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WAVELET BASED LIFTING SCHEMES FOR THE NUMERICAL SOLUTION OF BURGERS-FISHER EQUATIONS

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Abstract. There are many areas of practical mathematics that use partial differential equations (PDEs), such as quantum physics, hydrodynamics, elasticity, and electromagnetic theory. The analytical behavior of these equations is a rather involved process and requires the application of advanced mathematical methods. The wavelet is a powerful mathematical tool that plays an important role in science and technology. The Burgers-Fisher equation is a non-linear partial differential equation and has important applications in financial mathematics, gas dynamics, traffic flow, number theory, heat conduction, and elasticity, among many other problems in applied mathematics and physics. In this paper, we presented a wavelet-based lifting scheme for the numerical solution of Burgers-Fisher equations using orthogonal and biorthogonal wavelet filter coefficients. The numerical results obtained by this scheme are compared with the exact solution to demonstrate the accuracy and also speed up convergence in less computational time as compared with the existing scheme. Some test problems are presented about the applicability and validity of the scheme.

1. Introduction

Most of the problems arisen in the nature are modeled by using the non-linear partial differential equations. In this connection, Burgers-Fisher equation is prevalent in various domains of applied physics. Notable applications include fluid dynamics, turbulence, the generation of shock waves, and financial mathematics. The Burgers-Fisher equation serves to elucidate a range of physical phenomena, particularly the interactions among reaction mechanisms, diffusion transport, and convection effects. It incorporates diffusion transport characteristics derived from the Fisher differential equation and convection attributes from the renowned Burgers differential equation. The analysis conducted by Burgers and Fisher [10] employed nonlinear partial differential equation techniques to address this

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equation. Furthermore, the diffusion, reaction, and convection terms in this equation combine to form a parabolic-hyperbolic kind of nonlinear partial differential equation. Numerous methodologies have been developed to find the solution to this problem. In recent decades, several researchers have employed various iterative techniques to obtain both analytical and numerical solutions for nonlinear partial differential equations. Notable methods include the Homotopy perturbation method [8] the Adomian decomposition method [2], the Biorthogonal wavelet based multigrid and Full approximation scheme [11] , as well as the Adaptive grid Haar wavelet collocation method [12].

Wavelet analysis emerged as a significant field in the 1980s due to its successful application in signal and image processing. The method involves the hierarchical translation and dilation of a single function, resulting in a smooth orthonormal basis that proved invaluable for the creation of compression algorithms tailored for signals and images within specified amplitude thresholds. Key advancements in this area encompass wavelet series expansion in applied mathematics, sub-band coding designed for voice and image compression, and multiresolution signal processing employed in computer vision.

The wavelet-based full approximation scheme (WFAS) has demonstrated significant effectiveness and advantages over the traditional full approximation scheme (FAS) in addressing various challenges within computational science and engineering. References [3]-[5] introduced a set of discrete wavelet transforms (DWT) alongside the FAS. The WFAS has been shown to be a highly efficient and beneficial method for a wide range of issues in the fields of computational science and engineering [4]. These techniques can function as either iterative solvers or preconditioning methods, often yielding superior performance compared to several contemporary and advanced FAS algorithms. Further investigations have been conducted to enhance the efficiency and capabilities of WFAS. To achieve this objective, a construction utilizing the orthogonal/biorthogonal discrete wavelet transform through a lifting scheme has been developed [6]. In [15], Sweldens introduced a wavelet-based lifting technique that allows for enhancements in the characteristics of current wavelet transforms. Additionally, Shiralashetti et al. [13] presented a wavelet-based numerical approach to address elasto-hydrodynamic lubrication issues using a lifting scheme.

In [1], the numerical analysis of the Burger-Fisher equation was examined utilizing Haar wavelet operational matrices through the integration of Haar wavelet bases. As the number of iterations increases, the computational time and costs also rise, making it challenging to formulate operational matrices for higher orders. To address this issue and reduce both error and CPU time consumption, filter coefficients are crucial. Therefore, this study focuses on the wavelet lifting scheme, employing the filter coefficients of both orthogonal and biorthogonal wavelets.

Filter coefficients serve as essential instruments in image processing, facilitating the acquisition of clear images. This principle is similarly employed by numerous researchers in the field of numerical analysis. In this study, the wavelet lifting scheme, recognized for its efficiency in error minimization, is utilized. This approach incorporates both orthogonal and biorthogonal wavelet filter coefficients to reduce error and computational time for the problem at hand.

The lifting scheme commences with a collection of established filters, followed by the application of lifting steps aimed at enhancing the characteristics of the associated wavelet decomposition. This approach offers several mathematical advantages, including a decrease in the number of operations, which is crucial for iterative solvers. Furthermore, this paper demonstrates the utilization of the lifting scheme in the numerical resolution of Burger-Fisher equations.

This paper is structured in the following manner: Section 2 discusses the preliminary wavelet filter coefficients and the lifting scheme. The solution methodology is outlined in Section 3. Section 4 presents numerical simulations of the test problems. Lastly, Section 5 concludes the proposed work.

