Gamma\left(\frac{2+\theta}{\theta}, \gamma c^\theta (1+j) \right) - \Gamma\left(\frac{2+\theta}{\theta}, \gamma d^\theta (1+j) \right) \right\}, \tag{6.5}$$

$$\mu'_{3_{TZW}} = \frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^\theta}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^\theta}\right)^2}} \left[\frac{1}{\gamma(1+j)} \right]^{\frac{3+\theta}{\theta}} \times \sum_{i=0}^{\infty} \frac{\alpha^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{2i+1}{j} \left\{ \Gamma\left(\frac{3+\theta}{\theta}, \gamma c^\theta (1+j) \right) - \Gamma\left(\frac{3+\theta}{\theta}, \gamma d^\theta (1+j) \right) \right\}, \tag{6.6}$$

Table 2. Mean, median, variance, skewness, kurtosis, ID and CV of DTZ-W $(\underline{\psi})$ for different values of the parameters

α	θ	γ	Mean	Median	Variance	Skewness	Kurtosis	ID	CV
0.2	0.5	3	0.1437	0.1039	0.0157	1.0496	3.5103	0.1092	0.8719
	0.6		0.1994	0.1498	0.0261	0.9551	3.0182	0.1309	0.8102
	0.8		0.2930	0.2612	0.0374	0.6984	2.5070	0.1276	0.6600
2	0.5	0.5	0.5810	0.6225	0.1376	0.1491	1.6584	0.2368	0.6384
3.5			0.5839	0.5787	0.1322	0.1476	1.8521	0.2264	0.6227
4.5			0.5875	0.5526	0.1209	0.0102	1.8624	0.2058	0.5918
0.7	0.9	2.2	0.4476	0.4165	0.0512	0.3063	2.1861	0.1144	0.5055
		3.3	0.3428	0.3346	0.0401	0.3211	2.0778	0.1170	0.5841
		5	0.2438	0.2302	0.0216	0.3299	2.0312	0.0886	0.6028

and

$$\begin{aligned}\mu^{`}_{4_{TZW}} =& \frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}}\left[\frac{1}{\gamma(1+j)}\right]^{\frac{4+\theta}{\theta}} \\ &\times\sum_{i=0}^{\infty}\frac{\alpha^{i}}{i!}\sum_{j=0}^{\infty}(-1)^{j}\binom{2i+1}{j}\left\{\Gamma\left(\frac{4+\theta}{\theta},\gamma c^{\theta}(1+j)\right)-\Gamma\left(\frac{4+\theta}{\theta},\gamma d^{\theta}(1+j)\right)\right\}.\end{aligned} \tag{6.7}$$

The variance of the DTZ-W $(\underline{\psi})$ distribution is

$$\begin{aligned}V_{TZW}(x) =& \Bigg[\frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}}\left[\frac{1}{\gamma(1+j)}\right]^{\frac{2+\theta}{\theta}} \\ &\times\sum_{i=0}^{\infty}\frac{\alpha^{i}}{i!}\sum_{j=0}^{\infty}(-1)^{j}\binom{2i+1}{j}\left\{\Gamma\left(\frac{2+\theta}{\theta},\gamma c^{\theta}(1+j)\right)-\Gamma\left(\frac{2+\theta}{\theta},\gamma d^{\theta}(1+j)\right)\right\}\Bigg] \\ &-\Bigg[\frac{2\alpha\gamma}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}}\left[\frac{1}{\gamma(1+j)}\right]^{\frac{1+\theta}{\theta}}\times\sum_{i=0}^{\infty}\frac{\alpha^{i}}{i!}\sum_{j=0}^{\infty}(-1)^{j}\binom{2i+1}{j} \\ &\Gamma\left(\frac{1+\theta}{\theta},\gamma c^{\theta}(1+j)\right)-\Gamma\left(\frac{1+\theta}{\theta},\gamma d^{\theta}(1+j)\right)\Bigg]^{2}.\end{aligned} \tag{6.8}$$

The coefficient of variation (CV) and the index of dispersion (ID) are given, respectively, by:

$$CV = \frac{(\mu_2)^{1/2}}{\mu}, \quad \text{and} \quad ID = \frac{\mu_2}{\mu}. \tag{6.9}$$

Numerical results of the mean, median, variance, SK, Kur, ID and CV of DTZ-W distribution for different values of the parameters are presented in Table 2.

One can observe from Table 2 that:

- For fixed values of α and γ when θ increases, the mean, median and the variance increase, skewness and kurtosis decrease. When the parameter α increases, and fixed values of θ and γ; the mean increases, while the median and the variance decrease, skewness decreases and kurtosis increases. For fixed values of α and θ when γ increases, the mean, median and the variance decrease, skewness increases and kurtosis decreases.

- Since the mean is greater than the variance, the DT-ZW ($\underline{\psi}$) distribution is suitable for analyzing under-dispersed data sets.

- The proposed distribution is suitable for modeling positively skewed data and can accommodate either leptokurtic (Kur > 3) or platykurtic (Kur < 3) data sets.

- Since the CV is smaller than one, it indicates that the standard deviation (a measure of variability) is less than the mean, implying relatively low variability in the data.

6.3. Order statistic of doubly truncated Zubair-Weibull distribution

The pdf of the r^{th} order statistic of the DTZ-W ($\underline{\psi}$) is

$$
\begin{aligned}
f_{r:n_{TZW}}\left(x;\underline{\psi}\right) =& \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{2\alpha\theta\gamma x_{(r)}^{\theta-1}e^{-\gamma x_{(r)}^{\theta}}\left(1-e^{-\gamma x_{(r)}^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x_{(r)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right] \\
&\times\left[\frac{e^{\alpha\left(1-e^{-\gamma x_{(r)}^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]^{r-1}\left[\frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma x_{(r)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]^{n-r}, c < x_{(r)} < d.
\end{aligned}
\tag{6.10}
$$

The pdf of the smallest order statistic is as follows:

$$
f_{1:n_{TZW}}\left(x;\underline{\psi}\right) = n\left[\frac{2\alpha\theta\gamma x_{(1)}^{\theta-1}e^{-\gamma x_{(1)}^{\theta}}\left(1-e^{-\gamma x_{(1)}^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x_{(1)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]\left[\frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma x_{(1)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]^{n-1}, c < x_{(1)} < d.
\tag{6.11}
$$

The pdf of the largest order statistic is given below

$$
f_{n:n_{TZW}}\left(x;\underline{\psi}\right) = n\left[\frac{2\alpha\theta\gamma x_{(n)}^{\theta-1}e^{-\gamma x_{(n)}^{\theta}}\left(1-e^{-\gamma x_{(n)}^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x_{(n)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]\left[\frac{e^{\alpha\left(1-e^{-\gamma x_{(n)}^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right]^{n-1}, c < x_{(n)} < d.
\tag{6.12}
$$

6.4. Entropy measures

Rényi entropy for the DTZ-W distribution can be defined as

$$En_{\delta_{TZW}}(x) = \frac{1}{(1-\delta)} ln\left[\int_c^d \frac{(2\alpha\theta\gamma)^{\delta} x^{\delta(\theta-1)} e^{-\delta\gamma x^{\theta}} \left(1-e^{-\gamma x^{\theta}}\right)^{\delta} e^{\delta\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\delta\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\delta\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}} dx\right], \ \delta\neq 1, \delta > 0, \tag{6.13}$$

As $\delta \to 1$, Rényi entropy tends to Shannon entropy.

