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Picard and Adomian Solutions of RLC Electrical Circuits

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Abstract

This study focuses on the solution of RLC series circuits. We use three methods to solve them: ADM, PM, and Laplace. The existence and uniqueness of the solution are proved. The convergence of the ADM and PM solutions are discussed. Then, numerical examples are solved to compare the results of the three methods.

Keywords: *RLC circuit; Adomian decomposition method; existence and uniqueness; Picard method; convergence; error estimation.*

1. Introduction

In this paper, we aim to study one of the applications of differential equations, which is electrical circuits. We use three methods to solve the problem. They are the Adomian decomposition method (ADM), the Picard method (PM) and the Laplace transform method (LTM). They are widely used for different applications. The LTM solution is considered as the exact solution. After proving the convergence of the first two methods, we compare the results of ADM and PM by the LTM solution, and then we discuss the advantages of each method.

2. RLC Electrical Circuit

The oscillating electrical circuit where the resistance R, inductance L, and capacitance C are connected with the voltage source v_s. They can be connected in different ways, but here we study the following series RLC circuit.

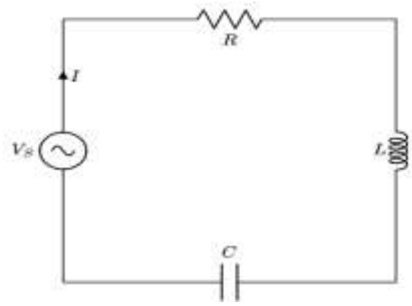

Figure 1

In the context of the RLC circuit, the parameters are defined as follows:

- V_s: the voltage source measured in volts,
- R: the resistance measured in ohms,
- L: the inductance measured in hernias,
- C: the capacitance measured in farads.

These components are integral to the behavior of the circuit and are used in the formulation of the following integro-differential equation (IDE) that describes the dynamics of the RLC series circuit:

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0) = v_s(t),$$

$$i(0) = I_{0,} \qquad (1)$$

where,

$$L\frac{di(t)}{dt}: \textit{the voltage across the inductor}$$

$$Ri(t): \textit{the voltage acrosse the resistor}$$

$$\frac{1}{C}\int_0^t i(t)dt : the\ voltage\ across\ the\ capacitor$$

$$v_0: the\ initial\ voltage\ across\ the\ capacitor$$

3. Methods of Solution

3.1. Adomian decomposition method (ADM)

i. The solution algorithm

From (1), we have

$$L\frac{di(t)}{dt} = (v_s - v_0) - Ri(t) - \frac{1}{C}\int_0^t i(\tau)d\tau \quad (2)$$

By integrating both sides of equation (2), we get

$$i(t) = I_0 + \frac{1}{L}\int_0^t (v_s - v_0)d\tau - \frac{R}{L}\int_0^t i(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i(\tau)d\tau d\tau \quad (3)$$

Decomposing $i(t) = \sum_{n=0}^{\infty} i_n(t)$ and substitute in equation (3), we get the following recursive relations that represent the ADM algorithm:

$$i_0\ (t) = I_0 + \frac{1}{L}\int_0^t (v_s - v_0)d\tau\ , \quad (4)$$

$$i_n(t) = -\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i_{n-1}(\tau)d\tau d\tau\,. \quad (5)$$

Finally, the ADM solution of (1) is

$$i(t) = \sum_{n=0}^{\infty} i_n(t). \quad (6)$$

ii. Convergence analysis

- **Existence and uniqueness of the solution**

Define the mapping $F: E \to E$ where E is the Banach space, $(C[I], \|\cdot\|)$ is the space of which consists of all continuous functions defined on the interval I with the norm $\|i(t)\| = \max_{t\in I}|i(t)|$, $\forall\ 0 \le \tau \le t \le T$.

