Electronic Journal of Mathematical Analysis and Applications
Vol. 13(1) Jan 2025, No. 8
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

RELATIVE (α, β, γ)-ORDER OF MEROMORPHIC FUNCTION WITH RESPECT TO ENTIRE FUNCTION

CHINMAY BISWAS, BABLU CHANDRA DAS

Abstract. The growth investigation of meromorphic function has usually been done through the Nevanlinna's characteristic function comparing with the exponential function. Order and type are the classical growth indicators which are generalized by several authors during the past decades. Belaïdi et al. [3] have introduced the concepts of (α, β, γ)-order and (α, β, γ)-lower order of a meromorphic function taking $\alpha \in L_1$-class, $\beta \in L_2$-class, $\gamma \in L_3$-class. But if one is paying attention to evaluate the growth rates of any meromorphic function with respect to a entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. In order to make some progresses in the study of growth analysis of meromorphic functions, here in this paper, we have introduced the definitions of the relative (α, β, γ)-order and relative (α, β, γ)-lower order of a meromorphic function with respect to an entire function as well as their integral representations. We have also investigated some growth properties of meromorphic functions on the basis of relative (α, β, γ)-order and relative (α, β, γ)-lower order as compared to the growth of their corresponding left and right factors.

1. Introduction

The standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions are available in [4, 6, 7, 8, 9], so we do not explain those in details. For $x \in [0, +\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. For meromorphic function f,

2010 *Mathematics Subject Classification.* 30D30, 30D35.
Key words and phrases. Meromorphic function, relative (α, β, γ)-order, relative (α, β, γ)-lower order, integral representation.
Submitted Oct. 09, 2024, Accepted Jan. 01, 2024.

the Nevanlinna's characteristic function $T_f(r)$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

where $m_f(r)$ and $N_f(r)$ are respectively called as the proximity function of f and the counting function of poles of f in $|z| \leq r$. For details about $T_f(r)$, $m_f(r)$ and $N_f(r)$ one may see [4, p.4]. If f is an entire function, then the Nevanlinna's characteristic function $T_f(r)$ is defined as

$$T_f(r) = m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Moreover, if f is non-constant entire function, then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0), +\infty) \to (0, +\infty)$ exists and is such that $\lim_{s\to+\infty} T_f^{-1}(s) = +\infty$. To start our paper, we just recall the following definition:

Definition 1.1. *The order ρ_f and the lower order λ_f of a meromorphic function f are defined as:*

$$\rho_f = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log r}.$$

Now first of all, let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t)\cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, we obtain

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$. Throughout this paper we assume $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [5] have introduced the concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [5]. Later on Belaïdi et al. [3] have extended the above idea and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order of a meromorphic function f, which are as follows:

Definition 1.2. [3] *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$, of a meromorphic function f are defined as:*

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))} \\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))}. \end{aligned}$$

Mainly the growth investigation of meromorphic function has usually been done through the Nevanlinna's characteristic function comparing with the exponential function. But if one is paying attention to evaluate the growth rates of any meromorphic function with respect to a entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progresses in the study of relative order of meromorphic function, one may introduce the definitions of relative (α, β, γ)-order and relative (α, β, γ)-lower order of a meromorphic function with respect to an entire function in the following way:

Definition 1.3. *The relative (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]_h$ of a meromorphic function f with respect to an entire function h is defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f]_h = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

Definition 1.4. *The relative (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ of a meromorphic function f with respect to an entire function h is defined as:*

$$\lambda_{(\alpha,\beta,\gamma)}[f]_h = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

Remark 1.1. *An entire function f is said to have regular relative (α, β, γ)-order with respect to an entire function h if $\rho_{(\alpha,\beta,\gamma)}[f]_h = \lambda_{(\alpha,\beta,\gamma)}[f]_h$.*

Definition 1.5. *The growth indicator $\rho_{(\alpha,\beta,\gamma)}[f]_h$ is alternatively defined as: The integral $\int_{r_0}^{+\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp[\beta(\log(\gamma(r)))]]^{t+1}} dr$ $(r_0 > 0)$ converges when $t > \rho_{(\alpha,\beta,\gamma)}[f]_h$ and diverges when $t < \rho_{(\alpha,\beta,\gamma)}[f]_h$.*

Definition 1.6. *The growth indicator $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ is alternatively defined as: The integral $\int_{r_0}^{+\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp[\beta(\log(\gamma(r)))]]^{t+1}} dr$ $(r_0 > 0)$ converges when $t > \lambda_{(\alpha,\beta,\gamma)}[f]_h$ and diverges when $t < \lambda_{(\alpha,\beta,\gamma)}[f]_h$.*

Here in this paper, we have introduced integral representations of the relative (α, β, γ)-order and relative (α, β, γ)-lower order of a meromorphic function with respect to an entire function. We are also investigating some basic properties of entire and meromorphic functions on the basis of relative (α, β, γ)-order and relative (α, β, γ)-lower order as compared to the growth of their corresponding left and right factors.