Shannon entropy measures the uncertainty or randomness of the rv X and is as follows:

$$H = -\int_c^d \left(\frac{2\alpha\theta\gamma x^{\theta-1} e^{-\gamma x^{\theta}}\left(1-e^{-\gamma x^{\theta}}\right) e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right) ln\left[\frac{2\alpha\theta\gamma x^{\theta-1} e^{-\gamma x^{\theta}}\left(1-e^{-\gamma x^{\theta}}\right) e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right] dx\,, \tag{6.14}$$

Tsallis entropy is parameterized by a real number ρ and is given by

$$T_{\rho_{TZW}}(x) = \frac{1}{\rho-1}\left[1-\int_c^d \left(\frac{(2\alpha\theta\gamma)^{\rho} x^{\rho(\theta-1)} e^{-\rho\gamma x^{\theta}} \left(1-e^{-\gamma x^{\theta}}\right)^{\rho} e^{\rho\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\rho\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\rho\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}\right) dx\right], \ \rho \neq 1. \tag{6.15}$$

6.5. Mean of residual life, Mean past lifetime and Mean time to failure

The MRL, MPL and MTTF for the DTZ-W are given, respectively, by

$$m_{TZW}(t) = \frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma t^{\theta}}\right)^2}} \int_t^d \frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}} dx, \tag{6.16}$$

$$m^*_{TZW}(t) = \frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma t^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}} \int_c^t \frac{e^{\alpha\left(1-e^{-\gamma x^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}} dx, \tag{6.17}$$

and

$$MTTF_{TZW} = \int_c^d \frac{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma t^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}} dt. \tag{6.18}$$

7. Maximum likelihood estimation for the doubly truncated Zubair-Weibull distribution

The unknown parameters of the DTZ-W ($\underline{\psi}$) can be estimated using the ML method as follows:

$$L_{TZW}\left(\underline{\psi}\,;\underline{x}\right) = \prod_{i=1}^{n} \frac{2\alpha\theta\gamma {x_{(i)}}^{\theta-1} e^{-\gamma x_{(i)}^{\theta}}\left(1-e^{-\gamma x_{(i)}^{\theta}}\right) e^{\alpha\left(1-e^{-\gamma x_{(i)}^{\theta}}\right)^2}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}}, \tag{7.1}$$

where $\underline{\psi} = \left(\alpha,\ \underline{\xi}, c, d\right)$ *and* $\underline{\xi} = (\theta, \gamma)$.

The log likelihood function (ln L) can be written as

$$
\begin{aligned}
l_{TZW} \equiv lnL_{TZW}\left(\underline{\psi};\underline{x}\right) &= \sum_{i=1}^{n} ln\left[2\alpha\theta\gamma x_{(i)}{}^{\theta-1}e^{-\gamma x_{(i)}{}^{\theta}}\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}}\right] \\
&\quad - nln\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right] \\
&= \sum_{i=1}^{n}\left[ln\left(2\right)+ln\left(\alpha\right)+ln\left(\theta\right)+ln\left(\gamma\right)+\left(\theta-1\right)ln(x_{(i)})-\gamma x_{(i)}{}^{\theta}+ln\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)+\alpha\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}\right] \\
&\quad - nln\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right].
\end{aligned}
\tag{7.2}
$$

The ML estimators of the truncation points c and d are $\hat{c}_{ML} = \arg\max L_{TZW}\left(\underline{\psi};\underline{x}\right) = x_{(1)}$, $\hat{d}_{ML} = \arg\max L_{TZW}\left(\underline{\psi}\,;\underline{x}\right) = x_{(n)}$, can be obtained by differentiating the log LF with respect to the parameters α, θ and γ as follows:

$$
\frac{\partial l_{TZW}}{\partial\alpha} = \sum_{i=1}^{n}\left[\frac{1}{\alpha}+\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}\right] - \frac{n\left\{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\left(1-e^{-\gamma d^{\theta}}\right)^{2}\right]-\left[e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\left(1-e^{-\gamma c^{\theta}}\right)^{2}\right]\right\}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}}, \tag{7.3}
$$

$$
\begin{aligned}
\frac{\partial l_{TZW}}{\partial\theta} &= \sum_{i=1}^{n}\left[\frac{1}{\theta}+ln\left(x_{(i)}\right)-\gamma x_{(i)}{}^{\theta}ln\left(x_{(i)}\right)+\frac{\left(\gamma x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)ln\left(x_{(i)}\right)}{\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)}\right. \\
&\quad \left.+2\alpha\left(\gamma x_{(i)}{}^{\theta}\right)ln\left(x_{(i)}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\right] \\
&\quad - \frac{2\alpha n\left\{\begin{array}{l}\left[\left(\gamma d^{\theta}\right)\ln\left(\mathrm{d}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right] \\ -\left[\left(\gamma c^{\theta}\right)ln(c)\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]\end{array}\right\}}{e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}},
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
\frac{\partial l_{TZW}}{\partial\gamma} &= \sum_{i=1}^{n}\left[\frac{1}{\gamma}-x_{(i)}{}^{\theta}+\frac{\left(x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)}{\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)}+2\alpha\left(x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\right] \\
&\quad - \frac{2\alpha n\left\{\begin{array}{l}\left[\left(d^{\theta}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right] \\ -\left[2\alpha\left(c^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]\end{array}\right\}}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]}.
\end{aligned}
\tag{7.5}
$$

The ML estimators of the parameters α, θ and γ can be obtained through equating equations (7.3), (7.4) and (7.5) to zero and solving numerically.

The ML estimators of the rf and hrf can be obtained using the invariance property of the ML estimators to by replacing the parameters $\underline{\psi} = \left(\alpha, \ \underline{\xi}, c, d\right)$ *and* $\underline{\xi} = (\theta, \gamma)$ in (5.5) and (5.6) with their ML estimators, then the ML estimators of $R_{TZW}\left(x; \underline{\psi}\right)$ and $h_{TZW}\left(x; \underline{\psi}\right)$ are given, respectively, by:

$$\hat{R}_{TZW}\left(x; \widehat{\underline{\psi}}\right) = \frac{e^{\widehat{\alpha}\left(1-e^{-\widehat{\gamma}\hat{d}^{\widehat{\theta}}}\right)^2} - e^{\widehat{\alpha}\left(1-e^{-\widehat{\gamma}x^{\widehat{\theta}}}\right)^2}}{e^{\widehat{\alpha}\left(1-e^{-\widehat{\gamma}\hat{d}^{\widehat{\theta}}}\right)^2} - e^{\widehat{\alpha}\left(1-e^{-\widehat{\gamma}\hat{c}^{\widehat{\theta}}}\right)^2}}, \quad 0 < \hat{c} < x < \hat{d} < \infty, \tag{7.6}$$

$$\hat{h}_{TZW}\left(x; \widehat{\underline{\psi}}\right) = \frac{2\widehat{\alpha}\widehat{\theta}\widehat{\gamma}x^{\widehat{\theta}-1}e^{-\widehat{\gamma}x^{\widehat{\theta}}}(1 - e^{-\widehat{\gamma}x^{\widehat{\theta}}})e^{\widehat{\alpha}(1-e^{-\widehat{\gamma}x^{\widehat{\theta}}})^2}}{e^{\widehat{\alpha}(1-e^{-\widehat{\gamma}\hat{d}^{\widehat{\theta}}})^2} - e^{\widehat{\alpha}(1-e^{-\widehat{\gamma}x^{\widehat{\theta}}})^2}}, \quad 0 < \hat{c} < x < \hat{d} < \infty. \tag{7.7}$$