Theorem 1:

The problem (1) has a unique solution whenever $0 < \beta < 1$ *where,* $\beta = \frac{T}{L}\left[R + \frac{T}{2C}\right]$

Proof:

The mapping $F: E \rightarrow E$ is defined as,

$$Fi(t) = I_0 + \frac{1}{L}\int_0^t (v_s - v_0)d\tau - \frac{R}{L}\int_0^t i(\tau)d\tau - \frac{1}{LC}\int_0^t \int_0^t i(\tau)d\tau d\tau$$

Let: $i(t), z(t) \in E$

$$\|Fi - Fz\| = \max_{t\in I}\left|-\frac{R}{L}\int_0^t i(\tau)d\tau - \frac{1}{LC}\int_0^t \int_0^t i(\tau)d\tau d\tau + \frac{R}{L}\int_0^t z(\tau)d\tau + \frac{1}{LC}\int_0^t \int_0^t z(\tau)d\tau d\tau\right|$$

$$= \max_{t\in I}\left|[\frac{R}{L}\int_0^t i(\tau)d\tau - \frac{R}{L}\int_0^t z(\tau)d\tau\,] + [\frac{1}{LC}\int_0^t \int_0^t i(\tau)d\tau d\tau - \frac{1}{LC}\int_0^t \int_0^t z(\tau)d\tau d\tau\right|$$

$$\leq \max_{t\in I}\left|\frac{R}{L}\int_0^t [i(\tau) - z(\tau)]d\tau\right| + \max_{t\in I}\left|\frac{1}{LC}\int_0^t \int_0^t [i(\tau) - z(\tau)]d\tau d\tau\right|$$

$$\leq \frac{R}{L}\max_{t\in I}|i(t) - z(t)|\left|\int_0^t d\tau\right| + \frac{1}{LC}\max_{t\in I}|i(t) - z(t)|\left|\int_0^t \int_0^t d\tau d\tau\right|$$

$$\leq \frac{R}{L}\max_{t\in I}|i(t) - z(t)|T + \frac{1}{LC}\frac{T^2}{2}\max_{t\in I}|i(t) - z(t)|$$

$$\leq \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]\|i - z\|$$

$$\leq \frac{T}{L}\left[R + \frac{T}{2C}\right]\|i - z\|$$

$$\leq \beta\|i - z\|$$

Under the condition, $0 < \beta < 1$, the mapping F is a contraction. Hence, there exists a unique solution of the problem (1) and this completes the proof.

- **Proof of convergence**

Theorem 2:

The series solution (6) of the problem (1) using ADM converges if $|i_1(t)| < \infty$ *and* $0 < \beta < 1$, $\beta = \frac{T}{L}\left[R + \frac{T}{2C}\right]$.

Proof:

Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution.

Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\|S_n - S_m\| = \max_{t\in I}|S_n - S_m| = \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right|$$

$$= \max_{t\in I}\left|\sum_{k=m+1}^{n} \frac{R}{L}\int_0^t i_k(\tau)d\tau + \frac{1}{LC}\int_0^t\int_0^t i_k(\tau)d\tau d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t \sum_{k=m+1}^{n} i_k(t)d\tau + \frac{1}{LC}\int_0^t\int_0^t \sum_{k=m+1}^{n} i_k(t)d\tau d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t \sum_{k=m}^{n-1} i_k(t)d\tau + \frac{1}{LC}\int_0^t\int_0^t \sum_{k=m}^{n-1} i_k(t)d\tau d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t [S_{n-1} - S_{m-1}]d\tau + \frac{1}{LC}\int_0^t\int_0^t [S_{n-1} - S_{m-1}]d\tau d\tau\right|$$

$$\le \max_{t\in I}\frac{R}{L}\int_0^t |S_{n-1} - S_{m-1}|d\tau + \max_{t\in I}\frac{1}{LC}\int_0^t\int_0^t |S_{n-1} - S_{m-1}|d\tau d\tau$$

$$\le \frac{R}{L}T \max_{t\in I}|S_{n-1} - S_{m-1}| + \frac{1}{LC}\frac{T^2}{2}\max_{t\in I}|S_{n-1} - S_{m-1}|$$

$$\le \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]\|S_{n-1} - S_{m-1}\|$$

$$\le T\left[\frac{R}{L} + \frac{1}{LC}\frac{T}{2}\right]\|S_{n-1} - S_{m-1}\|$$

$$\le \beta\|S_{n-1} - S_{m-1}\|$$

Let $n = m + 1$ then,

$\|S_{m+1} - S_m\| \le \beta\|S_m - S_{m-1}\| \le \beta^2\|S_{m-1} - S_{m-2}\| \le \cdots \le \beta^m\|S_1 - S_0\|$

