Nile Journals

Nile Journal of Basic Science

Volume 4, Number 1, December 2024

Journal Webpage: https://privatewebgallery.com

Analytical Solution of RL and RC Electrical Circuits

S. R. E. Ibrahim [1], F. I. I. Mansy [2]

[1] Misr Higher Institute for Commerce and Computers, Mansoura, Egypt.

[2] Nile Higher Institute for Engineering and Technology, Mansoura, Egypt.

Emails: shimaarefaat310@gmail.com, fatmamansy99@gmail.com

Abstract

The paper provides a detailed study of the Adomian decomposition method (ADM) and the Picard method (PM) for solving ordinary differential equations (ODEs) related to electric circuits, specifically focusing on RC and RL circuits. It clearly establishes the existence and uniqueness of solution, while exploring how the series solutions converge and conducting a careful error analysis. This examination not only strengthens the theoretical understanding of these methods but also offers useful insights into their practical applications in electrical engineering and circuit analysis.

Keywords: *Adomian method; Picard method; existence; uniqueness; error analysis; RC circuits; RL circuits.*

1. Introduction

Differential equations are essential in many areas of engineering and science, including electrical networks, fluid dynamics, control theory, fractal theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, and optical and neural network systems. This paper focuses on applying the Adomian decomposition method (ADM) and the Picard method (PM) to solve equations related to electric circuits, specifically RLC circuits. The study examines the convergence of the series solutions and conducts a thorough error analysis. Furthermore, it presents numerical examples and practical applications, including the series

RLC circuit equation and various specific cases derived from this circuit such as RC and RL circuits.

2. RC Electrical Circuit

The RC electrical circuit consists of a resistor (R) and a capacitor (C) connected in series with a voltage source (V_S). This type of circuit is known for its oscillatory behavior, where the capacitor charges and discharges over time in response to the applied voltage.

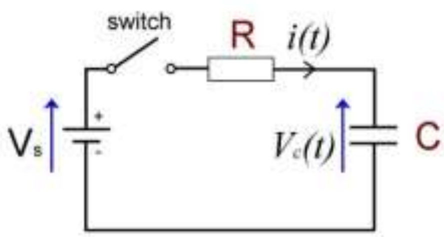

Figure 2.1

Where:

In the context of the RC circuit, the parameters are defined as follows:

- V_s: the voltage source measured in volts,
- R: the resistance measured in ohms,
- C: the capacitance measured in Farad.

These components are integral to the behavior of the circuit are used in the formulation of the differential equation (DE) that describes the dynamics of the RC series circuit:

From Kirchoff's voltage law (KVL) we get,

$$V_C + V_R = V_S$$

Where,

- $V_c(t)$: *the voltage across the capacitor*
- $V_R(t)$: *the voltage acrosse the resistor*

Then,

$$V_R = i(t)R$$

$$V_C = \frac{1}{C}\int_0^t i(t)dt$$

Then we have,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_C(0) = V_S$$

Therefore:

$$Ri(t) = V_S - \frac{1}{C}\int_0^t i(t)dt - v_C(0)$$

Then,

$$i(t) = \frac{V_S}{R} - \frac{v_C(0)}{R} - \frac{1}{RC}\int_0^t i(\tau)d\tau \tag{2.1}$$

2.1 Methods of Solution

2.1.2 Adomian decomposition method (ADM)

i. Solution algorithm

From equation (2.1),

With initial condition:

$$i_0(t) = \frac{V_S}{R} - \frac{v_C(0)}{R} \tag{2.2}$$

Recursive relation:

$$i_n(t) = -\frac{1}{RC}\int_0^t i_{n-1}(\tau)d\tau \tag{2.3}$$

Finally, the ADM solution of (2.1) is

$$i(t) = \sum_{n=0}^{\infty} i_n(t) \tag{2.4}$$

ii. Convergence analysis of ADM:

- **Existence and uniqueness of the solution**

Define the mapping $F: E \to E$ where E is the Banach space, $(C[I], \|\cdot\|)$ is the space of which consists of all continuous functions defined on the interval I with the norm $\|i(t)\| = \max_{t\in I} |i(t)|$, $\forall\ 0 \le \tau \le t \le T$.

<u>Theorem 2.1</u>:

The problem (2.1) has a unique solution whenever $0 < \beta < 1$ *where,* $\beta = \frac{T}{CR}$.