To obtain the confidence intervals (CIs) of the parameters $\underline{\psi} = \alpha, \theta$ *and* γ of the DTZ-W distribution, the distributions of the ML estimators $\widehat{\underline{\psi}} = \widehat{\alpha}, \widehat{\theta}$ *and* $\widehat{\gamma}$ are needed. Since these estimators do not have a closed form, it is not possible to obtain their exact distributions. Therefore, the approximate CIs can be constructed using the asymptotic distribution of the ML estimators as the results of the asymptotically normal with mean α, θ *and* γ and the asymptotic variance-covariance matrix which is the inverse of the asymptotic Fisher information matrix as:

$$\tilde{I}\left(\widehat{\underline{\psi}}\right) = \left[\begin{array}{ccc} -\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & -\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) & -\left(\frac{\partial^2 l}{\partial\alpha\partial\gamma}\right) \\ -\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & -\left(\frac{\partial^2 l}{\partial\theta^2}\right) & -\left(\frac{\partial^2 l}{\partial\theta\partial\gamma}\right) \\ -\left(\frac{\partial^2 l}{\partial\gamma\partial\alpha}\right) & -\left(\frac{\partial^2 l}{\partial\gamma\partial\theta}\right) & -\left(\frac{\partial^2 l}{\partial\gamma^2}\right) \end{array}\right]\Bigg|_{(\widehat{\alpha},\widehat{\theta}\ \ \widehat{\gamma})}, \tag{7.8}$$

which can be written as

$$\tilde{I}\left(\widehat{\underline{\psi}}\right) \approx -\left[\frac{\partial^2 l_{TZW}}{\partial\psi_i\partial\psi_j}\right] \approx \left[I_{ij}\right]\Big|_{\widehat{\underline{\psi}}}, i, j = 1, 2, 3, \tag{7.9}$$

with the elements as given below

$$I_{11} = \frac{-\partial^2 l_{TZW}}{\partial\alpha^2} = \sum_{i=1}^{n}\left[\frac{1}{\alpha^2}\right] + \left(\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2}\right]^2}\right) \times \left\{e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2}\left[\begin{array}{c} 2\left(1 - e^{-\gamma c^{\theta}}\right)^2\left(1 - e^{-\gamma d^{\theta}}\right)^2 \\ -\left(1 - e^{-\gamma c^{\theta}}\right)^4\left(1 - e^{-\gamma d^{\theta}}\right)^4 \end{array}\right]\right\}, \tag{7.10}$$

$$I_{12} = \frac{-\partial^2 l_{TZW}}{\partial\alpha\partial\theta} = \sum_{i=1}^{n} \left[-2\left(1 - e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-\gamma x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \right]$$
$$+ \left(\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2} \right]^2} \right)$$
$$\times \left\{ 2\left(\gamma d^{\theta}\right)\ln(d)\ \left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \right] \right. \tag{7.11}$$
$$+ 2\left(\gamma c^{\theta}\right)\ln(c)\ \left(e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \right]$$
$$- 2\alpha\gamma e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \left[\left(c^{\theta}\right)\ln(c)\left(e^{-\gamma c^{\theta}}\right)\ \left(1 - e^{-\gamma c^{\theta}}\right)^3 + \left(d^{\theta}\right)\ln(d)\left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)^3 \right.$$
$$\left. +2\alpha\gamma e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \left[\begin{array}{c} \left(c^{\theta}\right)\ln(c)\left(e^{-\gamma c^{\theta}}\right)\ \left(1 - e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)^2 \\ +\left(d^{\theta}\right)\ln(d)\left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)^2\left(1 - e^{-\gamma d^{\theta}}\right) \end{array} \right] \right\},$$

$$I_{13} = \frac{-\partial^2 l_{TZW}}{\partial\alpha\partial\gamma}$$
$$= \sum_{i=1}^{n} \left[-2\alpha\left(1 - e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right) \right] + \left(\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2} \right]^2} \right)$$
$$\times \left\{ 2\left(d^{\theta}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \right] \right. \tag{7.12}$$
$$+ 2\left(c^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \right]$$
$$- 2\alpha e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \left[\left(c^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)^3 + \left(d^{\theta}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)^3 \right]$$
$$\left. + 2\alpha e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2 + \alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} \left[\begin{array}{c} \left(c^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)\left(1 - e^{-\gamma d^{\theta}}\right)^2 \\ +\left(d^{\theta}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1 - e^{-\gamma c^{\theta}}\right)^2\left(1 - e^{-\gamma d^{\theta}}\right) \end{array} \right] \right\},$$

$$I_{22} = \frac{-\partial^2 l_{TZW}}{\partial\theta^2} = \sum_{i=1}^{n} \left\{ \frac{1}{\theta^2} + \gamma\ln\left(x_{(i)}\right)\left(x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \right.$$
$$- \frac{\left\{ \begin{array}{c} \left(1 - e^{-\gamma x_{(i)}{}^{\theta}}\right)\left[\gamma\ln\left(x_{(i)}\right)\right]\left[x_{(i)}{}^{\theta}\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-\gamma x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) + \left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \right] \\ -\left[\left(\gamma x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\ln\left(x_{(i)}\right)\ \left(-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-\gamma x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \right] \end{array} \right\}}{\left(1 - e^{-\gamma x_{(i)}{}^{\theta}}\right)^2}$$

$$-2\alpha\gamma \ln\left(x_{(i)}\right)\left[\begin{array}{c} x_{(i)}{}^{\theta}\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-\gamma x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \\ +x_{(i)}{}^{\theta}\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(-\gamma x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \\ +\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right) \end{array}\right]\Bigg\}$$

$$+\left(\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]^{2}}\right)$$

$$\times\Bigg\{2\alpha\gamma^{2}\ \left(d^{2\theta}\right)[\ln(\mathrm{d})]^{2}\left(e^{-2\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$+2\alpha\gamma\left(d^{\theta}\right)[\ln(\mathrm{d})]^{2}\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$-2\alpha\gamma^{2}\ \left(d^{2\theta}\right)[\ln(\mathrm{d})]^{2}\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$+2\alpha\gamma^{2}\ \left(c^{2\theta}\right)[\ln(\mathrm{c})]^{2}\left(e^{-2\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$+2\alpha\gamma\left(c^{\theta}\right)[\ln(\mathrm{c})]^{2}\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$-2\alpha\gamma^{2}\ \left(c^{2\theta}\right)[\ln(\mathrm{c})]^{2}\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$

$$-4\alpha^{2}\gamma^{2}e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\left[\left(c^{2\theta}\right)[\ln(\mathrm{c})\ ]^{2}\left(e^{-2\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\left(d^{2\theta}\right)[\ln(\mathrm{d})\ ]^{2}\left(e^{-2\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)^{2}\right]$$

$$\left.\begin{array}{c} +8\alpha^{2}\gamma^{2}\left(c^{\theta}\right)\left(d^{\theta}\right)[\ln(\mathrm{c})\ ][\ln(\mathrm{d})\ ]\left(e^{-\gamma c^{\theta}}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right) \\ \left(1-e^{-\gamma d^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}} \end{array}\right\}, \tag{7.13}$$

$$I_{23}=\frac{-\partial^{2}l_{TZW}}{\partial\theta\partial\gamma}$$

$$=\sum_{i=1}^{n}\left\{\left[x_{(i)}{}^{\theta}\ln\left(x_{(i)}\right)\ \right]-\frac{\begin{array}{c}\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}\left\{\left(x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right)\ \left[\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)-(\gamma)\left(x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\right]\right\} \\ -\left\{\gamma\left(x_{(i)}{}^{2\theta}\right)\ln\left(x_{(i)}\right)\ \left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\right\}\end{array}}{\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}}\right.$$

$$-2\alpha\left(x_{(i)}{}^{\theta}\right)\ln\left(x_{(i)}\right)\left[\begin{array}{c}(\gamma)\left(x_{(i)}{}^{\theta}\right)\left(e^{-2\gamma x_{(i)}{}^{\theta}}\right)-(\gamma)\left(x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right) \\ +\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\end{array}\right]\Bigg\}$$