From the triangle inequality we have,

$$\begin{aligned}\|S_n - S_m\| &\le \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\| \\ &\le [\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}]\|S_1 - S_0\| \\ &\le \beta^m[1 + \beta + \cdots + \beta^{n-m-1}]\|S_1 - S_0\| \\ &\le \beta^m \left[\frac{1-\beta^{n-m}}{1-\beta}\right]\|i(t)\|\end{aligned}$$

Since $0 < \beta < 1$, and $n > m$, then $(1 - \beta^{n-m}) \le 1$. Consequently,

$$\begin{aligned}\|S_n - S_m\| &\le \frac{\beta^m}{1-\beta}\|i_1(t)\| \\ &\le \frac{\beta^m}{1-\beta}\max_{t\in I}|i_1(t)|\end{aligned}$$

Nevertheless, $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges, and this statement concludes the proof.

- **Error analysis**

For the Adomian Decomposition Method (ADM), we can assess the maximum absolute truncation error of the series solution as outlined in the subsequent theorem

Theorem 3:

The maximum absolute truncation error of the series solution (6) to the problem (1) is estimated to be

$$\max_{t\in I}\left|y(t) - \sum_{k=0}^{m} i_k(t)\right| \le \frac{\beta^m}{1-\beta}\max_{t\in I}|i_1(t)|$$

Proof: From theorem 2 we have,

$$\|S_n - S_m\| \le \frac{\beta^m}{1-\beta}\max_{t\in I}|i_1(t)|$$

But, $S_n = \sum_{i=0}^{n} i_k(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \le \frac{\beta^m}{1-\beta}\max_{t\in I}|i_1(t)|$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t\in I}\left|i(t)-\sum_{i=0}^{m} i_k(t)\right| \le \frac{\beta^m}{1-\beta}\max_{t\in I}|i_1(t)|$$

In addition, this completes the proof.

3.2. Successive approximation method (PM)

i. Solution algorithm

Applying PM to IDE (3), the solution is

$$i_0(t) = I_0 + \frac{1}{L}\int_0^t (v_s - v_0)d\tau \tag{7}$$

$$i_n(t) = i_0(t) - \frac{R}{L}\int_0^t i(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i(\tau)d\tau d\tau. \tag{8}$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences.

$$i_n(t) = i_0(t) + \sum_{k=1}^{n} i_n(t) - i_{n-1}(t)$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.

The final PM solution takes the form

$$i(t) = \lim_{n\to\infty} i_n(t).$$

ii. Convergence analysis

We can deduce that if the series $\sum_{k=1}^{n} i_k(t) - i_{k-1}(t)$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove that the sequence $\{i_n(t)\}$ is convergent, consider the associated series,

$$\sum_{k=0}^{\infty} i_k(t) - i_{k-1}(t)$$

For *k=1*, we get

$$i_1(t) - i_0(t) = -\frac{R}{L}\int_0^t i_0(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i_0(\tau)d\tau d\tau$$

$$|i_1(t) - i_0(t)| = \left|-\frac{R}{L}\int_0^t i_0(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i_0(\tau)d\tau d\tau\right|$$

$$\le \left|\frac{R}{L}\int_0^t i_0(\tau)d\tau\right| + \left|\frac{1}{LC}\int_0^t\int_0^t i_0(\tau)d\tau d\tau\right|$$

$$\le |i_0(t)|\left[\frac{R}{L}\int_0^t d\tau + \frac{1}{LC}\int_0^t\int_0^t i_0(\tau)d\tau d\tau\right]$$

$$\le |i_0(t)|\left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]$$

$$\leq \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]\eta \leq \varphi_1$$

Where $|i_0(t)| \leq \eta \; and \; \varphi_1 = \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]\eta.$

Now, we will get an estimate for $i_n(t) - i_{n-1}(t)$, $n \geq 2$

$i_n(t) - i_{n-1}(t) =$

$$-\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau - \frac{1}{LC}\int_0^t \int_0^t i_{n-1}(\tau)d\tau d\tau + \frac{R}{L}\int_0^t i_{n-2}(\tau)d\tau + \frac{1}{LC}\int_0^t \int_0^t i_{n-2}(\tau)d\tau d\tau$$