2. Preliminaries of wavelet filter coefficients and Lifting scheme

The lifting scheme starts with a set of well-known filters; thereafter, lifting steps are used in an attempt to improve the properties of corresponding wavelet decomposition. Now, we have discussed different wavelet filters as follows:

2.1. **Haar wavelet filter coefficients.** We know that low pass filter $[h_0, h_1]^T = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ and high pass filter coefficients $[g_0, g_1]^T = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$ play an important role in decomposition. Thus, it is natural to wonder if it is possible to model the decomposition in terms of linear transformations, i.e., matrices. Moreover, since digital signals and images are composed of discrete data, we need a discrete analog of the decomposition algorithm so that we can process signal and image data.

2.2. **Daubechies wavelet filter coefficients.** Daubechies introduced scaling functions having the shortest possible support. The scaling function ϕ_N has a support $[0, N-1]$, while the corresponding wavelet Ψ_N has support in the interval $[1-\frac{N}{2}, \frac{N}{2}]$.
We have low pass filter coefficients $[h_0, h_1, h_2, h_3]^T = \left[\frac{1+\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, \frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{1-\sqrt{3}}{4\sqrt{2}}\right]^T$.
and high pass filter coefficients $[g_0, g_1, g_2, g_3]^T = \left[\frac{1-\sqrt{3}}{4\sqrt{2}}, -\frac{3-\sqrt{3}}{4\sqrt{2}}, \frac{3+\sqrt{3}}{4\sqrt{2}}, -\frac{1+\sqrt{3}}{4\sqrt{2}}\right]^T$.

2.3. **Biorthogonal (CDF (2,2)) wavelets.** Lets consider the $(5,3)$ biorthogonal spline wavelet filter pair are
$[h_0, h_1, h_2]^T = \left(\frac{1}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, and $\left[\tilde{h}_{-2}, \tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1, \tilde{h}_2\right]^T = \left(\frac{-1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{-1}{4\sqrt{2}}\right)$
Similarly, high pass filters: $g_k = (-1)^k \tilde{h}_{4-k}$ and $g_k = (-1)^{k+1}\tilde{h}_{2-k}$

2.4. **Foundations of lifting scheme.** The wavelet transform utilizes averages and differences, leading us to the concept of the lifting procedure. The operations of averaging and differencing can be regarded as specific instances of broader operations. When two data points are nearly identical, the difference is minimal, suggesting that the first data point serves as a reasonable prediction for the second.

This prediction is deemed effective if the difference remains small. Additionally, we computed the average of the two data points, which can be interpreted in two ways: either as an operation that retains certain characteristics of the original data or as a means of extracting a fundamental property of the data. The final perspective emphasizes that the pair-wise average values encapsulate the overall structure of the data while utilizing only half the original data points. The lifting procedure has three steps i.e. split, prediction and update. Finally, a wavelet transform built through lifting consists of three steps: split. Predict and update as given in the figure 1 [9]. **Split:** Splitting the signal into two disjoint sets of samples.
Predict: If the signal contains some structure, then we can expect a correlation between a sample and its nearest neighbors, i.e. $d_{j-1} = odd_{j-1} - P(even_{j-1})$
Update: Given an even entry, we have predicted that the next odd entry has the same value and stored the difference. We then update our even entry to reflect our knowledge of the signal, i.e. $s_{j-1} = even_{j-1} + U(d_{j-1})$
The general lifting stages for decomposition and reconstruction of a signal are given in figure 2.

The detailed algorithm using different wavelets is given in the next section.

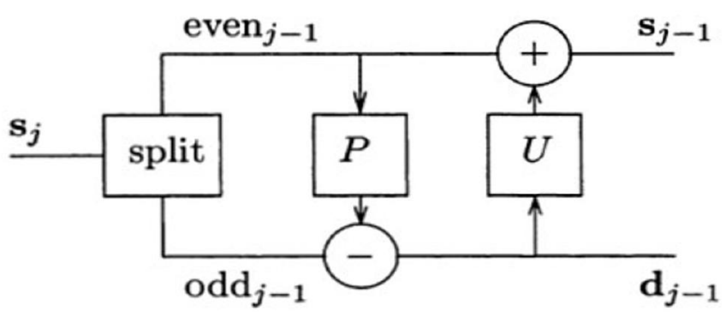


Figure 1. Steps in lifting scheme

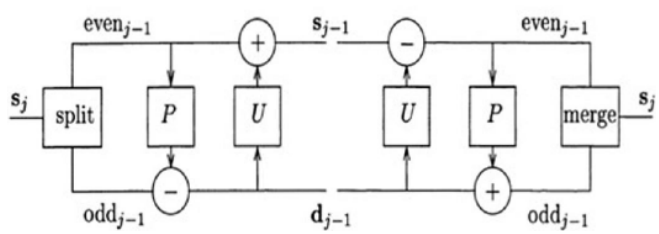


Figure 2. Steps in lifting scheme

3. Method of solution

Consider the generalized Burger-Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha u^n \frac{\partial u}{\partial x} + \beta u\Big(1 - u^n\Big), 0 \leq x \leq 1, t > 0. \tag{1}$$