$$+\left(\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]^{2}}\right)$$

$$\times\left\{2\alpha\left(\gamma\right)\left(d^{2\theta}\right)\ln\left(\mathrm{d}\right)\left(e^{-2\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]\right.$$
$$-2\alpha\left(\gamma\right)\left(d^{2\theta}\right)\ln\left(\mathrm{d}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$
$$+2\alpha d^{\theta}\ln\left(\mathrm{d}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$
$$+2\alpha\left(\gamma\right)\left(c^{2\theta}\right)\ln\left(\mathrm{c}\right)\left(e^{-2\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$
$$-2\alpha\left(\gamma\right)\left(c^{2\theta}\right)\ln\left(\mathrm{c}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$
$$+2\alpha c^{\theta}\ln\left(\mathrm{c}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left[e^{2\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]$$
$$-4\alpha^{2}\left(\gamma\right)\left(c^{2\theta}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\left[c^{2\theta}\ln\left(c\right)\left(e^{-2\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)^{2}+d^{2\theta}\ln(d)\left(e^{-2\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)^{2}\right]$$
$$\left.+4\alpha^{2}\left(\gamma\right)\left(c^{\theta}\right)\left(d^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\left[\ln\left(c\right)-\ln\left(d\right)\right]\right\}, \tag{7.14}$$

$$\begin{aligned}I_{33}&=\frac{-\partial^{2}l_{TZW}}{\partial\gamma^{2}}\\&=\sum_{i=1}^{n}\left[\begin{array}{c}\frac{1}{\gamma^{2}}+\frac{\left(x_{(i)}{}^{\theta}\right)\left\{\left[\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\right]-\left[\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\right]\right\}}{\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)^{2}}\\-\left\{\left[2\alpha\left(x_{(i)}{}^{\theta}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\right]+\left[\left(1-e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(e^{-\gamma x_{(i)}{}^{\theta}}\right)\left(x_{(i)}{}^{\theta}\right)\right]\right\}\end{array}\right]\\&\quad+\frac{n}{\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}\right]^{2}}\\&\quad\times\left\{2\alpha\left(d^{2\theta}\right)\left(e^{-2\gamma d^{\theta}}\right)\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]\right.\\&\quad-2\alpha\left(d^{2\theta}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)\left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]\\&\quad+2\alpha\left(c^{2\theta}\right)\left(e^{-2\gamma c^{\theta}}\right)\left[e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]\\&\quad-2\alpha\left(c^{2\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left[e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}}-e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right]\\&\quad-4\alpha^{2}e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\left[\left(c^{2\theta}\right)\left(e^{-2\gamma c^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\left(d^{2\theta}\right)\left(e^{-2\gamma d^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)^{2}\right]\\&\quad\left.+8\alpha^{2}\left(c^{\theta}\right)\left(d^{\theta}\right)\left(e^{-\gamma c^{\theta}}\right)\left(e^{-\gamma d^{\theta}}\right)\left(1-e^{-\gamma c^{\theta}}\right)\left(1-e^{-\gamma d^{\theta}}\right)e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^{2}+\alpha\left(1-e^{-\gamma d^{\theta}}\right)^{2}}\right\}.\end{aligned} \tag{7.15}$$

8. Simulation Study

In this section, a simulation study is presented to examine the accuracy and efficiency of the ML estimates of the parameters, rf and hrf of the DTZ-W ($\underline{\psi}$) for different samples of size (n =30, 60, 100, 200 and 300) using number of replications (NR)=1000. The computations are performed using Mathematica 11.

The steps of the simulation procedure are:

a. Using two different combinations of population parameter values

I: (α= 2, γ= 0.5, θ= 0.3, c= 0.2 and d= 2.5),
and
II: (α= 3, γ= 0.3, θ= 0.2, c= 0.1 and d= 4).

b. Generating 1000 random samples of size (n =30, 60, 100, 200 and 300) from DTZ-W($\underline{\psi}$) distribution using the following formula:

$$x_{u_{TZW}} = \left\{ \frac{-1}{\gamma} \ln \left[1 - \left(\frac{\ln \left\{ u \left[e^{\alpha\left(1-e^{-\gamma d^{\theta}}\right)^2} - e^{\alpha\left(1-e^{-\gamma c^{\theta}}\right)^2} \right] + e^{\alpha(1-e^{-\gamma c^{\theta}})^2} \right\}}{\alpha} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{\theta}}, 0 < u < 1,$$

where u are random samples from the uniform distribution.

c. Computing the averages, relative errors (REs), variances, relative absolute biases (RABs) and mean square errors (MSEs) of the ML estimates, as follows:

Average = $\frac{\sum_{i=1}^{NR} estimated\ value}{NR}$,

$Estimated\ risk(ER) = \frac{\sum_{i=1}^{NR} (estimated\ value - true\ value)^2}{NR}$

RE = $\frac{\sqrt{ER\ (estimated\ value)}}{true\ value}$,

$variance = ER\,(estimated\ value) - bais^2\,(estimated\ value)$.

$bias = estimatedvalue - true\ value$

RAB = $\frac{|estimatedvalue - true\ value|}{true\ value}$,

and

MSE= $variance\,(estimated\ value) + bais^2\,(estimated\ value)$.

d. Calculating the averages of the ML estimates, RABs, REs, MSEs, biases and variances of the parameters, rf and hrf for the model parameters and for each sample size.

Table 3- 6 shows the averages of the ML, RABs, REs, MSEs, biases, variances and 95% CIs of the unknown parameters α, γ, θ, c and ***d*** under different sample sizes for the DTZ-W distribution.

Table 3. Averages of the ML estimates, relative absolute biases, relative errors, mean square errors, biases, variances and 95% confidence intervals of the parameters of the DTZ-W distribution for different samples of size $\boldsymbol{n}$ and the number of replications $\boldsymbol{NR}$ =1000, ($\boldsymbol{\alpha}$= 2, $\boldsymbol{\gamma}$= 0.5, $\boldsymbol{\theta}$= 0.3, $\boldsymbol{c}$= 0.2 and $\boldsymbol{d}$= 2.5)

n	Parameters	Averages	RABs	REs	MSEs	Biases	Variances	UL	LL	Length
30	α	1.8798	0.0601	0.1924	0.148	0.0144	0.1336	2.5962	1.1634	1.4327
	γ	0.51	0.0201	0.4392	0.0482	0.0001	0.0481	0.9399	0.0801	0.8598
	θ	0.2816	0.0613	0.4903	0.0216	0.0003	0.0213	0.5676	0.0998	0.5676
	c	0.2356	0.1781	0.2624	0.0027	0.0013	0.0015	0.3111	0.1601	0.151
	d	2.3899	0.044	0.0609	0.0232	0.0121	0.0111	2.5964	2.1835	0.4129
60	α	1.9128	0.0436	0.1716	0.1178	0.0076	0.1102	2.5634	1.2622	1.3012
	γ	0.4964	0.0072	0.3318	0.0275	1.3×10^{-5}	0.0275	0.8215	0.1713	0.6502
	θ	0.2977	0.0077	0.3414	0.0105	5.4×10^{-6}	0.0105	0.4983	0.097	0.4013
	c	0.2184	0.0921	0.131	0.0007	0.0003	0.0003	0.2549	0.1819	0.073
	d	2.4431	0.0227	0.0318	0.0063	0.0034	0.0031	2.552	2.3342	0.2178
100	α	1.9219	0.039	0.1171	0.0549	0.0061	0.0488	2.3548	1.489	0.8658
	γ	0.4935	0.013	0.2655	0.0176	4.2×10^{-5}	0.0176	0.7534	0.2336	0.5198
	θ	0.3039	0.0131	0.259	0.006	0.000015	0.006	0.456	0.1518	0.3042
	c	0.2113	0.0568	0.0809	0.0003	0.0001	0.0001	0.2339	0.1888	0.0451
	d	2.4649	0.014	0.0196	0.0024	0.0012	0.0012	2.5323	2.3975	0.1348
200	α	1.9495	0.0252	0.1044	0.0436	0.0025	0.0411	2.3469	1.5521	0.7948
	γ	0.5013	0.0025	0.2047	0.0105	1.6×10^{-5}	0.0105	0.7018	0.3007	0.4011
	θ	0.2979	0.0068	0.1832	0.003	4.1×10^{-6}	0.003	0.4056	0.1903	0.2153
	c	0.2051	0.0256	0.0356	0.000051	0.000026	0.000024	0.2148	0.1954	0.0194
	d	2.4817	0.0073	0.0103	0.0007	0.0003	0.0003	2.5172	2.4462	0.071
300	α	1.9578	0.021	0.0969	0.0376	0.0018	0.0358	2.3288	1.5869	0.7418
	γ	0.498	0.004	0.1833	0.0084	4.1×10^{-6}	0.0084	0.6776	0.3183	0.3593
	θ	0.2991	0.0028	0.1489	0.002	3×10^{-7}	0.002	0.3867	0.2116	0.1751
	c	0.2036	0.0183	0.0255	0.000026	0.000013	0.000012	0.2106	0.1967	0.0139
	d	2.4881	0.0047	0.0066	0.0003	0.0001	0.0001	2.5105	2.4657	0.0448

