The Extended Log-Logistic Distribution :

Properties and Application

By

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Abstract

In this article, we study a new extension of the log-logistic model called the Kumaraswamy alpha-power log-logistic (KAPLL) distribution, an extension of the log-logistic model. It investigates some of their mathematical and statistical properties, including reliability properties (survival function, hazard rate function (HRF), moments, quantile functions (QF), and moment generating functions), emphasizing their utility in modeling diverse aging and failure criteria. One key advantage of the KAPLL distribution lies in its capacity to represent its density as a blend of log-logistic densities, offering both symmetric and asymmetric shapes for greater modeling flexibility. The estimation of KAPLL parameters is achieved through maximum likelihood estimation (MLE), a widely used statistical method. The study presents comprehensive simulation results to assess the effectiveness of the proposed estimation technique. Furthermore, a practical application on real-world data is conducted to showcase the adaptability and versatility of the KAPLL distribution when compared with other extensions of the log-logistic model.

Keywords: Kumaraswamy alpha-power class; log-logistic distribution.

1. Introduction

The quality of the procedures in a statistical analysis depends heavily on the assumed probability distribution. Hence, considerable efforts have been expended for developing generalized classes of distributions along with relevant statistical methodologies. In practice, probability distributions are applied in many fields including the actuarial science and insurance, risk analysis, investment, market research, business and economic research, reliability engineering, chemical engineering, medicine, sociology, demography, among others.

The statistical literature contains several new generated classes of univariate continuous distributions by introducing additional shape parameter(s) to a baseline model. The extended distributions have attracted several statisticians to develop new models due to their flexibility and ability to model monotone and non-monotone real-life data. Some notable classes include the Marshall-Olkin-G (MO-G) [16], Kumaraswamy Marshal-Olkin-G (KMO-G) [3], Weibull Marshall Olkin-G [14], Marshall-Olkin alpha power-G (MOAP-G) [19], and Kumaraswamy alpha power-G (KAP-G) [17], among others.

The log-logistic (LL) distribution is also known as the Fisk distribution in the income distribution literature [8, 10, 22]. Additionally, Arnold [4] referred to it as the Pareto type III distribution and included an additional location parameter to it. Furthermore, the LL distribution is a special case of the Burr-XII distribution [5], and Kappa distribution [18], which have been applied to stream flow and precipitation data. Further details about the LL model can be found in[13].

Several authors have studied different generalized forms of the LL distribution to improve its capability and flexibility. The LL model can be considered the probability distribution of a random variable whose logarithm has a logistic distribution, and it is an alternative to the log-normal distribution since it presents a hazard rate function (HRF) that increases initially and decreases later. Some improved versions of the LL model include the alpha power transformed-LL [1], beta-LL [15], MO-LL [11], extended-LL [2], Zografos-Balakrishnan LL [20], and odd Lomax LL distributions [7]. This article investigates a new extended form of the LL distribution called the Kumaraswamy alpha-power log-logistic (KAPLL) distribution, which can provide more flexibility in modeling reallife data than other competing LL models. The proposed model is obtained using the KAp-G family [17]. The motives of the KAPLL distribution including the following: (i) The KAPLL model is capable of modeling increasing, J-shape, decreasing, reversed J-shape, bathtub, modified bathtub, and unimodal HRF shapes; (ii) the KAPLL distribution can be viewed as a suitable model for modeling skewed real-life data, which may not be properly modeled by other known distributions; (iii) it can be applied in different fields including survival analysis, public health, industrial reliability, biomedical studies, reliability, and engineering; and (iv) The KAPLL distribution outperforms many well-known LL distributions with respect to real-life data examples.

The rest of the paper is organized into seven sections. The KAPLL distribution is investigated in Section 2. In Section 3, some key properties of the KAPLL distribution are explored. Inference about the KAPLL parameters is presented In Section 4. Section 5 provides a simulation study. A real-life data application is presented in Section 6. Section 7 gives some conclusions.