$|i_n(t) - i_{n-1}(t)|$

$$= \left|-\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau - \frac{1}{LC}\int_0^t \int_0^t i_{n-1}(\tau)d\tau d\tau + \frac{R}{L}\int_0^t i_{n-2}(\tau)d\tau + \frac{1}{LC}\int_0^t \int_0^t i_{n-2}(\tau)d\tau d\tau\right|$$

$$\leq \left|\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau - \frac{R}{L}\int_0^t i_{n-2}(\tau)d\tau\right| + \left|\frac{1}{LC}\int_0^t \int_0^t i_{n-1}(\tau)d\tau d\tau - \frac{1}{LC}\int_0^t \int_0^t i_{n-2}(\tau)d\tau d\tau\right|$$

$$\leq \left[\frac{R}{L}\int_0^t d\tau + \frac{1}{LC}\int_0^t \int_0^t d\tau d\tau\right]|i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]|i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \beta|i_{n-1}(t) - i_{n-2}(t)|$$

In the above equation, if we put *n=2*

$$|i_2(t) - i_1(t)| \leq \left[\frac{R}{L}T + \frac{1}{LC}\frac{T^2}{2}\right]|i_1(t) - i_0(t)|$$

$$|i_2(t) - i_1(t)| \leq \beta\varphi_1$$

Doing the same for n=3, 4,…

$$|i_3(t) - i_2(t)| \leq \beta|i_2(t) - i_1(t)| \leq \beta^2\varphi_1,$$

$$|i_4(t) - i_3(t)| \leq \beta|i_3(t) - i_2(t)| \leq \beta^3\varphi_1,$$

$$\vdots$$

Then the general solution will be,

$$|i_n(t) - i_{n-1}(t)| \leq \beta^{n-1}\varphi_1$$

Since $\beta < 1$, so the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty}\left(-\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i_{n-1}(\tau)d\tau d\tau\right)$$

$$i(t) = -\frac{R}{L}\int_0^t i(\tau)d\tau - \frac{1}{LC}\int_0^t\int_0^t i(\tau)d\tau d\tau$$

4. Numerical Examples

For the circuit of Figure 2, i(0) = 5A , v(0)= 2.5V and the 0.5 Ω resistor represents the resistance

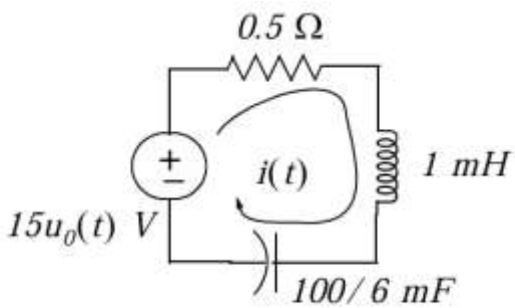

Figure 2

of the inductor. Compute and sketch *i(t)* for *t > 0*.

Solution

$$i(t) = 5 + 12500\int_0^t 1\,d\tau - 500\int_0^t i(\tau)d\tau - 60{,}000\int_0^t\int_0^t i(\tau)\,d\tau d\tau$$

i. We can get the exact solution of (1) by applying Laplace transform.

First, we get the DE form of (1)

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

Then we apply Laplace transform and get the exact solution

$$i(t) = 115e^{-200t} - 110e^{-300t}$$

ii. From (4) and (5) we get,

$i_0(t) = 5 + 12500\int_0^t 1\,d\tau,$

$i_n(t) = -500\int_0^t i_{n-1}(\tau)d\tau - 60{,}000\int_0^t\int_0^t i_{n-1}(\tau)\,d\tau d\tau, \quad n \geq 1.$

Hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t)$$

iii. From (7) and (8) we get,

$$i_0(t) = 5 + 12500 \int_0^t 1 \, d\tau ,$$

$$i_n(t) = 5 + 12500 \int_0^t 1 \, d\tau - 500 \int_0^t i_{n-1}(\tau) \, d\tau - 60{,}000 \int_0^t \int_0^t i_{n-1}(\tau) \, d\tau d\tau , \quad n \geq 1.$$

Hence,

$$i(t) = \lim_{n \to \infty} i_n(t).$$

Figures illustrate a comparison among the exact solution, the ADM, and the PM. These visuals demonstrate that as the number of terms *n* increases, the accuracy of the solution improves, ultimately converging to the exact solution.