Table 4. Averages of the ML estimates, relative absolute biases, relative errors, mean square errors, biases, variances and 95% confidence intervals of the parameters of the DTZ-W distribution for different samples of size ***n*** and the number of replications ***NR*** =1000, **(α=3, γ=0.3, θ=0.2, c=0.1 and d=4)**

n	Parameters	Averages	RABs	REs	MSEs	Biases	Variances	UL	LL	Length
30	α	2.8542	0.0486	0.1477	0.1962	0.0212	0.175	3.6741	2.0343	1.6398
	γ	0.3098	0.0327	0.4285	0.0165	9.6×10^{-5}	0.0164	0.561	0.0585	0.5025
	θ	0.1842	0.0788	0.5037	0.0101	2.5×10^{-4}	0.0099	0.3792	0.0000	0.3792
	c	0.1294	0.2938	0.4372	0.0019	8.6×10^{-5}	0.001	0.1928	0.0659	0.1269
	d	3.7434	0.0641	0.0867	0.1202	0.0658	0.0544	4.2007	3.2861	0.9146
60	α	2.9131	0.0289	0.1383	0.1722	0.0075	0.1646	3.7084	2.1179	1.5905
	γ	0.3039	0.0131	0.3268	0.0096	1.5×10^{-5}	0.0096	0.4959	0.1119	0.384
	θ	0.1955	0.0223	0.3498	0.0049	2.0×10^{-5}	0.0049	0.3324	0.0587	0.2737
	c	0.1137	0.1369	0.1972	0.0004	1.9×10^{-4}	0.0002	0.1415	0.0859	0.0556
	d	3.8728	0.0318	0.0442	0.0313	0.0162	0.0151	4.1138	3.6319	0.4819
100	α	2.9416	0.0195	0.1311	0.1547	0.0034	0.1513	3.7039	2.1792	1.5247
	γ	0.2981	0.0064	0.2748	0.0068	3.7×10^{-6}	0.0068	0.4596	0.1365	0.3231
	θ	0.1986	0.007	0.2671	0.0028	1.9×10^{-6}	0.0028	0.3033	0.0939	0.2094
	c	0.1081	0.0808	0.1165	0.0001	6.5×10^{-5}	7.0×10^{-5}	0.1245	0.0916	0.0329
	d	3.9232	0.0192	0.0269	0.0116	0.0059	0.0057	4.071	3.7754	0.2956
200	α	2.9476	0.0175	0.1157	0.1205	0.0027	0.1177	3.62	2.2751	1.3449
	γ	0.3043	0.0143	0.2268	0.0046	1.8×10^{-5}	0.0046	0.4374	0.1712	0.2662
	θ	0.1979	0.0107	0.1924	0.0015	4.6×10^{-6}	0.0015	0.2732	0.1225	0.1507
	c	0.1041	0.0408	0.0572	3.3×10^{-5}	1.7×10^{-5}	1.6×10^{-5}	0.1119	0.0962	0.0157
	d	3.9616	0.0096	0.0136	0.003	0.0015	0.0015	4.0373	3.8859	0.1514
300	α	2.9502	0.0166	0.113	0.115	0.0025	0.1125	3.6078	2.2926	1.3151
	γ	0.3011	0.0039	0.2044	0.0038	1.4×10^{-6}	0.0037	0.4213	0.181	0.2403
	θ	0.199	0.0048	0.1548	0.0009	9.3×10^{-7}	0.0009	0.2597	0.1384	0.1213
	c	0.1026	0.0266	0.0381	1.4×10^{-5}	7.1×10^{-6}	7.4×10^{-6}	0.108	0.0973	0.0107
	d	3.9734	0.0066	0.0093	0.0014	0.0007	0.0007	4.0247	3.9221	0.1026

Table 5. Averages of the ML estimates, relative absolute biases, relative errors, mean square errors, biases, variances and 95% confidence intervals of the reliability and hazard rate functions at ($\boldsymbol{x_0 = 0.4}$) of the DTZ-W distribution for different samples of size $\boldsymbol{n}$ and the number of replications $\boldsymbol{NR}$ =1000, **(α=2, γ=0.5, θ=0.3, c=0.2 and d=2.5)**

n	rf and hrf	Averages	RABs	REs	MSEs	Biases	Variances	UL	LL	Length
30	$\boldsymbol{R(x_0)}$	0.856	0.0169	0.0653	0.003	0.0002	0.0028	0.9602	0.7518	0.2084
	$\boldsymbol{h(x_0)}$	0.8984	0.1083	0.3237	0.0688	0.0077	0.0611	1.383	0.4138	0.9692
60	$\boldsymbol{R(x_0)}$	0.8502	0.01	0.044	0.0014	7.1×10^{-5}	0.0013	0.9208	0.7795	0.1413
	$\boldsymbol{h(x_0)}$	0.8467	0.0445	0.2131	0.0298	0.0013	0.0285	1.1778	0.5154	0.6624
100	$\boldsymbol{R(x_0)}$	0.8483	0.0077	0.0327	0.0007	4.2×10^{-5}	0.0007	0.9007	0.7959	0.1048
	$\boldsymbol{h(x_0)}$	0.8261	0.0191	0.1573	0.0162	0.0002	0.016	1.0742	0.5779	0.4963
200	$\boldsymbol{R(x_0)}$	0.8431	0.0016	0.0222	0.0003	1.8×10^{-6}	0.0003	0.8798	0.8065	0.0733
	$\boldsymbol{h(x_0)}$	0.8254	0.0183	0.1105	0.008	0.0002	0.0078	0.9986	0.6522	0.3464
300	$\boldsymbol{R(x_0)}$	0.8429	0.0013	0.0184	0.0002	1.3×10^{-6}	0.0002	0.8732	0.8127	0.0605
	$\boldsymbol{h(x_0)}$	0.8201	0.0117	0.0892	0.0052	0.0001	0.0051	0.9606	0.6795	0.2811

8.1. Concluding Remarks

- From Tables 1 and 2, one can observe that the ML averages are close to the parameter values as the sample size increases, demonstrating the asymptotic properties of the estimators such as the consistency and the efficiency of the ML estimators where the estimator uses the information from the data most effectively. Also, in most cases the RABs, REs and variances of the ML estimates of the parameters $(\alpha, \theta, \gamma, c, d)$ decrease when the sample size increases indicating improved precision and accuracy.
- From Tables 3 and 4, it is noticed that the RABs, REs and variances of the rf and hrf decrease when the sample size increases, which confirms that larger sample sizes enhance the reliability of parameter estimates.
- The lengths of the confidence intervals of the parameters, rf and hrf decrease when the sample size increases, reflecting the increased robustness of the model when having with more data and information.