2. The KAPLL Distribution

The KAPLL model and its special cases are presented in this section. The cumulative distribution function (CDF) of the two-parameter LL model has the form

$$G(x) = \left(1 + \frac{\lambda}{x^{\beta}}\right)^{-1},\ x > 0, \lambda, \beta > 0, \tag{1}$$

where λ and β are the scale and shape parameters, respectively. The LL probability density function (PDF) reduces to

$$g(x) = \lambda\beta x^{-\beta-1}\left(1 + \frac{\lambda}{x^{\beta}}\right)^{-2}. \tag{2}$$

The KAPLL distribution is constructed based on the KAP-G family, which is specified by the CDF

$$F(x) = \begin{cases} 1 - \left\{1 - \left[\dfrac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^{a}\right\}^{b} & \text{if } \alpha, a, b > 0, \alpha \neq 1, \\ G(x) & \text{if } \alpha = 1, \end{cases} \tag{3}$$

where α, a and b are shape parameters. The corresponding PDF of the KAP-G class is expressed by

$$f(x) = \frac{ab\ln\alpha}{\alpha - 1} g(x)\alpha^{G(x)}\left[\frac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^{a-1}\left\{1 - \left[\frac{\alpha^{G(x)} - 1}{\alpha - 1}\right]^{a}\right\}^{b-1}. \tag{4}$$

Further information about the KAP-G family can be explored in [17]. Inserting 1 in Equation 3, the CDF of the KAPLL distribution follows as

$$F(x) = 1 - \left[1 - \left(\frac{\alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} - 1}{\alpha - 1}\right)^{a}\right]^{b}. \quad (5)$$

The PDF corresponding to Equation 5 takes the form

$$f(x) = \frac{ab\lambda\beta\ln\alpha}{\alpha - 1} x^{-\beta-1}\left(1 + \frac{\lambda}{x^{\beta}}\right)^{-2} \alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} \left(\frac{\alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} - 1}{\alpha - 1}\right)^{a-1}$$

$$\times \left[1 - \left(\frac{\alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} - 1}{\alpha - 1}\right)^{a}\right]^{b-1}. \quad (6)$$

The HRF of the KAPLL distribution reduces to

$$h(x) = \frac{ab\lambda\beta\ln\alpha}{\alpha - 1} x^{-\beta-1}\left(1 + \frac{\lambda}{x^{\beta}}\right)^{-2} \alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} \left(\frac{\alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} - 1}{\alpha - 1}\right)^{a-1}$$

$$\times \left[1 - \left(\frac{\alpha^{\left(1+\frac{\lambda}{x^{\beta}}\right)^{-1}} - 1}{\alpha - 1}\right)^{a}\right]^{-1}. \quad (7)$$

Table 1 provides five important special sub-models of the new KAPLL distribution. Figure 1 shows some forms of the KAPLL PDF for some selected parameter values. Figures 2 displays the HRF plots of the KAPLL distribution for some parametric values. The density plots show that the KAPLL model can be symmetric and asymmetric density shapes. Furthermore, the HRF plots show the capability of the proposed model in modeling different failure rate shapes including both monotone and non-monotone shapes.

Table 1: Sub-models of the KAPLL distribution

λ	β	α	a	B	Sub-model	Authors
λ	β	α	a	1	EAP-LL distribution	Special case
λ	β	1	a	B	K-LL distribution	De Santana et al. [9]
λ	β	1	a	1	E-LL distribution	Rosaiah et al. [21]
λ	β	α	1	1	AP-LL distribution	Aldahlan [1]
λ	β	1	1	1	LL distribution	Fisk [10]