Where β are any constants. After discretizing the equation 1 through the finite difference method (FDM), we get systems of algebraic equations. Through this system, we can write the system as

$$Au = b \tag{2}$$

where A is $N \times N$ coefficient matrix, b is $N \times N$ matrix and u is $N \times N$ matrix to be determined. where $N = 2^J$, N is the number of grid points and J is the level of resolution. Solve equation 2 through the iterative method, and we get an approximate solution. An approximate solution contains some error; therefore, the required solution equals the sum of the approximate solution and error. There are many methods to minimize such errors to get an accurate solution. Some of them are HWLS, DWLS, BWLS, etc. Recently, lifting schemes have been useful in signal analysis and image processing in the area of science and engineering. But currently it extends to approximations in the numerical analysis [4]. Here, we are discussing the algorithm of the lifting schemes as follows:

3.1. **Haar wavelet Lifting scheme (HWLS).** In [6], Daubechies and Sweldens have shown that every wavelet filter can be decomposed into lifting steps. More details of the advantages as well as other important structural advantages of the lifting technique can be available in [15]. The representation of Haar wavelet via lifting form is presented as;
Decomposition: Consider an approximate solution, $S = \tilde{P}_j$ like a signal, and then apply the HWLS decomposition (finer to coarser) procedure as,

$$\begin{aligned} d_1 &= S2j - S(2j-1), \\ s_1 &= S(2j-1) + \frac{1}{2}d_1, \\ S_1 &= \sqrt{2}s_1, \\ D &= \frac{1}{\sqrt{2}}d_1. \end{aligned} \tag{3}$$

In this stage finally, we get new approximation as,

$$S = [S_1 D] \tag{4}$$

Reconstruction: Consider equation 2 and then apply the HWLS reconstruction (coarser to finer) procedure as,

$$\begin{aligned} d_1 &= \sqrt{2}D, \\ s_1 &= \frac{1}{\sqrt{2}}S_1, \\ S(2j-1) &= s_1 - \frac{1}{2}d_1, \\ S(2j) &= d_1 + S(2j-1). \end{aligned} \tag{5}$$

which is the required solution of the given equation.

3.2. **Daubechies wavelet Lifting scheme (DWLS).** As discussed in the previous section 3.1, we followed the same procedure but used a different wavelet, i.e., Daubechies 4th order wavelet coefficient. The DWLS procedure is as follows:
Decomposition:

$$\left.\begin{aligned} s_1 &= S(2j-1) + \sqrt{3}S(2j), \\ d_1 &= S(2j) - \tfrac{\sqrt{3}}{4}s_1(j) - \tfrac{\sqrt{3}-2}{4}s_1(j-1), \\ s_2 &= s_1 - d_1 j + 1, \\ S_1 &= \tfrac{\sqrt{3}-1}{\sqrt{2}}s_2, \\ D &= \tfrac{\sqrt{3}+1}{\sqrt{2}}d_1. \end{aligned}\right\} \tag{6}$$

Here, we get a new approximation as,

$$S = [S_1 D] \tag{7}$$

Reconstruction: Consider equation 5, then apply the DWLS reconstruction (coarser to finer) procedure as,

$$\left.\begin{aligned} d_1 &= \tfrac{\sqrt{2}}{\sqrt{3}+1}D, \\ s_2 &= \tfrac{\sqrt{2}}{\sqrt{3}-1}S_1, \\ s_1 &= s_2 + d_1(j+1), \\ S(2j) &= d_1 + \tfrac{\sqrt{3}}{4}s_1 + \tfrac{\sqrt{3}-2}{4}s_1(j-1), \\ S(2j-1) &= s_1 - \sqrt{3}S(2j). \end{aligned}\right\} \tag{8}$$

which is the required solution of the given equation.

3.3. **Biorthogonal wavelet Lifting scheme (BWLS).** As discussed in the previous sections 3.1 and 3.2, we follow the same procedure here; we used another wavelet, i.e., a biorthogonal wavelet (CDF (2,2)). The BWLS procedure is as follows:
Decomposition:

$$\left.\begin{aligned} &d_1 = S(2j) - \tfrac{1}{2}[S(2j-1) + S(2j+2)],\\ &s_1 = S(2j-1) + \tfrac{1}{4}[d_1(j-1) + d_1],\\ &D = \tfrac{1}{\sqrt{2}} d_1,\\ &S = \sqrt{2} s_1. \end{aligned}\right\} \tag{9}$$

In this stage, finally, we get a new signal as,

$$S = [S_1 D] \tag{10}$$

Reconstruction: Consider equation 10, and then apply the DWLS reconstruction (coarser to finer) procedure as

$$\left.\begin{aligned} &s_1 = \tfrac{1}{2} S_1,\\ &d_1 = \sqrt{2} D,\\ &S(2j-1) = s_1 - \tfrac{1}{4}[d_1(j-1) + d_1],\\ &S(2j) = d_1 + \tfrac{1}{2}[S(2j-1) + S(2j+2)]. \end{aligned}\right\} \tag{11}$$

which is the required solution of the given equation.
The coefficients $s_1^{(j)}$ and $d_1^{(j)}$ are the average and detailed coefficients, respectively, of the approximate solution u_a. The new approaches are tested through some of the numerical problems, and the results are shown in the next section.