<u>Proof:</u>

The mapping $F: E \to E$ is defined as,

$$Fi(t) = \frac{V_s - v_0}{R} - \frac{1}{CR}\int_0^t i(\tau)d\tau$$

Let: $i(t), z(t) \in E$

$$\|Fi - Fz\| = \max_{t\in I}\left| \frac{V_s - v_0}{R} - \frac{1}{CR}\int_0^t i(\tau)d\tau - \frac{V_s - v_0}{R} + \frac{1}{CR}\int_0^t z(\tau)d\tau \right|$$

$$= \max_{t\in I}\left| -\frac{1}{CR}\int_0^t i(\tau)d\tau + \frac{1}{CR}\int_0^t z(\tau)d\tau \right|$$

$$\le \max_{t\in I}\left| -\frac{1}{CR}\int_0^t [i(\tau) - z(\tau)]d\tau \right|$$

$$\le \frac{1}{CR}\max_{t\in I}|i(t) - z(t)|\left|\int_0^t 1\, d\tau\right|$$

$$\le \frac{1}{CR}\max_{t\in I}|i(t) - z(t)|T$$

$$\le \left[\frac{1}{CR}T\right]\|i - z\|$$

$$\le \frac{T}{CR}\|i - z\|$$

$$\le \beta\|i - z\|$$

Under the condition $0 < \beta < 1$, the mapping F is a contraction, hence, there exists a unique solution of the problem (2.1) and this completes the proof.

- **Proof of convergence**

Theorem 2.2:

The series solution (2.4) of the problem (2.1) using ADM converges if $|i_1| < \infty$ *and* $0 < \beta < 1$, $\beta = \frac{T}{CR}$.

Proof:

Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution.

Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\|S_n - S_m\| = \max_{t\in I}|S_n - S_m| = \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right|$$

$$= \max_{t\in I}\left|\sum_{k=m+1}^{n}\left(-\frac{1}{CR}\int_0^t i_k(\tau)d\tau\right)\right|$$

$$= \max_{t\in I}\left|-\frac{1}{CR}\int_0^t \sum_{k=m+1}^{n} i_k(t)d\tau\right|$$

$$= \max_{t\in I}\left|-\frac{1}{CR}\int_0^t \sum_{k=m}^{n-1} i_k(t)d\tau\right|$$

$$= \max_{t\in I}\left|-\frac{1}{CR}\int_0^t [S_{n-1} - S_{m-1}]d\tau\right|$$

$$\leq \max_{t\in I}\frac{1}{CR}\int_0^t |S_{n-1} - S_{m-1}|d\tau$$

$$\leq \frac{1}{CR}T \max_{t\in I}|S_{n-1} - S_{m-1}|$$

$$\leq \left[\frac{T}{CR}\right]\|S_{n-1} - S_{m-1}\|$$

$$\leq \beta\|S_{n-1} - S_{m-1}\|$$

Let $n = m + 1$ then,

$$\|S_{m+1} - S_m\| \leq \beta\|S_m - S_{m-1}\| \leq \beta^2\|S_{m-1} - S_{m-2}\| \leq \cdots \leq \beta^m\|S_1 - S_0\|$$

From the triangle inequality we have,

$$\|S_n - S_m\| \leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\|$$

$$\leq [\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}]\|S_1 - S_0\|$$
$$\leq \beta^m[1 + \beta + \cdots + \beta^{n-m-1}]\|S_1 - S_0\|$$
$$\leq \beta^m \left[\frac{1-\beta^{n-m}}{1-\beta}\right] \|i(t)\|$$

Since $0 < \beta < 1$, and $n > m$, then $(1 - \beta^{n-m}) \leq 1$. Consequently,

$$\|S_n - S_m\| \leq \frac{\beta^m}{1-\beta} \|i_1(t)\| \ \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

However, $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges, and this statement concludes the proof.

- **Error analysis**

For the Adomian decomposition method (ADM), we can assess the maximum absolute truncation error of the series solution as outlined in the subsequent theorem.

Theorem 2.3:

The maximum absolute truncation error of the series solution (2.4) to the problem (2.1) is estimated to be

$$\max_{t \in I} \left| y(t) - \sum_{k=0}^{m} i_k(t) \right| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

Proof: From theorem 2.2 we have,

$$\|S_n - S_m\| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

But, $S_n = \ \sum_{i=0}^{n} i_k\,(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t \in I} |i(t) - \textstyle\sum_{i=0}^{m} i_k\,(t)| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

In addition, this completes the proof.

2.1.3 Picard Method (PM)

i. Solution algorithm

Applying PM to the IE (2.1), the solution is

$$i_0(t) = \frac{V_S}{R} - \frac{v_C(0)}{R} \tag{2.5}$$

$$i_n(t) = i_0(t) - \frac{1}{RC}\int_0^t i_{n-1}(\tau)d\tau \tag{2.6}$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences.

$$i_n(t) = i_0(t) + \sum_{k=1}^{n} i_n(t) - i_{n-1}(t)$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.