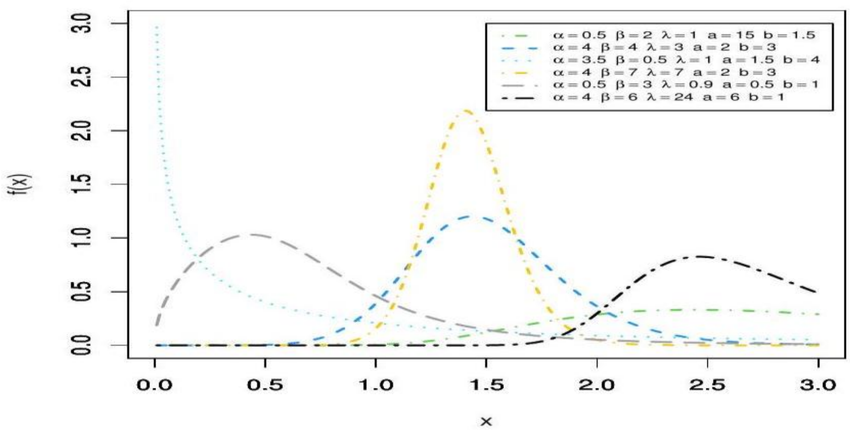

Figure 1: Shapes of the KAPLL density for various parametric values

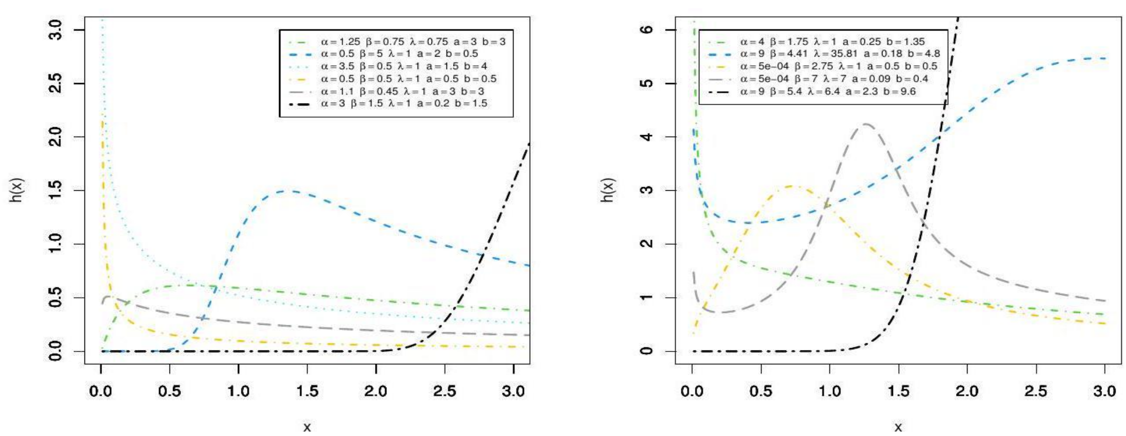

Figure 2: Shapes of the KAPLL HRF for various parametric values

3. Properties of the KAPLL Distribution

This section provides some key features of the KAPLL model.

3.1 Linear Representation

Mead et al. [17] provided a useful mixture representation of the PDF of the KAP-G class. According to [17], the KAP-G density reduces to

$$f(x) = ab\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(\ln\alpha)^{k+1}}{k!\,[a(1+i)-j]^{-k}(\alpha-1)^{a(1+i)}}g(x)(G(x))^{k}\binom{b-1}{i}\binom{a(1+i)-1}{j}.$$

Using the PDF and CDF of the LL model and after some algebra, the KAPLL density takes the form

$$f(x) = ab\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(\ln\alpha)^{k+1}[a(1+i)-j]^{k}}{k!\,(\alpha-1)^{a(1+i)}}\lambda\beta x^{-\beta-1}\left(1+\frac{\lambda}{x^{\beta}}\right)^{-2}$$
$$\times\left(1+\frac{\lambda}{x^{\beta}}\right)^{-k}\binom{b-1}{i}\binom{a(1+i)-1}{j}.$$

It can also be rewritten simply as follows

$$f(x) = \sum_{k=0}^{\infty} d_k\zeta_{k+1}(x), \qquad (8)$$

where $\zeta_{k+1}(x) = (k+1)g(x)(G(x))^{k}$ is the exponentiated-LL density with power parameter and $(k+1) > 0$, and

$$d_k = ab\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\ln\alpha)^{k+1}[a(1+i)-j]^{k}}{(k+1)!\,(\alpha-1)^{a(1+i)}}\binom{b-1}{i}\binom{a(1+i)-1}{j}.$$

3.2 Quantile Function

The quantile function (QF) of the KAPLL distribution follows by inverting Equation 5 as

$$Q(u) = \left[\frac{-1}{\lambda}\left(1 - \frac{\ln\alpha}{\ln(1+\xi)}\right)\right]^{\frac{-1}{\beta}}, \quad (9)$$

where $\xi = \left\{(\alpha-1)\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}$, u follows the uniform (0,1) distribution.