4. Numerical simulation

In this section, we implemented the lifting scheme to numerically solve the Burgers-Fisher equations, demonstrating the effectiveness and applicability of HWLS, DWLS, and BWLS. The error is calculated using the formula $E_{max} = max|u_e - u_a|$, where u_e and u_a represent the exact and approximate solutions, respectively. I will now focus on Fishers equation to explain the detailed methodology employed in solving it through the lifting scheme with various wavelets.
Problem 4.1 Now, we consider the Fishers equation [7]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \tag{12}$$

subject to I.C.

$$u(x,0) = \frac{1}{(1+e^x)^2} \tag{13}$$

and to B.C.s:

$$u(0,t) = \frac{1}{(1+e^{-5t})^2}, u(1,t) = \frac{1}{(1+e^{1-5t})^2} \tag{14}$$

which has the exact solution

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2} \tag{15}$$

Solving equation 12 using finite difference, we get the solution as

$$u = \begin{bmatrix} 0.486 & 0.730 & 0.876 & 0.948 & 0.435 & 0.685 & 0.848 & 0.934 \\ 0.380 & 0.639 & 0.822 & 0.922 & 0.319 & 0.589 & 0.797 & 0.913 \end{bmatrix}^T_{16\times 1}$$

The wavelet-based numerical solutions of equation 12 are obtained as per the procedure explained in Section 3.1 and are as follows:
Assume $S = u$, and then apply the HWLS as explained in Section 3.1 as,
Decomposition:

$$d_1 = S(1:2:15) - S(2:2:16)$$
$$= \begin{bmatrix} 0.244 & 0.072 & 0.250 & 0.085 & 0.259 & 0.100 & 0.270 & 0.115 \end{bmatrix}$$

$$s_1 = S(2:2:16) + \frac{d_1}{2}$$
$$= \begin{bmatrix} 0.608 & 0.912 & 0.560 & 0.891 & 0.509 & 0.872 & 0.454 & 0.855 \end{bmatrix}$$

$$S_1 = \sqrt{2}s_1 = \begin{bmatrix} 0.860 & 1.290 & 0.792 & 1.260 & 0.720 & 1.233 & 0.642 & 1.209 \end{bmatrix}$$

$$D = \frac{1}{\sqrt{2}}d_1 = \begin{bmatrix} 0.172 & 0.051 & 0.177 & 1.060 & 0.183 & 0.071 & 0.191 & 0.081 \end{bmatrix}$$

We get new approximation

$$S = \begin{bmatrix} S_1 D \end{bmatrix} = \begin{bmatrix} 0.860 & 1.290 & 0.792 & 1.260 & 0.720 & 1.233 & 0.642 & 1.209 \\ 0.172 & 0.051 & 0.177 & 0.060 & 0.183 & 0.071 & 0.191 & 0.081 \end{bmatrix}^T_{16\times 1}$$

Reconstruction: Then apply the HWLS reconstruction procedure as,

$$d_1 = \sqrt{2}D = \begin{bmatrix} 0.244 & 0.072 & 0.250 & 0.085 & 0.259 & 0.100 & 0.270 & 0.115 \end{bmatrix}$$

$$s_1 = \frac{1}{2}S_1 = \begin{bmatrix} 0.608 & 0.912 & 0.560 & 0.891 & 0.509 & 0.872 & 0.454 & 0.855 \end{bmatrix}$$

$$S(1:2:15) = s_1 - \frac{d_1}{2}$$
$$= \begin{bmatrix} 0.486 & 0.876 & 0.435 & 0.848 & 0.380 & 0.822 & 0.319 & 0.797 \end{bmatrix}$$

$$S(2:2:16) = d_1 + S(1:2:15)$$
$$= \begin{bmatrix} 0.730 & 0.948 & 0.685 & 0.934 & 0.639 & 0.922 & 0.589 & 0.913 \end{bmatrix}$$

Therefore

$$= \begin{bmatrix} 0.860 & 0.730 & 0.876 & 0.948 & 0.435 & 0.685 & 0.848 & 0.934 \\ 0.380 & 0.639 & 0.822 & 0.922 & 0.319 & 0.589 & 0.797 & 0.913 \end{bmatrix}^T_{16\times 1}$$

This is the required HWLS solution of the given equation.