2. Lemma

In this section, we establish a lemma which will be needed in the sequel.

Lemma 2.1. *If the integral* $\int_{r_0}^{+\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp[\beta(\log(\gamma(r)))]]^{t+1}} dr$ $(r_0 > 0)$ *is convergent for* $0 < t < +\infty$*, then*

$$\lim_{r \to +\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^t} = 0.$$

Proof. As the integral $\int_{r_0}^{+\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp[\beta(\log(\gamma(r)))]]^{t+1}} dr$ converges for $0 < t < +\infty$, so for given ε (> 0) there exists a number $n = n(\varepsilon)$ such that

$$\int_{r_0}^{+\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^{t+1}} dr < \varepsilon \text{ for } r_0 > n,$$

i.e., for $r_0 > n$,

$$\int_{r_0}^{r_0+r} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^{t+1}} dr < \varepsilon.$$

Since $\exp\left[\beta\left(\log(\gamma(r))\right)\right]$ a increasing function of r, so

$$\int_{r_0}^{r_0+\exp[\beta(\log(\gamma(r_0)))]} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^{t+1}} dr$$

$$\geq \quad \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r_0)))\right]}{[\exp\left[\beta\left(\log(\gamma(r_0))\right)\right]]^{t+1}} \cdot \exp\left[\beta\left(\log(\gamma(r_0))\right)\right],$$

$$\text{i.e.,} \quad \int_{r_0}^{r_0+\exp[\beta(\log(\gamma(r_0)))]} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^{t+1}} dr$$

$$\geq \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r_0)))\right]}{[\exp\left[\beta\left(\log(\gamma(r_0))\right)\right]]^{t}} \text{ for } r_0 > n,$$

$$\text{i.e., } \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r_0)))\right]}{[\exp\left[\beta\left(\log(\gamma(r_0))\right)\right]]^{t}} < \varepsilon \text{ for } r_0 > n,$$

from which it is clear that

$$\lim_{r \to +\infty} \frac{\exp\left[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))\right]}{[\exp\left[\beta\left(\log(\gamma(r))\right)\right]]^t} = 0.$$

This proves the lemma. □

3. Main results

In this section, we present the main results of the paper.

Theorem 3.1. *The Definition 1.3 implies and is implied by Definition 1.5, i.e., they are equivalent.*

Proof. **Case 1**. $\rho_{(\alpha,\beta,\gamma)}[f]_h = +\infty$.

Definition 1.3 ⇒ Definition 1.5.

Since $\rho_{(\alpha,\beta,\gamma)}[f]_h = +\infty$, by Definition 1.3 for arbitrary positive K, we have a sequence of real numbers r tending to infinity that

$$\alpha(\log^{[2]} T_h^{-1}(T_f(r))) > K \cdot \beta\left(\log(\gamma(r))\right),$$

$$\text{i.e., } \exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))] > [\exp \beta\left(\log(\gamma(r))\right)]^K. \tag{1}$$

Let us suppose that the integral $\int\limits_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta(\log(\gamma(r)))]^{K+1}} dr$ $(r_0 > 0)$ be convergent. Then by using Lemma 2.1,

$$\limsup_{r\to+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta\left(\log(\gamma(r))\right)]^K} = 0.$$

So for all sufficiently large values of r,

$$\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r))) < [\exp \beta\left(\log(\gamma(r))\right)]^K. \tag{2}$$

Now from (1) and (2) we reach at a contradiction.

Hence $\int\limits_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta(\log(\gamma(r)))]^{K+1}} dr$ $(r_0 > 0)$ is divergent whenever K is finite, which is Definition 1.5.

Definition 1.5 ⇒ Definition 1.3.