The final PM solution takes the form

$$i(t) = \lim_{n\to\infty} i_n(t).$$

ii. Convergence analysis

We can deduce that if the series $\sum_{k=1}^{n} i_k(t) - i_{k-1}(t)$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove that the sequence $\{i_n(t)\}$ is convergent, consider the associated series,

$$\sum_{k=0}^{\infty} i_k(t) - i_{k-1}(t)$$

For *k=1*, we get

$$i_1(t) - i_0(t) = -\frac{1}{RC}\int_0^t i_0(\tau)d\tau$$

$$|i_1(t) - i_0(t)| = \left|-\frac{1}{RC}\int_0^t i_0(\tau)d\tau\right|$$

$$\leq \left|-\frac{1}{RC}\int_0^t i_0(\tau)d\tau\right|$$

$$\leq |i_0(t)|\left[-\frac{1}{RC}\int_0^t 1\,d\tau\right]$$

$$\leq |i_0(t)|\left[\frac{1}{RC}T\right]$$

$$\leq \left[\frac{T}{RC}\right]\eta \leq \varphi_1$$

Where $|i_0(t)| \leq \eta\ and\ \varphi_1 = \left[\frac{T}{RC}\right]\eta.$

Now, we will get an estimate for $i_n(t) - i_{n-1}(t)$, $n \geq 2$

$$i_n(t) - i_{n-1}(t) = -\frac{1}{RC}\int_0^t i_{n-1}(\tau)d\tau + \frac{1}{RC}\int_0^t i_{n-2}(\tau)d\tau$$

$$= \left|-\frac{1}{RC}\int_0^t i_{n-1}(\tau)d\tau + \frac{1}{RC}\int_0^t i_{n-2}(\tau)d\tau\right|$$

$$\leq \left[\frac{1}{RC}\int_0^t 1\, d\tau\right] |i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \left[\frac{1}{RC}T\right] |i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \beta |i_{n-1}(t) - i_{n-2}(t)|$$

In the above equation, if we put n = 2

$$|i_2(t) - i_1(t)| \leq \left[\frac{T}{RC}\right] |i_1(t) - i_0(t)|$$

$$|i_2(t) - i_1(t)| \leq \beta\varphi_1$$

Doing the same for n = 3, 4, …

$$|i_3(t) - i_2(t)| \leq \beta |i_2(t) - i_1(t)| \leq \beta^2\varphi_1,$$

$$|i_4(t) - i_3(t)| \leq \beta |i_3(t) - i_2(t)| \leq \beta^3\varphi_1,$$

$$\vdots$$

Then the general solution will be,

$$|i_n(t) - i_{n-1}(t)| \leq \beta^{n-1}\varphi_1$$

Since $\beta < 1$, so the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty}\left(-\frac{1}{RC}\int_0^t i_{n-1}(\tau)d\tau\right)$$

$$i(t) = -\frac{1}{RC}\int_0^t i(\tau)d\tau$$

2.2 Numerical examples:

In the circuit of figure (2.2) , $V_S = 1\ V$, $R = 1000\ \Omega$, $C = 0.1\ mF, v_c(0) = 1\ V$. Compute and sketch i(t) for t > 0 when : (1) S_1 is closed and S_2 is open. (2) S_1 is open and S_2 is closed.

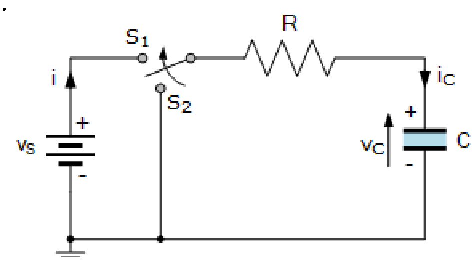

Figure 2.2

- **Solution:**

(1) When S_1 is closed and S_2 is open

$$i(t) = \frac{V_S}{R} - \frac{v_c(0)}{R} - \frac{1}{RC}\int_0^t i(\tau)d\tau$$

i. We can get the exact solution of (2.1) by applying Laplace transform.