Similarly, in DWLS, as discussed in Section 3.2, we follow the same procedure as;
Decomposition:

$$S(1:2:15)+\sqrt{3}S(1:2:16)$$
$$=\begin{bmatrix}1.750 & 2.518 & 1.622 & 2.465 & 1.486 & 2.419 & 1.338 & 2.378\end{bmatrix}$$

$$d_1=S(2:2:16)-\frac{\sqrt{3}}{4}s_1-\frac{\sqrt{3}-2}{4}[s_1(8)s_1(1:7)]$$
$$=\begin{bmatrix}0.131 & -0.025 & 0.152 & -0.025 & 0.160 & -0.026 & 0.171 & -0.027\end{bmatrix}$$

$$s_2=s_1-[d_1(2:8)d_1]$$
$$=\begin{bmatrix}1.776 & 2.366 & 1.647 & 2.305 & 1.512 & 2.248 & 1.366 & 2.477\end{bmatrix}$$

$$S_1=\frac{\sqrt{3}-1}{\sqrt{2}}s_2$$
$$=\begin{bmatrix}0.919 & 1.225 & 0.853 & 1.193 & 0.783 & 1.163 & 0.707 & 1.163\end{bmatrix}$$

$$D=\frac{\sqrt{3}+1}{\sqrt{2}}d_1$$
$$=\begin{bmatrix}0.253 & -0.049 & 0.293 & -0.049 & 0.310 & -0.050 & 0.331 & -0.053\end{bmatrix}$$

We get new approximation

$$S=\begin{bmatrix}S_1D\end{bmatrix}=\begin{bmatrix}0.919 & 1.225 & 0.853 & 1.193 & 0.783 & 1.163 & 0.707 & 1.163\\ 0.253 & -0.049 & 0.293 & -0.049 & 0.310 & -0.050 & 0.331 & -0.053\end{bmatrix}$$

Reconstruction: Then apply the DWLS reconstruction procedure as,

$$d_1=\frac{\sqrt{2}}{\sqrt{3}+1}D=\begin{bmatrix}0.131 & -0.025 & 0.152 & -0.025 & 0.160 & -0.026 & 0.171 & -0.027\end{bmatrix}$$

$$s_2=\frac{\sqrt{2}}{\sqrt{3}-1}S_1=\begin{bmatrix}1.776 & 2.366 & 1.647 & 2.305 & 1.512 & 2.248 & 1.366 & 2.247\end{bmatrix}$$

$$s_1=s_2+\begin{bmatrix}d_1(8)d_1(2:7)d_1(1)\end{bmatrix}$$
$$=\begin{bmatrix}1.750 & 2.518 & 1.622 & 2.465 & 1.486 & 2.419 & 1.388 & 2.378\end{bmatrix}$$

$$s(2:2:16)=d_1+\frac{\sqrt{3}}{4}s_1+\frac{\sqrt{3}-2}{4}\begin{bmatrix}s_1(2:8)s_1(1)\end{bmatrix}$$
$$=\begin{bmatrix}0.730 & 0.948 & 0.685 & 0.934 & 0.639 & 0.922 & 0.589 & 0.913\end{bmatrix}$$

$$S(1:2:16)=s_1-\sqrt{3}S(2:2:16)$$
$$=\begin{bmatrix}0.486 & 0.876 & 0.435 & 0.848 & 0.380 & 0.822 & 0.319 & 0.797\end{bmatrix}$$

Therefore

$$= \begin{bmatrix} 0.486 & 0.730 & 0.876 & 0.948 & 0.435 & 0.685 & 0.848 & 0.934 \\ 0.380 & 0.639 & 0.822 & 0.922 & 0.319 & 0.589 & 0.797 & 0.913 \end{bmatrix}^T_{16\times 1}$$

This is the required DWLS solution to the given equation. Also, BWLS is explained in Sections 3.3; we follow the similar procedure as follows:

Decomposition:

$$d_1 = S(2:2:16) - \frac{1}{2}\Big(\Big[S(1:2:15)\Big] + \Big[S(3:8)S(1:2)\Big]\Big)$$

$$= \Big[0.144 \quad 0.043 \quad 0.148 \quad 0.049 \quad 0.154 \quad 0.054 \quad 0.064 \quad 0.040\Big]$$

$$s_1 = S(1:2:15) + \frac{1}{4}\Big(\Big[d_1(8)d_1(2:7)d_1(1)\Big] + d_1\Big)$$

$$= \Big[0.532 \quad 0.923 \quad 0.483 \quad 0.897 \quad 0.431 \quad 0.874 \quad 0.348 \quad 0.824\Big]$$

$$D = \frac{1}{\sqrt{2}}d_1 = [0.102 \quad 0.030 \quad 0.105 \quad 0.034 \quad 0.109 \quad 0.039 \quad 0.046 \quad 0.028]$$

$$S_1 = \sqrt{2}s_1 = [0.735 \quad 1.305 \quad 0.683 \quad 1.269 \quad 0.609 \quad 1.236 \quad 0.493 \quad 1.165]$$