We choose any positive number K. As $\rho_{(\alpha,\beta,\gamma)}[f]_h = +\infty$, from Definition 1.5 the divergence of the integral $\int\limits_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta(\log(\gamma(r)))]^{K+1}} dr$ $(r_0 > 0)$ implies that for any arbitrarily chosen positive number ε and for a sequence of real numbers r tending to infinity,

$$\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r))) > [\exp \beta\left(\log(\gamma(r))\right)]^{K-\varepsilon},$$

$$i.e,\ \alpha(\log^{[2]} T_h^{-1}(T_f(r))) > (K-\varepsilon) \cdot \beta\left(\log(\gamma(r))\right).$$

This gives that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \geq (K-\varepsilon).$$

As $K > 0$ is arbitrarily chosen, it implies that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} = +\infty.$$

Thus Definition 1.3 follows.

Case 2. $0 \leq \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$.

Definition 1.3 ⇒ Definition 1.5.

Subcase (I). $0 < \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$.

If $0 < \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$, then for any arbitrarily chosen $\varepsilon(> 0)$ and for all sufficiently large values of r,

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} < \rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon,$$

$$\begin{aligned}
&i.e,\ \exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r))) < [\exp\beta\left(\log(\gamma(r))\right)]^{(\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon)},\\
&i.e,\ \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta\left(\log(\gamma(r))\right)]^t} < \frac{[\exp\beta\left(\log(\gamma(r))\right)]^{(\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon)}}{[\exp\beta\left(\log(\gamma(r))\right)]^t},\\
&i.e,\ \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta\left(\log(\gamma(r))\right)]^t} < \frac{1}{[\exp\beta\left(\log(\gamma(r))\right)]^{t-(\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon)}}.
\end{aligned}$$

Therefore $\int_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta(\log(\gamma(r)))]^{t+1}} dr$ $(r_0 > 0)$ is convergent when $t > \rho_{(\alpha,\beta,\gamma)}[f]_h$ and divergent when $t < \rho_{(\alpha,\beta,\gamma)}[f]_h$.

Subcase (II).

When $\rho_{(\alpha,\beta,\gamma)}[f]_h = 0$, Definition 1.3 gives for all sufficiently large values of r that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \le \varepsilon.$$

Then as previous we get that $\int_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta(\log(\gamma(r)))]^{t+1}} dr$ $(r_0 > 0)$ is convergent when $t > 0$ and divergent when $t < 0$.

By Subcase (I) and Subcase (II), we get Definition 1.5.

Definition 1.5 ⇒ Definition 1.3.

By Definition 1.5, for arbitrary ε (> 0) the integral $\int_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta(\log(\gamma(r)))]^{\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon+1}} dr$ converges. Then using Lemma 2.1, we get

$$\limsup_{r\to+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta\left(\log(\gamma(r))\right)]^{\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon}} = 0,$$

i.e, for all sufficiently large values of r and for any arbitrarily chosen $\varepsilon_0(> 0)$,

$$\begin{aligned}
&\frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp\beta\left(\log(\gamma(r))\right)]^{\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon}} < \varepsilon_0,\\
&i.e,\ \exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r))) < \varepsilon_0 \cdot [\exp\beta\left(\log(\gamma(r))\right)]^{\rho_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon},\\
&i.e,\ \alpha(\log^{[2]} T_h^{-1}(T_f(r))) < \log\varepsilon_0 + (\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon) \cdot \beta\left(\log(\gamma(r))\right),\\
&i.e,\ \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} < \frac{\log\varepsilon_0}{\beta\left(\log(\gamma(r))\right)} + (\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon),\\
&i.e,\ \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \le \rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon.
\end{aligned}$$

Since ε (> 0) is arbitrarily chosen, from above we get

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \le \rho_{(\alpha,\beta,\gamma)}[f]_h. \tag{3}$$

As the integral $\int\limits_{r_0}^{+\infty} \frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta(\log(\gamma(r)))]^{\rho_{(\alpha,\beta,\gamma)}[f]_h-\varepsilon+1}} dr$ is divergent, so from Definition 1.5 we have a sequence of values of r tending to infinity for which

$$\frac{\exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{[\exp \beta(\log(\gamma(r)))]^{\rho_{(\alpha,\beta,\gamma)}[f]_h-\varepsilon+1}} > \frac{1}{[\exp \beta(\log(\gamma(r)))]^{1+\varepsilon}},$$

$$i.e, \ \exp[\alpha(\log^{[2]} T_h^{-1}(T_f(r))) > [\exp \beta(\log(\gamma(r)))]^{\rho_{(\alpha,\beta,\gamma)}[f]_h-2\varepsilon},$$

$$i.e, \ \alpha(\log^{[2]} T_h^{-1}(T_f(r))) > (\rho_{(\alpha,\beta,\gamma)}[f]_h - 2\varepsilon) \cdot \beta(\log(\gamma(r))),$$

$$i.e, \ \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta(\log(\gamma(r)))} > (\rho_{(\alpha,\beta,\gamma)}[f]_h - 2\varepsilon).$$