3.3 Moment

The r th moment of X can be obtained from Equation 8 as

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} d_k \int x^r \zeta_{k+1}(x)dx,$$

hence,

$$= \sum_{k=0}^{\infty} d_k \lambda\beta \int_0^{\infty} x^{r-\beta-1}\left(1+\frac{\lambda}{x^{\beta}}\right)^{-k-2} dx,$$

after calculating the integration, you get μ_r' as follows

$$\mu_r' = \sum_{k=0}^{\infty} \frac{d_k \lambda^{\frac{r}{\beta}} \Gamma\left(1-\frac{r}{\beta}\right)\Gamma\left(\frac{r}{\beta}+k+1\right)}{\Gamma(k+2)}, \frac{r}{\beta} < 1. \quad (10)$$

3.4 Order Statistics

Let $X_1, X_2, \ldots\ldots X_n$ be a random sample of size n and let $X_{1:n}, \ldots, X_{n:n}$ be this associated order statistic. Then, the PDF of the i th order statistics, say, $X_{i:n}$, which is denoted by $f_{x_{i:n}}(x)$ is given by:

$$f_{x_{i:n}}(x) = \frac{n!\, f(x)}{(n-i)!\,(i-1)!}[F(x)]^{i-1}[1-F(x)]^{n-i}. \#(11)$$

Substituting Equation 5 and Equation 6 in Equation 11 ,if $\alpha \neq 1$, the i th order statistic of the KAPLL distribution reduces

$$f_{x_{i:n}}(x) = \frac{ab\lambda\beta n!\ \ln\alpha(\alpha-1)}{(n-i)!\,(i-1)!} x^{-\beta-1} d_i^{2\ 2} \alpha^{d_i} \eta_i^{\alpha-1} [1-\eta_i^{\alpha}]^{b(n-i+1)-1},$$

where $d_i = \left(1+\frac{\lambda}{x_i^{\beta}}\right)^{-1}$ and $\eta_i = (\alpha^{d_i}-1)/(\alpha-1)$.

$$f_{x_{i:n}}(x) = \sum_{k=0}^{i-1} \sum_{j,m=0}^{\infty} \frac{ab(\ln\alpha)^{s+1}(-1)^{k+j+m}[a(j+1)-m]^{s}}{\beta_{(i,n-i+1)} s!\,(\alpha-1)^{a(j+1)}} \lambda\beta x^{-\beta-1} d_i^{s+2}$$
$$\times \binom{i-1}{k} \binom{b(n+k-i+1)-1}{j} \binom{a(j+1)-1}{m}.$$

Or simply in the form

$$f_{i:n}(x) = \sum_{s=0}^{\infty} d_s h_{s+1}(x),$$

where $h_{s+1}(x) = (s+1)g(x)[G(x)]^{s}$ is the exponentiated-LL density with power parameter $(s+1) > 0$, and

$$d_s = \sum_{k=0}^{i-1} \sum_{j,m=0}^{\infty} \frac{ab(\ln\alpha)^{s+1}(-1)^{k+j+m}[a(j+1)-m]^{s}}{\beta_{(i,n-i+1)} (s+1)!\,(\alpha-1)^{a(j+1)}} \binom{i-1}{k}$$
$$\times \binom{b(n+k-i+1)-1}{j} \binom{a(j+1)-1}{m}.$$

4. Estimation of the Parameters

In this section, we use the maximum likelihood (ML) method to estimate the KAPLL parameter.

Let $x_1, x_2, \ldots, x_n$ be a random sample from KAPLL distribution then the logarithm of the likelihood function (ℓ), becomes

$$\ell = n[\ln a + \ln b + \ln\lambda + \ln\beta] + n\ln\left(\frac{\ln\alpha}{\alpha - 1}\right) + (\beta - 1)\sum_{i=1}^{n} \ln(x_i) -$$

$$\lambda\sum_{i=1}^{n} x_i^{\beta} + \ln(\alpha)\sum_{i=1}^{n} d_i + (a-1)\sum_{i=1}^{n} \ln(\eta_i) + (b-1)\sum_{i=1}^{n} \ln(1-\eta_i^{a}),$$

where $d_i = \left(1 + \frac{\lambda}{x_i^{\beta}}\right)^{-1}$ and $\eta_i = (\alpha^{d_i} - 1)/(\alpha - 1)$.