9. Applications to Real Data Set

This section is devoted to illustrate the ability and flexibility of the DTZ-W distribution in real life. Two applications are used to demonstrate the importance of the DTZ-W distribution compared with some distributions such as Zubair-Weibull (Z-W), doubly truncated exponentiated inverse Weibull distribution (DTEIW), truncated Weibull power Lomax (TWPL), truncated log-logistic-Weibull (TLLW) and truncated exponential Marshall Olkin Weibull (TEMOW). By using some criteria such as Kolmogorov-Smirnov (K-S) test and its P-value, -2 log likelihood function (-2lnL), Akaike information criterion (AIC), Bayesian information criterion (BIC) and corrected Akaike information criterion (CAIC), also named AIC with correction, where

$$AIC = 2k - 2ln(L),$$

Table 6. Averages, relative absolute biases, relative errors, mean square errors, biases, variances of the ML estimates and 95% confidence intervals of the reliability and hazard rate functions at ($\boldsymbol{x_0 = 0.4}$) of the DTZ-W distribution for different samples of size $\boldsymbol{n}$ and the number of replications $\boldsymbol{NR}$=1000, **(α=3, γ=0.3, θ=0.2, c=0.1 and d=4)**

n	rf and hrf	Average	RABs	REs	MSE	Bias	Variance	UL	LL	Length
30	$\boldsymbol{R(x_0)}$	0.7784	0.0066	0.0845	0.0043	2.6×10^{-5}	0.0042	0.9062	0.6507	0.2555
	$\boldsymbol{h(x_0)}$	0.7556	0.1134	0.2929	0.0395	0.0059	0.0336	1.1148	0.3964	0.7184
60	$\boldsymbol{R(x_0)}$	0.7767	0.0044	0.0582	0.002	1.1×10^{-5}	0.002	0.8647	0.6887	0.176
	$\boldsymbol{h(x_0)}$	0.7103	0.0466	0.1838	0.0155	0.001	0.0145	0.9468	0.4737	0.4731
100	$\boldsymbol{R(x_0)}$	0.7762	0.0037	0.044	0.0012	8.1×10^{-6}	0.0011	0.8427	0.7097	0.133
	$\boldsymbol{h(x_0)}$	0.695	0.0241	0.1354	0.0084	0.0003	0.0082	0.8722	0.5178	0.3544
200	$\boldsymbol{R(x_0)}$	0.7739	0.0007	0.0313	0.0005	3.1×10^{-7}	0.0006	0.8213	0.7265	0.0948
	$\boldsymbol{h(x_0)}$	0.6894	0.0158	0.0959	0.0042	0.0001	0.0041	0.8152	0.5635	0.2517
300	$\boldsymbol{R(x_0)}$	0.7738	0.0006	0.0251	0.0004	2.9×10^{-7}	0.0004	0.8119	0.7358	0.0761
	$\boldsymbol{h(x_0)}$	0.6852	0.0097	0.0758	0.0026	0.1×10^{-5}	0.0026	0.7852	0.5852	0.2

$$CAIC = AIC + 2\frac{k(k+1)}{n-k-1},$$

and

$$BIC = kln(2) - 2ln(L) \ ,$$

where k denotes the number of distribution parameters, n is the sample size, and $ln(L)$ is the log-likelihood function evaluated at the ML estimates. The distribution with the lowest value of these statistics and the largest P-value for the K-S test is the best fit for the data.

9.1. Application I:

The first application was provided by Murthy et al. [18]. The data refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Table 7 shows the ML estimates of the parameters and standard errors (SEs), K-S statistic, P-value, $-2ln(L)$ statistic, AIC, BIC and CAIC. From the results, one can observe that the DTZ-W distribution provides a better fit to this data compared with other distributions.

Figure 3 displays the TTT-plot, fitted pdf, P-P, Q-Q and histogram plots indicate that the DTZ-W distribution provides a better fit to real data set, Also, the total time test (TTT) plot of the first real data set which shows that this data has an increasing hrf which is one of the hrf shapes of the DTZ-W distribution given in Figure 2(a). It means that as the item ages or undergoes more use, it becomes more likely to fail. This is characteristic of the ”wear-out” phase, commonly observed in repairable systems or mechanical components. This indicates that precautionary maintenance or replacement strategies should be considered to reduce the risk of failure as the item continues to age.

9.2. Application II:

The second application was given by Lawless [17]. The data consists of a number of cycles divided by 1000 up to the failure for 60 electrical appliances in a life test. The data is: 0.014, 0.034 ,0.059,

Table 7. ML estimates, SEs, AIC, BICs, CAIC, K-S statistics and P-values for fitted distributions

Model	Parameter	Estimate	SEs	K-S	P-value	-2lnL	AIC	BIC	CAIC
DTZ-W	α	1.2295	0.1976	0.1333	0.9560	82.4705	88.4705	92.6741	89.3936
	γ	0.5986	0.2222						
	θ	0.2990	0.2377						
	c	0.1100	0.2478						
	d	4.7300	0.3382						
Z-W	α	1.1609	0.1994	0.1667	0.8039	92.5821	98.5821	102.786	99.5052
	γ	1.303	0.1961						
	θ	0.5477	0.2247						
DTEIW	α	2.1563	0.2073	0.3333	0.0709	89.1206	95.1206	99.3242	96.0437
	γ	0.0406	0.2516						
	θ	0.4312	0.2307						
	c	0.1100	0.2478						
	d	4.7300	0.3382						
TWPL	α	2.9199	0.2448	0.3333	0.0693	90.5631	98.5631	104.168	100.163
	β	0.8814	0.2092						
	λ	1.6864	0.1947						
	γ	4.5089	0.3276						
TLLW	α	1.5942	0.1940	0.3000	0.1234	106.166	114.166	119.771	115.766
	γ	0.6103	0.2216						
	β	2.6641	0.2312						
	λ	0.3188	0.2366						
TEMOW	θ	1.1067	0.2010	0.2667	0.2355	179.01	187.01	192.614	188.61
	λ	1.4213	0.1944						
	α	0.1836	0.2438						
	β	0.9937	0.2048						

0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100, 4.106, 4.116, 4.315, 4.510, 4.580, 5.267, 5.299, 5.583, 6.065 and 9.701.

Table 8 presents the ML estimates of the parameters, SEs, K-S statistic, P-value, $-2ln(L)$ statistic, AIC, BIC and CAIC. From the results, one can observe that the DTZ-W distribution provides a better fit to this data compared with other distributions.

Figure 4 indicates that the TTT-plot, P-P plot, Q-Q plot, histogram and the fitted pdf show that the DTZ-W distribution fits the data very well. The TTT plot of the second data set represents the bathtub hrf which is one of the hrf shapes of the DTZ-W distribution given in Figure 2(a) and 2(b). The bathtub-shaped TTT plot reflects the typical lifecycle of many products, including electrical appliances, where there is a phase of early failures, followed by a period of stability (useful life), and then a wear-out

Table 8. The ML estimates, SEs, AIC, BICs, CAIC, K-S statistics and its p-values for fitted distributions

Model	Parameter	Estimate	SEs	K-S	P-value	-2lnL	AIC	BIC	CAIC
DTZ-W	α	0.9825	0.1424	0.0833	0.9866	26.3745	32.3745	36.5781	33.2975
	γ	1.4706	0.1770						
	θ	0.4149	0.1446						
	c	0.014	0.1732						
	d	1.649	0.1909						
Z-W	α	0.8639	0.1373	0.15	0.5130	107.371	113.371	117.575	114.294
	γ	0.7007	0.1349						
	θ	0.5506	0.1381						
DTEIW	α	2.3731	0.2429	0.1667	0.3777	32.1318	38.1318	42.3354	39.0548
	γ	0.4596	0.1421						
	θ	0.3955	0.1457						
	c	0.014	0.1732						
	d	1.649	0.1909						
TWPL	α	2.7716	0.2682	0.2167	0.1197	54.3423	62.3423	67.9471	63.9423
	β	0.6414	0.1356						
	λ	1.4897	0.1785						
	γ	2.5933	0.2571						
TLLW	α	2.6258	0.2592	0.1333	0.6647	51.762	59.762	65.3668	61.362
	γ	0.5151	0.1395						
	β	2.9954	0.2816						
	λ	0.9556	0.1411						
TEMOW	θ	0.7186	0.1349	0.1833	0.2671	132.683	140.683	146.287	142.283
	λ	4.2723	0.3491						
	α	0.2612	0.1546						
	β	1.8919	0.2092						

phase as the appliances age. This indicates that precautionary measures could be considered during the early stage to address defects and extend the operational lifespan, and maintenance strategies may be needed towards the end to reduce the impact of wear-out failures.