We get new approximation

$$S = [S_1 D] = \begin{bmatrix} 0.735 & 1.305 & 0.683 & 1.269 & 0.609 & 1.236 & 0.493 & 1.165 \\ 0.102 & 0.030 & 0.105 & 0.034 & 0.109 & 0.039 & 0.046 & 0.028 \end{bmatrix}^T_{16\times 1}$$

Reconstruction: Then apply the BWLS reconstruction procedure as,

$$s_1 = \frac{1}{\sqrt{2}}S_1 = \Big[0.532 \quad 0.932 \quad 0.483 \quad 0.897 \quad 0.431 \quad 0.874 \quad 0.348 \quad 0.824\Big]$$

$$d_1 = \sqrt{2}D = \Big[0.144 \quad 0.043 \quad 0.148 \quad 0.049 \quad 0.154 \quad 0.154 \quad 0.064 \quad 0.040\Big]$$

$$S(1:2:15) = s_1 - \frac{1}{4}\Big([d_1(8)d_1(1:7)\Big)\Big] + \Big[d_1(1)\Big]$$

$$= \Big[0.486 \quad 0.876 \quad 0.435 \quad 0.848 \quad 0.380 \quad 0.822 \quad 0.319 \quad 0.797\Big]$$

$$S(2:2:16) = d_1 - \frac{1}{2}\Big([S(1:2:15)] + \Big[S(3:8)S(1:2)\Big]\Big)$$

$$= \Big[0.730 \quad 0.948 \quad 0.685 \quad 0.934 \quad 0.639 \quad 0.922 \quad 0.589 \quad 0.913\Big]$$

Therefore

$$= \begin{bmatrix} 0.486 & 0.730 & 0.876 & 0.948 & 0.435 & 0.685 & 0.848 & 0.934 \\ 0.380 & 0.639 & 0.822 & 0.922 & 0.319 & 0.589 & 0.797 & 0.913 \end{bmatrix}^T_{16\times 1}$$

This is the required BWLS solution to the given equation.

The obtained numerical solutions and compared with the exact solutions are presented in Table 1, and the maximum absolute errors with CPU time of the methods are presented in Table 2.

TABLE 1. Comparison of numerical solutions with exact solution of the problem 4.1.

x	t	FDM	HWLS	DWLS	BWLS	Exact
0.2	0.2	0.486395	0.486395	0.486395	0.486395	0.476065
0.2	0.4	0.729769	0.435155	0.435155	0.435155	0.416872
0.2	0.6	0.876191	0.379759	0.379759	0.379759	0.358427
0.2	0.8	0.947800	0.318678	0.318678	0.318678	0.302317
0.4	0.2	0.435155	0.729769	0.729769	0.729769	0.736420
0.4	0.4	0.685101	0.685101	0.685101	0.685101	0.692255
0.4	0.6	0.848250	0.638659	0.638659	0.638659	0.643499
0.4	0.8	0.933705	0.588617	0.588617	0.588617	0.590630
0.6	0.2	0.379759	0.876191	0.876191	0.876191	0.888638
0.6	0.4	0.638659	0.848250	0.848250	0.848250	0.866503
0.6	0.6	0.822053	0.822053	0.822053	0.822053	0.840572
0.6	0.8	0.921850	0.797432	0.797432	0.797432	0.810449
0.8	0.2	0.318678	0.947800	0.947800	0.947800	0.956716
0.8	0.4	0.588617	0.933705	0.933705	0.933705	0.947513
0.8	0.6	0.797432	0.921850	0.921850	0.921850	0.936452
0.8	0.8	0.912665	0.912665	0.912665	0.912665	0.923203

TABLE 2. Maximum error and CPU time (in seconds) of the methods of the problem 4.1.

N×N	Method	E$_{max}$	Setup time	Running time	Total time
4×4	FDM	2.1332e-02	2.9144	0.0004	2.9148
4×4	HWLS	2.1332e-02	0.0011	0.0017	0.0028
4×4	DWLS	2.1332e-02	0.0010	0.0128	0.0138
4×4	BWLS	2.1332e-02	0.0009	0.0044	0.0053
64×64	FDM	2.6256e-03	10.1280	0.0034	10.1314
64×64	HWLS	2.6256e-03	0.0007	0.0012	0.0019
64×64	DWLS	2.6256e-03	0.0006	0.0083	0.0089
64×64	BWLS	2.6256e-03	0.0005	0.0029	0.0034

Problem 4.2 Now, Consider the Fishers equation (In equation 1: $\alpha = -1$, $n = 1$ and $\beta = 2$) [14]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + 2u\Big(1-u\Big), 0 \leq x \leq 1, t > 0 \tag{16}$$

subject to I.C.

$$u(x,0) = \frac{1}{2} + \frac{1}{2}tanh\Big(\frac{x}{4}\Big) \tag{17}$$

and to B.C.s:

$$u(0,t) = \frac{1}{2} + \frac{1}{4}tanh\Big(\frac{1}{2}\Big(\frac{9}{2}t\Big)\Big), \tag{18}$$
$$u(1,t) = \frac{1}{2} + \frac{1}{2}tanh\Big(\frac{1}{4}\Big(1+\frac{9}{2}t\Big)\Big)$$

which has the exact solution

$$u(x,t) = \frac{1}{2} + \frac{1}{2}tanh\Big(\frac{1}{4}\Big(x+\frac{9}{2}t\Big)\Big).$$