To obtain the MLE of a, b, α, λ and β, the first derivatives of ℓ are obtained with respect to a, b, α, λ and β. These derivatives are

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln(\eta_i) - (b-1)\sum_{i=1}^{n} \eta_i^{a}(1-\eta_i^{a})^{-1} \ln(\eta_i),$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(1-\eta_i^{a}),$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n[\alpha - 1 - \alpha\ln(\alpha)]}{\alpha(\alpha-1)\ln(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n} d_i + (a-1)\sum_{i=1}^{n} \left[\frac{(\alpha-1)d_i\alpha^{d_i-1} - (\alpha^{d_i}-1)}{(\alpha-1)(\alpha^{d_i}-1)}\right]$$

$$- a(b-1)\sum_{i=1}^{n} \eta_i^{a-1}(1-\eta_i^{a})^{-1}\left[\frac{(\alpha-1)d_i\alpha^{d_i-1} - (\alpha^{d_i}-1)}{(\alpha-1)^2}\right],$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\beta} - \ln(\alpha)\sum_{i=1}^{n} \frac{d_i}{x_i^{\beta}} - (a-1)\sum_{i=1}^{n} \frac{d_i^2\alpha^{d_i}\ln(\alpha)}{\eta_i x_i^{\beta}(\alpha-1)} + (b-1)\sum_{i=1}^{n} \frac{d_i^2\alpha^{d_i}\eta_i^{a-1}\ln(\alpha)}{x_i^{\beta}(\alpha-1)(1-\eta_i^{a})},$$

and

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(x_i) - \lambda\sum_{i=1}^{n} x_i^{\beta}\ln(x_i) + \ln(\alpha)\sum_{i=1}^{n} \lambda d_i^2 x_i^{-\beta}\ln(x_i) +$$

$$(a-1)\sum_{i=1}^{n} \frac{\lambda d_i^2\alpha^{d_i}\ln(\alpha)\ln x_i}{x_i^{\beta}(\alpha^{d_i}-1)} - a(b-1)\sum_{i=1}^{n} \frac{\lambda d_i{}^2\alpha^{d_i}\ln(\alpha)\ln(x_i)\eta_i^{a-1}}{x_i^{\beta}(\alpha-1)(1-\eta_i^{a})}.$$

5. Simulation Analysis

Now, we provided detailed simulation results to explore the performances of the ML estimation in estimating the parameters of the KAPLL model. We considered several sample sizes and different values of the parameters, that is, $n = \{50,100,250,500,750\}$ We generated $\mathrm{N} = 1000$ random samples using Equation 8. The behavior of the different estimates is compared with respect to their: average absolute biases $(|BIAS|), |BIAS| = \frac{1}{N}\sum_{i=1}^{N}|\hat{\theta} - \theta|$, mean square errors (MSE), $MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta} - \theta)^2$, and mean relative errors (MRE), MRE $= \frac{1}{N}\sum_{i=1}^{N}|\hat{\theta} - \theta|/\theta$. The Tables 2 and 3 show the simulation results, average ML estimates of the parameters, $|BIAS|$, MSE, and MRE, of the KAPLL parameters using different approaches. These results showed that estimates are very close to their true values and have small biases, MSE and MRE. The results illustrated that the biases, MSE, and MRE decrease as n increases, showing that the introduced estimators are consistent.

Table 2: The AEs, MSE, BIAS and MRE of the KAPLL parameters for different values of the parameter and n

n	α=0.5, β=1.5, λ=0.75, a=0.6, b=0.3											
50	AEs	2.1728	1.3551	10.0240	4.2151	3.7464	MSE	667.0271	0.6764	83232.9566	63.1576	8417.5848
100		0.9903	1.3400	0.9144	2.1318	0.5661		13.3704	0.4209	0.6706	16.1204	0.6286
250		0.6076	1.3619	0.9303	1.0080	0.3919		2.4015	0.1918	0.4558	1.6152	0.0577
500		0.4796	1.4246	0.9180	0.7292	0.3427		0.0703	0.1028	0.3060	0.1672	0.0170
750		0.4904	1.4257	0.8695	0.6889	0.3298		0.0545	0.0678	0.1913	0.0752	0.0081
50	|BIAS|	2.1098	0.6897	9.7455	3.7946	3.5305	MRE	4.2196	0.4598	12.9941	6.3243	11.7683
100		0.8430	0.5314	0.4917	1.6829	0.3292		1.6860	0.3543	0.6556	2.8049	1.0974
250		0.3796	0.3400	0.3747	0.5104	0.1329		0.7591	0.2267	0.4996	0.8507	0.4431
500		0.2007	0.2299	0.2849	0.2230	0.0757		0.4013	0.1533	0.3799	0.3717	0.2523
750		0.1667	0.1816	0.2192	0.1659	0.0560		0.3335	0.1211	0.2922	0.2764	0.1866