Notice: All calculations and graphical representations in the paper were performed using MATHEMATICA 5.2 software for the examples presented.

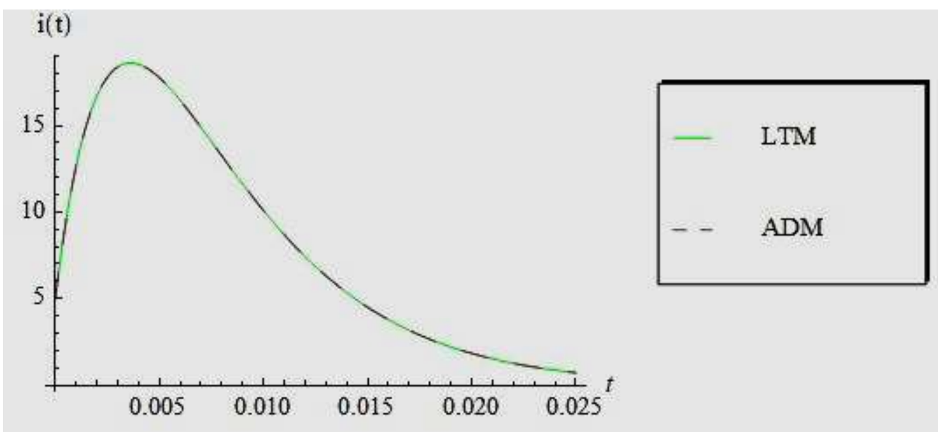

Figure 3: ADM and LTM solutions

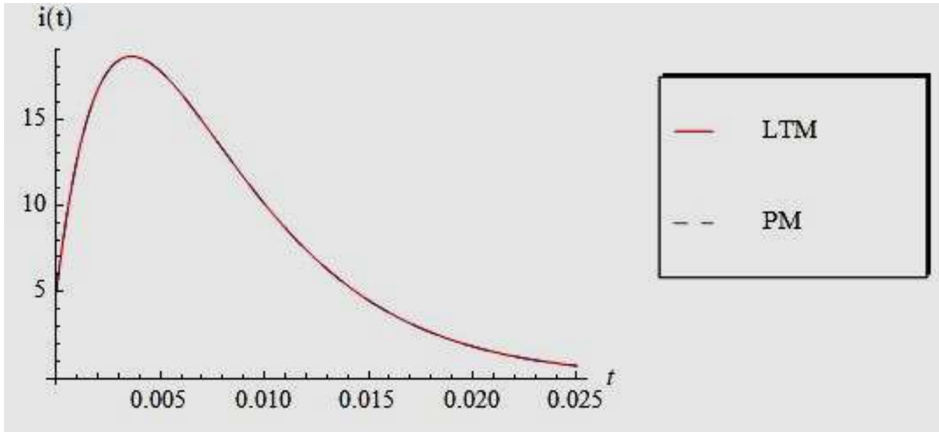

Figure 4: PM and LTM solutions

Table 1 shows the absolute relative error (ARE) of ADM and PM solutions. The time comparison between them is given in Table 2.

Table 1: ARE of ADM and PM solutions

t	$\left\lvert\frac{i_{ADM} - i_{Exact}}{i_{Exact}}\right\rvert$	$\left\lvert\frac{i_{PM} - i_{exact}}{i_{Exact}}\right\rvert$
0.005	3.61×10^{-16}	0
0.01	2.09×10^{-14}	2.82×10^{-15}
0.015	1.23×10^{-12}	1.51×10^{-13}
0.02	9.86×10^{-11}	2.09×10^{-13}
0.025	1.45×10^{-8}	1.12×10^{-8}

From Table 1, we can see that the two methods are close to each other, but PM gives solution that is more accurate.

Table 2: time comparison

ADM time	**PM time**
58.36 $sec.$	103.453 sec.

From Table 2, we deduce that the ADM gives results faster than PM.

5. Conclusion

In this paper, we discussed the RLC series circuit equation, and then we proved the convergence of ADM and PM solutions. We provided an example of the circuit and compared the ADM and PM by the Laplace solution. In addition, we compared between the ADM and PM, resulting that the ADM is faster, but the PM is more accurate.

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