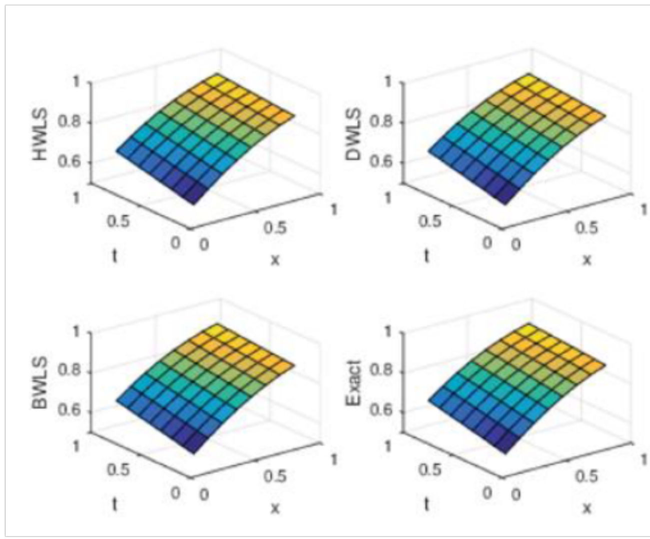
By applying the methods explained in the section 3 and in the problem 4.1, we obtain the numerical solutions and compared with exact solution are presented in Figure 3. The maximum absolute errors with CPU time of the methods are presented in Table 3.

Table 3. Maximum error and CPU time (in seconds) of the methods of problem 4.2.

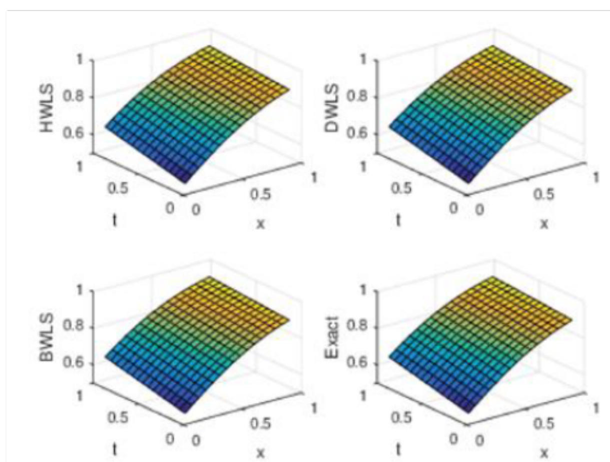
$N \times N$	Method	E$_{max}$	Setup time	Running time	Total time
4×4	FDM	5.6596e-03	3.2373	0.0020	3.2393
4×4	HWLS	5.6596e-03	0.0009	0.0029	0.0038
4×4	DWLS	5.6596e-03	0.0003	0.0097	0.0100
4×4	BWLS	5.6596e-03	0.0003	0.0040	0.0043
8×8	FDM	2.8678e-03	3.6739	0.0021	3.6960
8×8	HWLS	2.8678e-03	0.0010	0.0031	0.0041
8×8	DWLS	2.8678e-03	0.0003	0.0103	0.0106
8×8	BWLS	2.8678e-03	0.0004	0.0042	0.0046
32×32	FDM	7.6815e-04	5.3950	0.0027	5.3977
32×32	HWLS	7.6815e-04	0.0009	0.0030	0.0039
32×32	DWLS	7.6815e-04	0.0003	0.0096	0.0099
32×32	BWLS	7.6815e-04	0.0003	0.0047	0.0050
64×64	FDM	3.9016e-04	7.7931	0.0040	7.7971
64×64	HWLS	3.9016e-04	0.0009	0.0030	0.0039
64×64	DWLS	3.9016e-04	0.0003	0.0097	0.0100
64×64	BWLS	3.9016e-04	0.0003	0.0041	0.0044

Problem 4.3 Finally, consider another form of the Burger-Fisher equation (In equation 1: $\alpha = 1$, $n = 2$ and $\beta = 1$) [10]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^2\frac{\partial u}{\partial x} + u\Big(1-u^2\Big), 0 \le x \le 1, t > 0 \tag{19}$$



(a)



(b)

FIGURE 3. Comparison of numerical solutions with exact solution of problem 4.2 for (a) $N = 8 \times 8$, (b) $N = 16 \times 16$.

subject to I.C.

$$u(x,0) = \left[\frac{1}{2}\Big(1 - tanh\Big(\frac{x}{3}\Big)\Big)\right]^{\frac{1}{2}} \tag{20}$$

and to B.C.s:

$$u(0,t) = \left[\frac{1}{2}\Big(1 - tanh\Big(\frac{10}{9}t\Big)\Big)\right]^{\frac{1}{2}}, u(1,t) = \left[\frac{1}{2}\Big(1 - tanh\Big(\frac{1}{3} - \frac{10}{9}t\Big)\Big)\right]^{\frac{1}{2}} \tag{21}$$