First, we get

$$I(s) = \left(\frac{V_S}{RS}\right) - \left(\frac{v_C(0)}{RS}\right) - \left(\frac{1}{RC}\frac{I(s)}{S}\right)$$

Then we apply Laplace transform and get the exact solution

$$i(t) = 0.001\, e^{-10t}$$

ii. From (2.2) and (2.3) we get,

$$i_0(t) = \frac{1}{1000} = 0.001,$$

$$i_n(t) = -10 \int_0^t i_{n-1}(\tau)d\tau\,, \quad n \geq 1.$$

Hence, $$i(t) = \sum_{n=0}^{\infty} i_n(t)$$

iii. From (2.5) and (2.6) we get,

$$i_0(t) = 0.001,$$

$$i_n(t) = 0.001 - 10 \int_0^t i_{n-1}(\tau)\, d\tau\,, \quad n \geq 1.$$

Hence, $$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures illustrate a comparison among the exact solution, the ADM, and the PM. These visuals demonstrate that as the number of terms n increases, the accuracy of the solution improves, ultimately converging to the exact solution.

Notice: All calculations and graphical representations in the paper were performed using MATHEMATICA software for the examples presented.

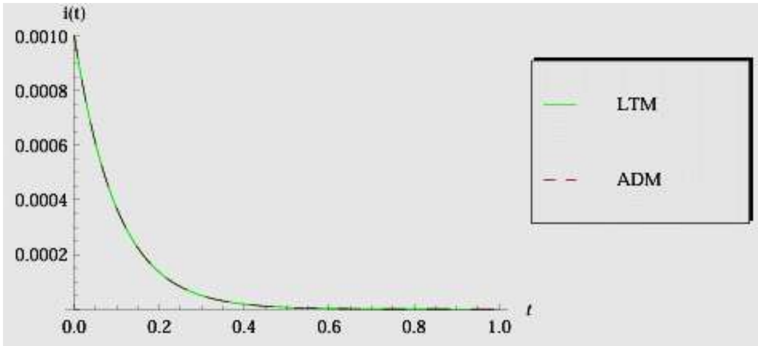

Figure 2.4: ADM and LTM Solutions

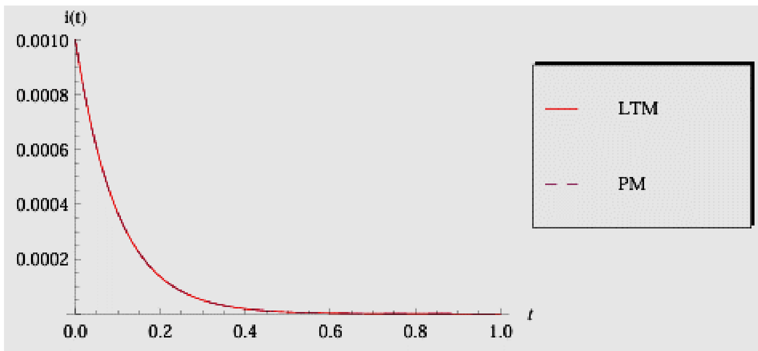

Figure 2.4: PM and LTM Solutions

Table 2.1 shows the absolute relative error (ARE) of ADM and PM solutions. The time comparison between them is given in Table 2.2.

Table 2.1: ARE of ADM and PM solutions

t	$\left\|\frac{i_{ADM} - i_{Exact}}{i_{Exact}}\right\|$	$\left\|\frac{i_{PM} - i_{exact}}{i_{Exact}}\right\|$
0.1	1.9525×10^{-16}	2.9472×10^{-16}
0.2	3.5083×10^{-15}	1.6023×10^{-15}
0.3	1.3531×10^{-14}	1.9599×10^{-14}
0.4	4.8071×10^{-14}	1.4207×10^{-13}
0.5	2.0327×10^{-14}	3.2182×10^{-13}
0.6	2.0564×10^{-12}	1.6621×10^{-12}
0.7	1.5189×10^{-11}	2.6157×10^{-11}
0.8	5.6159×10^{-10}	1.37294×10^{-9}
0.9	2.66524×10^{-7}	2.67731×10^{-7}
1	**0.0000531353**	**0.0000531437**

From Table 2.2, we can see that the two methods are close to each other, but PM gives solution that is more accurate.

Table 2.2: time comparison

ADM time	PM time
0.5	2.141

(2) S_1 is open and S_2 is closed ($v_S = 0$)

$$i(t) = -\frac{v_c(0)}{R} - \frac{1}{RC}\int_0^t i(\tau)d\tau$$

i. We can get the exact solution of (2.1) by applying Laplace transform.