Table 3: The AEs, MSE, BIAS and MRE of the KAPLL parameters for different values of the parameter and n

n	$\alpha=0.5,\ \beta=1.5,\ \lambda=0.75,\ a=0.6,\ b=0.5$											
50	AEs	2.9617	1.4261	3.4081	3.2683	2.8566	MSE	531.4093	1.0405	2290.3986	34.9617	228.9894
100		1.1988	1.3767	1.1114	1.7714	1.2585		25.9460	0.6199	1.8929	7.8881	5.3762
250		0.7363	1.3929	1.0509	0.8875	0.6662		3.6896	0.2461	0.7184	0.6184	0.3088
500		0.5135	1.4240	1.0839	0.7337	0.5643		0.1924	0.1429	0.6732	0.1285	0.0445
750		0.4944	1.4241	1.0377	0.7034	0.5482		0.1252	0.1103	0.5249	0.0694	0.0231
50	|BIAS|	2.9086	0.8259	3.1349	2.8713	2.5312	MRE	5.8171	0.5506	4.1799	4.7854	5.0624
100		1.0709	0.6445	0.7147	1.3459	0.8924		2.1418	0.4296	0.9530	2.2431	1.7848
250		0.5629	0.4023	0.5441	0.4075	0.2590		1.1257	0.2682	0.7255	0.6791	0.5181
500		0.3536	0.3056	0.5288	0.2327	0.1416		0.7071	0.2037	0.7051	0.3878	0.2832
750		0.3158	0.2725	0.4685	0.1908	0.1155		0.6317	0.1817	0.6247	0.3180	0.2310
n	$\alpha=0.5,\ \beta=1.5,\ \lambda=0.75,\ a=1.3,\ b=0.3$											
50	AEs	0.4163	1.4543	0.7934	9.9084	0.5527	MSE	1.1381	0.6305	1.1916	257.0448	0.5400
100		0.7821	1.3798	0.7522	7.0192	0.4733		158.9650	0.4087	0.4065	135.9476	0.1611
250		0.4473	1.3694	0.7352	3.7217	0.3905		0.0778	0.1964	0.1789	42.6789	0.0431
500		0.4780	1.4117	0.7785	2.0879	0.3510		0.0543	0.1082	0.1071	6.7132	0.0182
750		0.4843	1.4296	0.7685	1.6694	0.3346		0.0352	0.0698	0.0646	1.4057	0.0102
50	|BIAS|	0.4137	0.6691	0.6112	8.9559	0.3465	MRE	0.8274	0.4460	0.8149	6.8891	1.1549
100		0.6963	0.5342	0.4637	6.0416	0.2408		1.3926	0.3562	0.6182	4.6474	0.8026
250		0.2233	0.3423	0.2977	2.6339	0.1331		0.4465	0.2282	0.3969	2.0261	0.4436
500		0.1699	0.2341	0.2068	0.9501	0.0807		0.3398	0.1561	0.2757	0.7309	0.2689
750		0.1252	0.1741	0.1541	0.4910	0.0570		0.2504	0.1161	0.2055	0.3777	0.1899
n	$\alpha=0.5,\ \beta=1.5,\ \lambda=1,\ a=0.6,\ b=0.3$											
50	AEs	1.54103	1.33741	1.17181	4.63679	0.88468	MSE	113.47375	0.7528	1.57522	77.40649	2.95225
100		1.2575	1.34158	1.18359	2.17199	0.55937		64.0291	0.42283	0.92763	16.87478	0.53473
250		0.48213	1.36369	1.21855	1.0412	0.39197		0.1931	0.19078	0.55121	2.21197	0.05924
500		0.51814	1.39512	1.17976	0.76106	0.35004		0.72911	0.10082	0.35164	0.26772	0.0165
750		0.51525	1.42828	1.11597	0.68652	0.32986		0.10853	0.06584	0.20619	0.08561	0.00968
50	|BIAS|	1.47507	0.73413	0.80595	4.21671	0.66267	MRE	2.95014	0.48942	0.80595	7.02785	2.20889
100		1.10249	0.53028	0.61146	1.71458	0.32276		2.20498	0.35352	0.61146	2.85764	1.07587
250		0.26812	0.33627	0.44028	0.55082	0.13236		0.53625	0.22418	0.44028	0.91804	0.44121
500		0.22421	0.22833	0.32169	0.23835	0.07456		0.44842	0.15222	0.32169	0.39725	0.24853
750		0.1655	0.17027	0.22973	0.16061	0.05629		0.33099	0.11351	0.22973	0.26769	0.18764
n	$\alpha=0.5,\ \beta=1.5,\ \lambda=1,\ a=0.6,\ b=0.5$											
50	AEs	2.52779	1.42585	5.11456	3.174	3.25387	MSE	228.97008	1.0378	8210.41973	35.03124	824.43492
100		1.2029	1.38497	1.41674	1.69825	1.21188		22.31918	0.62929	2.89763	6.98005	4.48509
250		0.71353	1.39264	1.37229	0.88792	0.6665		3.78127	0.25117	1.0878	0.65997	0.32634
500		0.50497	1.42421	1.38712	0.7334	0.56317		0.19516	0.13963	0.91856	0.13128	0.04268
750		0.49602	1.41928	1.34203	0.70816	0.55033		0.12386	0.11336	0.75361	0.07309	0.02396
50	|BIAS|	2.46873	0.83438	4.72668	2.78063	2.92586	MRE	4.93746	0.55625	4.72668	4.63439	5.85172
100		1.07208	0.64446	0.8878	1.27283	0.84778		2.14417	0.42964	0.8878	2.12139	1.69555
250		0.54621	0.40508	0.68615	0.40668	0.2591		1.09243	0.27005	0.68615	0.67781	0.5182
500		0.34139	0.30145	0.62313	0.23341	0.13862		0.68278	0.20096	0.62313	0.38902	0.27725
750		0.31154	0.27433	0.56627	0.19312	0.11618		0.62307	0.18288	0.56627	0.32187	0.23237