As ε (> 0) is arbitrarily chosen, we have

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta(\log(\gamma(r)))} \geq \rho_{(\alpha,\beta,\gamma)}[f]_h. \tag{4}$$

Thus from (3) and (4) it follows that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta(\log(\gamma(r)))} = \rho_{(\alpha,\beta,\gamma)}[f]_h.$$

This is the Definition 1.3.

Hence by Case 1 and Case 2, we reach at the conclusion. □

As Theorem 3.1, we can state Theorem 3.2 without its proof.

Theorem 3.2. *The Definition 1.4 and Definition 1.6 are equivalent.*

Theorem 3.3. *Let f, g be meromorphic functions and h be an entire function such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f]_h \leq \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f \circ g]_h = +\infty$, then*

$$\lim_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \leq \Delta \cdot \alpha(\log^{[2]} T_h^{-1}(T_f(r))). \tag{5}$$

Again from the definition of $\rho_{(\alpha,\beta,\gamma)}[f]_h$, it follows for all sufficiently large values of r that

$$\alpha(\log^{[2]} T_h^{-1}(T_f(r))) \leq (\rho_{(\alpha,\beta,\gamma)}[f]_h + \epsilon)\beta(\log(\gamma(r))). \tag{6}$$

From (5) and (6), for a sequence of values of r tending to $+\infty$,we have

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f]_h + \epsilon)\beta(\log(\gamma(r))),$$

$$i.e., \ \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\beta(\log(\gamma(r)))} \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f]_h + \epsilon),$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\beta(\log(\gamma(r)))} < +\infty,$$

$$i.e., \ \lambda_{(\alpha,\beta,\gamma)}[f \circ g]_h < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 3.2. *If we take "$0 < \lambda_{(\alpha,\beta,\gamma)}[g]_h \leq \rho_{(\alpha,\beta,\gamma)}[g]_h < +\infty$" instead of "$0 < \lambda_{(\alpha,\beta,\gamma)}[f]_h \leq \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$" and other conditions remain same, the conclusion of Theorem 3.3 remains true with "$\alpha(\log T_h^{-1}(T_g(r)))$" in place of "$\alpha(\log T_h^{-1}(T_f(r)))$" in the denominator.*

Remark 3.3. *Theorem 3.3 and Remark 3.2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h = +\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h = +\infty$" and the other conditions remain the same.*

Theorem 3.4. *Let f, g be meromorphic functions and h be an entire function such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h \leq \rho_{(\alpha,\beta,\gamma)}[f\circ g]_h < +\infty$ and $0 < \lambda_{(\alpha,\beta,\gamma)}[f]_h \leq \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$, then*

$$\begin{aligned}\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h} &\leq \liminf_{r\to+\infty}\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}\\ &\leq \min\left\{\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}, \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h}\right\}\\ &\leq \max\left\{\frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}, \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}.\end{aligned}$$

Proof. From the definitions of $\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h$, $\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h$, $\lambda_{(\alpha,\beta,\gamma)}[f]_h$, $\rho_{(\alpha,\beta,\gamma)}[f]_h$ and for arbitrary positive ε, we have for all sufficiently large values of r,

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \geqslant \left(\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h - \varepsilon\right)\beta(\log(\gamma(r))), \tag{7}$$

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \leq \left(\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h + \varepsilon\right)\beta(\log(\gamma(r))), \tag{8}$$

$$\alpha(\log^{[2]} T_h^{-1}(T_f(r))) \geqslant \left(\lambda_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon\right)\beta(\log(\gamma(r))) \tag{9}$$

$$\text{and } \alpha(\log^{[2]} T_h^{-1}(T_f(r))) \leq \left(\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon\right)\beta(\log(\gamma(r))). \tag{10}$$