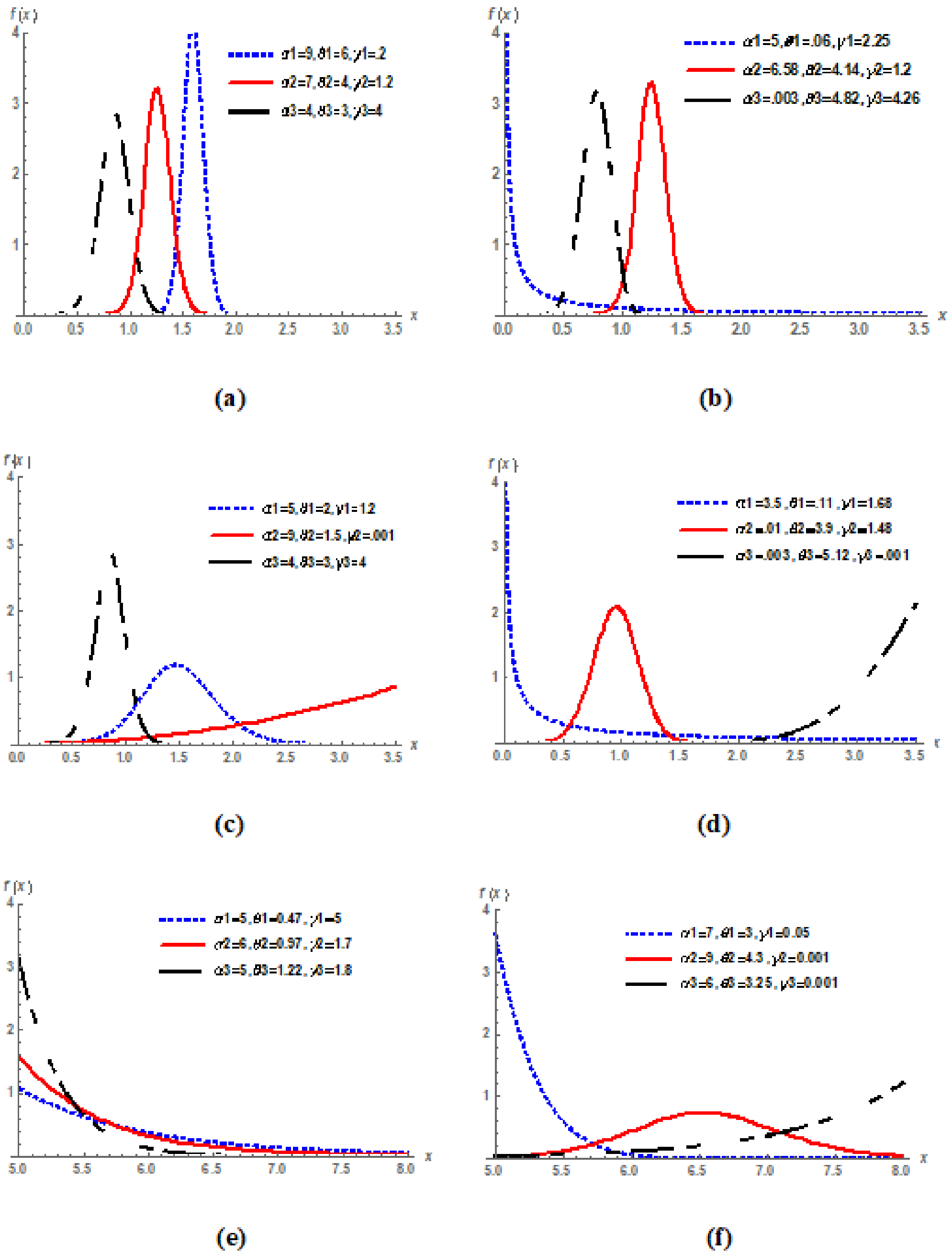

Figure 1. Plots of the pdf of the DTZ-W distribution

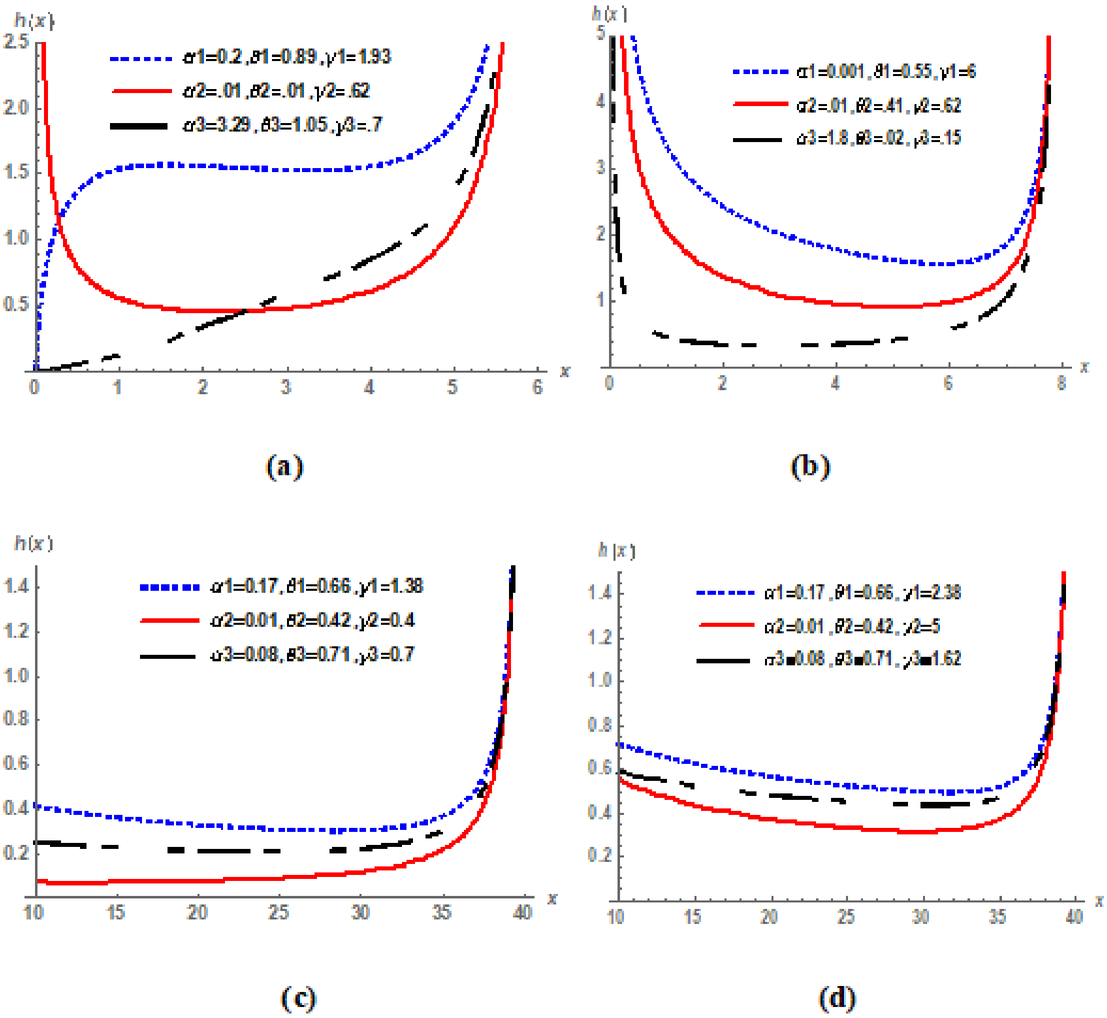

Figure 2. Plots of the hrf of the DTZ-W distribution

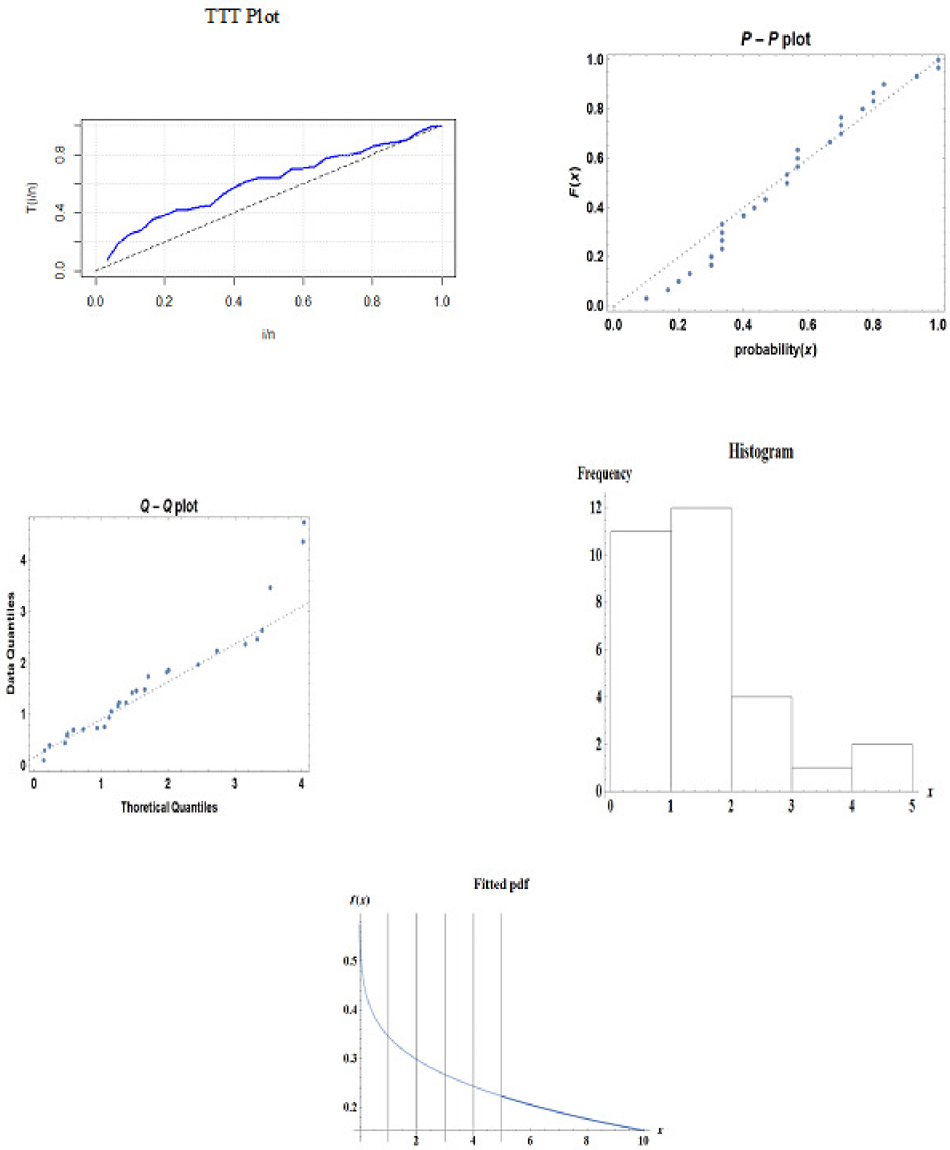

Figure 3. TTT-plot, P-P plot, Q-Q plot, histogram and the fitted pdf for the DTZ-W distribution for Application I

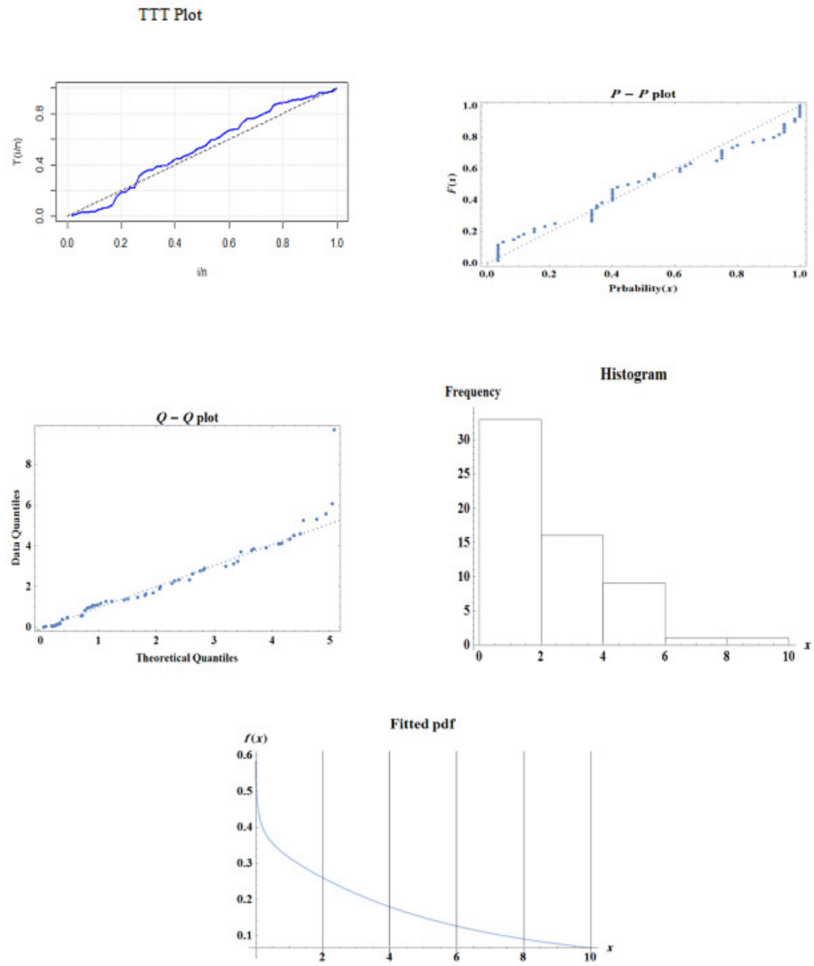

Figure 4. TTT-plot, P-P plot, Q-Q plot, histogram and the fitted pdf for the DTZ-W distribution for Application II

10. Conclusion

In this paper, a new truncated family of distributions called the DTZ-G family of distributions is presented, also the DTZ-W distribution as a sub model of this family is proposed. The DTZ-W distribution is flexible and has a variety of shapes of the pdf and hrf. Some of different statistical properties such as hrf, quantile function, moments, order statistics and entropies are derived. The ML method is used to estimate the unknown parameters. The simulation study is performed to investigate the effectiveness of the ML estimates of the DTZ-W distribution parameters, rf and hrf. Two real data sets are applied to show the flexibility and applicability of the DTZ-W compared with other distributions.

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