6. Real Data Application

In this section, we present a real-life data application to show the importance of the KAPLL distribution. The data set is studied by [24] and it represents set consists of 40 observations of time to failure $(10^3 h)$ of turbocharger of one type of engine. The data are as follows: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0 , 8.7, 8.8, 9.0.

The fits of the KAPLL distributions shall be compared with some rival distributions, namely: Log-logistic (LL), alpha power transformed log-logistic (APTLL) by [1], beta log-logistic (BLL) by [15], Kumaraswamy Marshall-Olkin log-logistic (KMOLL) by [6], McDonald log-logistic (McLL) by [23], additive Weibull log logistic (AWLL) by [12].

To compare the fitted distributions, we consider some criteria namely: The Akaike information criterion (AIC), Kolmogorov-Smirnov (KS) and p value (PV). The R program is used to obtain the numerical results in this section. Table 4 list the goodness-of-fit measures of the competing models and The MLEs and SEs for the data. It is observed, from Tables that the KAPLL distribution have the lowest values for goodness-of-fit criteria among all fitted models. So, it could be chosen as the best model for the analyzed data set.

Table 4: The ML estimates, SEs, and goodness-of-fit criteria for time to failure data

Distribution	AIC	KS	PV	Estimates					
KAPLL	167.85890	0.07307	0.98318	9658.12900	6.19488	999785.20000	0.43938	30.83448	
$(\alpha, \beta, \lambda, a, b)$				6823.38144	0.35984	6803.29770	0.11247	43.97041	
LL	181.41330	0.14369	0.38072	4.84162	6996.47772				
(β, λ)				0.70121	9447.15447				
APTLL	183.39570	0.14407	0.37745	1.93292	5.82120	4.81336			
(α, a, b)				5.62217	1.72831	0.69531			
BLL	182.67120	0.12612	0.54798	1693.51671	1.03733	6.98804	2322.59589		
(α, β, a, b)				2265.89606	0.44420	5.70121	2147.19265		
KMOLL	174.94480	0.10770	0.74232	28.71171	22.67472	5.66345	0.68373	2087.31877	
$(\alpha, \beta, \lambda, a, b)$				241.16458	984.37198	26.24854	3.16798	26900.79379	
McLL	183.85530	0.12411	0.56878	5.83242	77.11220	1.10483	79.95686	0.18127	
$(\alpha, \beta, \lambda, a, b)$				2.63637	3.15352	0.72072	51.18584	0.12886	
AWLL	176.95100	0.10770	0.74231	1.89384	41.17856	0.01666	4.73299	998.90351	2.04481
$(\alpha, \lambda, a, b, c, d)$				26.33612	341.11273	74.60581	76.26542	32032.35327	28.43561