Again for a sequence of values of r tending to infinity,

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \leq \left(\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h + \varepsilon\right)\beta(\log(\gamma(r))), \tag{11}$$

$$\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r))) \geqslant \left(\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h - \varepsilon\right)\beta(\log(\gamma(r))), \tag{12}$$

$$\alpha(\log^{[2]} T_h^{-1}(T_f(r))) \leq \left(\lambda_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon\right)\beta(\log(\gamma(r))) \tag{13}$$

$$\text{and } \alpha(\log^{[2]} T_h^{-1}(T_f(r))) \geqslant \left(\rho_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon\right)\beta(\log(\gamma(r))). \tag{14}$$

Now from (7) and (10) it follows for all sufficiently large values of r that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geqslant \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h - \varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geqslant \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h}. \tag{15}$$

Combining (9) and (11) , we have for a sequence of values of r tending to infinity that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h + \varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}. \tag{16}$$

Again from (7) and (13), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h - \varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}. \tag{17}$$

Now, it follows from (8) and (9) , for all sufficiently large values of r that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h + \varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\lambda_{(\alpha,\beta,\gamma)}[f]_h}. \tag{18}$$

Now from (8) and (14) , it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h + \varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h}. \tag{19}$$

So combining (10) and (12) , we get for a sequence of values of r tending to infinity that

$$\frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geqslant \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h - \varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]_h + \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_{f\circ g}(r)))}{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))} \geqslant \frac{\rho_{(\alpha,\beta,\gamma)}[f\circ g]_h}{\rho_{(\alpha,\beta,\gamma)}[f]_h}. \tag{20}$$

Thus the theorem follows from (15) , (16) , (17), (18) , (19) and (20) . □

Remark 3.4. *If we take "$0 < \lambda_{(\alpha,\beta,\gamma)}[g]_h \leq \rho_{(\alpha,\beta,\gamma)}[g]_h < +\infty$" instead of "$0 < \lambda_{(\alpha,\beta,\gamma)}[f]_h \leq \rho_{(\alpha,\beta,\gamma)}[f]_h < +\infty$" and other conditions remain same, the conclusion of Theorem 3.4 remains true with "$\lambda_{(\alpha,\beta,\gamma)}[g]_h$", "$\rho_{(\alpha,\beta,\gamma)}[g]_h$" and "$\alpha(\log^{[2]} T_h^{-1}(T_g(r)))$" in place of "$\lambda_{(\alpha,\beta,\gamma)}[f]_h$", "$\rho_{(\alpha,\beta,\gamma)}[f]_h$" and "$\alpha(\log^{[2]} T_h^{-1}(T_f(r)))$" respectively in the denominators.*

Acknowledgement. The authors are very much thankful to the respected editor and the renowned reviewer.

References

[1] L. Bernal-Gonzaléz, Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito, Doctoral Thesis, Universidad de Sevilla, Spain, 1984.
[2] L. Bernal, Orden relative de crecimiento de funciones enteras, Collect. Math., 39, 209-229, 1988.
[3] B. Belaïdi and T. Biswas, Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α, β, γ)-order, WSEAS Trans. Math., 21, 361-370, 2022, doi: 10.37394/23206.2022.21.43.
[4] W. K. Hayman, Meromorphic functions, The Clarendon Press, Oxford, 1964.
[5] J. Heittokangas, J. Wang, Z. T. Wen and H. Yu, Meromorphic functions of finite φ-order and linear q-difference equations, J. Difference Equ. Appl., 27, 9, 1280-1309, 2021, doi: 10.1080/10236198.2021.1982919.
[6] I. Laine, Nevanlinna theory and complex differential equations, De Gruyter Studies in Mathematics, 15 Walter de Gruyter & Co., Berlin, 1993.
[7] G. Valiron, Lectures on the general theory of integral functions, Chelsea Publishing Company, New York, 1949.
[8] L. Yang, Value distribution theory, Springer-Verlag, Berlin, 1993.
[9] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, Mathematics and its Applications, 557 Kluwer Academic Publishers Group, Dordrecht, 2003.

Chinmay Biswas
Department of Mathematics, Nabadwip Vidyasagar College,
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal, India.
Email address: chinmay.shib@gmail.com

Bablu Chandra Das
Department of Mathematics, Nabadwip Vidyasagar College,
Nabadwip, Dist.- Nadia, PIN-741302, West Bengal, India.
Email address: findbablu10@gmail.com