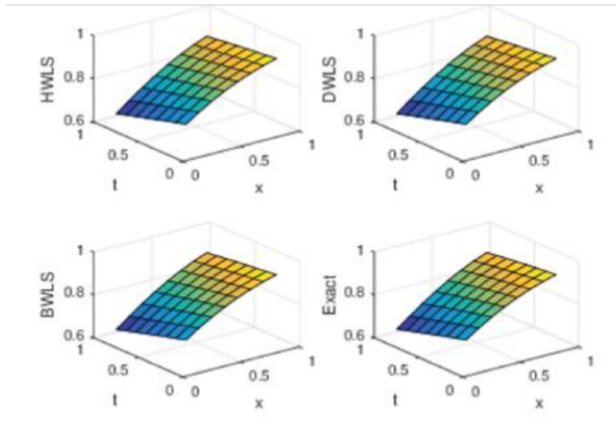
which has the exact solution

$$u(x,t) = \left[\frac{1}{2}\Big(1 - tanh\Big(\frac{x}{3} - \frac{10}{9}t\Big)\Big)\right]^{\frac{1}{2}}.$$

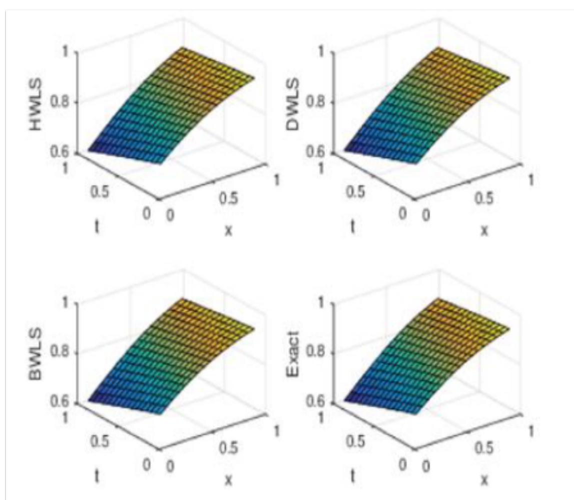
By applying the methods explained in the section 3 and in the problem 4.1, we obtain the numerical solutions and compared with exact solution are presented in Figure 4. The maximum absolute errors with CPU time of the methods are presented in Table4.

Table 4. Maximum error and CPU time (in seconds) of the methods of problem 4.3.

N×N	Method	E_{max}	Setup time	Running time	Total time
4×4	FDM	4.4208e-03	3.1919	0.0019	3.1938
4×4	HWLS	4.4208e-03	0.0009	0.0028	0.0037
4×4	DWLS	4.4208e-03	0.0003	0.0096	0.0099
4×4	BWLS	4.4208e-03	0.0003	0.0040	0.0043
8×8	FDM	2.0856e-03	4.6072	0.0019	4.6091
8×8	HWLS	2.0856e-03	0.0008	0.0028	0.0037
8×8	DWLS	2.0856e-03	0.0003	0.0096	0.0099
8×8	BWLS	2.0856e-03	0.0004	0.0039	0.0043
32×32	FDM	5.8218e-04	4.6153	0.0028	4.6181
32×32	HWLS	5.8218e-04	0.0009	0.0028	0.0037
32×32	DWLS	5.8218e-04	0.0003	0.0100	0.0103
32×32	BWLS	5.8218e-04	0.0004	0.0043	0.0047
64×64	FDM	2.9335e-04	7.5337	0.0041	7.5378
64×64	HWLS	2.9335e-04	0.0009	0.0029	0.0038
64×64	DWLS	2.9335e-04	0.0009	0.0098	0.0101
64×64	BWLS	2.9335e-04	0.0003	0.0041	0.0044



(a)



(b)

FIGURE 4. Comparison of numerical solutions with exact solution of problem 4.3 for (a) $N = 8 \times 8$, (b) $N = 16 \times 16$.

5. Conclusions

In this study, I utilized wavelet-based lifting schemes to numerically solve the Burgers-Fisher equations, employing various wavelet filters, including both orthogonal and biorthogonal wavelets. The analysis of the figures and tables presented indicates that.

- The numerical solutions obtained by different Lifting schemes are agrees with the exact solution.
- Convergence of the presented schemes is observed i.e. the error decreases when the level of resolution N increases.
- In addition the calculations involved in lifting schemes are simple, straight forward and low computation cost compared to classical method i.e. FDM.

Hence the presented lifting schemes in particular HWLS BWLS are very effective for solving non-linear partial differential equations. Moreover, the efficiency and reduced computational CPU time associated with the aforementioned wavelet lifting scheme can be applied to address divergent nonlinear ordinary and partial differential equations that arise in various domains of science and engineering.

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