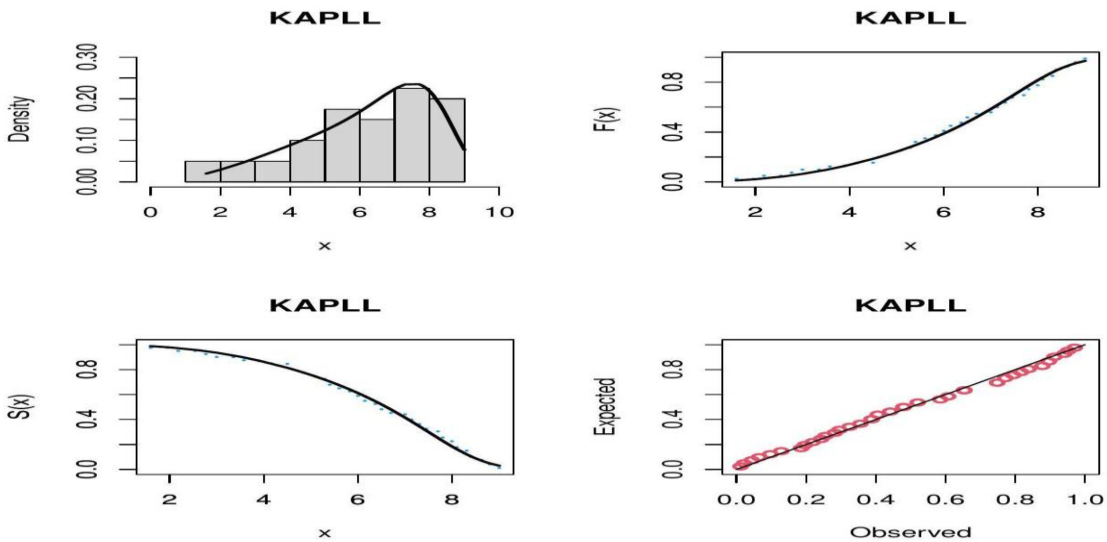

Figure 3: The fitted functions of the KAPLL model for time to failure data.

7. Conclusions

In this paper, we introduce a new five-parameter distribution called the Kumaraswamy alpha power log-logistic (KAPLL) distribution. The mathematical properties of the KAPLL model are derived. Further, the KAPLL parameters are estimated by the maximum likelihood method. A simulation study is conducted to explore the performance of the maximum likelihood method. Finally, the practical importance of the KAPLL distribution is studied by analyzing a real-life data set. Goodness-of-fit statistics for the analyzed data set showed that our KAPLL distribution provides a better fit in comparison with other rival distributions.

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الملخص العربى:

في هذه المقالة، ندرس امتدادًا جديدًا لنموذج اللوغاريتم اللوجستي يسمى توزيع اللوغاريتم اللوجستي بقوة ألفا كوماراسوامي (KAPLL) ، وهو امتداد لنموذج اللوغاريتم اللوجستي. ويبحث في بعض خصائصه الرياضية والإحصائية، بما في ذلك خصائص الموثوقية) دالة البقاء، ودالة معدل الخطر (HRF)، والعزوم، ووظائف الكميات(QF) ، ووظائف توليد العزوم (، مع التأكيد على فائدتها في نمذجة معايير الشيخوخة والفشل المتنوعة. تكمن إحدى المزايا الرئيسية لتوزيع KAPLL في قدرته على تمثيل كثافته كمزيج من الكثافات اللوغاريتمية اللوجستية، مما يوفر أشكالًا متماثلة وغير متماثلة لمزيد من مرونة النمذجة. يتم تقدير معلمات KAPLL من خلال تقدير الاحتمالية القصوى(MLE) ، وهي طريقة إحصائية مستخدمة على نطاق واسع. تقدم الدراسة نتائج محاكاة شاملة لتقييم فعالية تقنية التقدير المقترحة. علاوة على ذلك، يتم إجراء تطبيق عملي على البيانات في العالم الحقيقي لإظهار قابلية التكيف وتنوع توزيع KAPLL عند مقارنته بامتدادات أخرى لنموذج اللوغاريتم اللوجستي.

الكلمات الدالة : كوماراسوامي، قوة ألفا، التوزيع اللوجستي.