First, we get

$$I(s) = -\left(\frac{v_C(0)}{RS}\right) - \left(\frac{1}{RC}\frac{I(s)}{S}\right)$$

Then we apply Laplace transform and get the exact solution

$$i(t) = -0.001\, e^{-10t}$$

ii. From (2.2) and (2.3) we get,

$$i_0(t) = \frac{-1}{1000} = -0.001,$$

$$i_n(t) = -10 \int_0^t i_{n-1}(\tau)d\tau\,, \quad n \geq 1.$$

Hence, $$i(t) = \sum_{n=0}^{\infty} i_n\,(t)$$

iii. From (5) and (6) we get,

$$i_0(t) = -0.001,$$

$$i_n(t) = -\,0.001 - 10 \int_0^t i_{n-1}(\tau)\,d\tau\,, \quad n \geq 1.$$

Hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

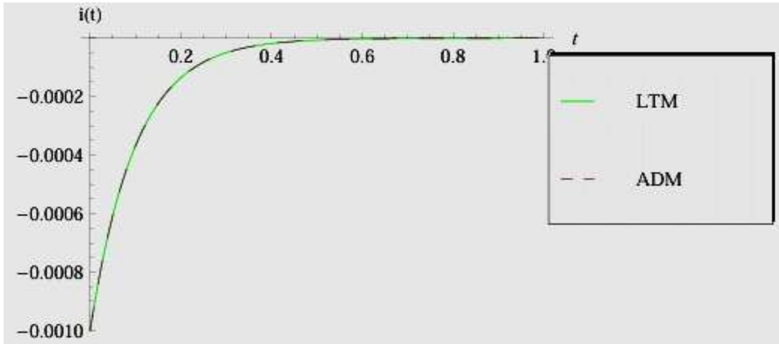

Figure 2.5: ADM and LTM Solutions

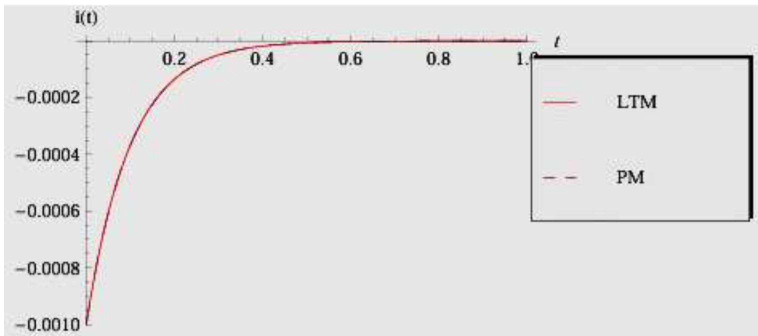

Figure 2.6: PM and LTM Solutions

Table 2.3 shows the absolute relative error (ARE) of ADM and PM solutions. The time comparison between them is given in Table 2.4.

Table 2.3: ARE of ADM and PM solutions

t	$\left\lvert\frac{i_{ADM} - i_{Exact}}{i_{Exact}}\right\rvert$	$\left\lvert\frac{i_{PM} - i_{exact}}{i_{Exact}}\right\rvert$
0.1	1.9525×10^{-16}	2.9472×10^{-16}
0.2	3.5083×10^{-15}	1.6023×10^{-15}
0.3	1.3531×10^{-14}	1.9599×10^{-14}
0.4	4.8071×10^{-14}	1.4207×10^{-13}
0.5	2.0327×10^{-14}	3.2182×10^{-13}
0.6	2.0564×10^{-12}	1.6621×10^{-12}
0.7	1.5189×10^{-11}	2.6157×10^{-11}
0.8	5.6159×10^{-10}	1.37294×10^{-9}
0.9	2.66524×10^{-7}	2.67731×10^{-7}
1	**0.0000531353**	**0.0000531437**

From Table 2.4, we can see that the two methods are close to each other, but PM gives solution that is more accurate.

Table 2.4: time comparison

ADM time	PM time
0.204	1.829

From table 2.3 and table 2.4, the results indicate that ADM is generally faster than PM, making it a more efficient choice for solving these types of equations.

3. RL Electrical Circuit

The oscillating electrical circuit consists of a voltage source Vs connected to a resistor R and an inductor L. While these components can be arranged in various configurations, this analysis focuses specifically on the series RL circuit. It is important to note that all these components are positive elements

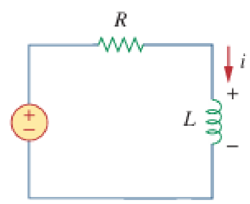

Figure 3.1

Where:

In the context of the RL circuit, the parameters are defined as follows:

- V_s: the voltage source measured in volts,
- R: the resistance measured in ohms,
- L: the inductance measured in Henry.

These components are integral to the behavior of the circuit and are used in the formulation of the differential equation (DE) that describes the dynamics of the RL series circuit:

$$Ri(t) + v_L(t) = V_s(t)$$

Where,

$$v_L(t): the\ voltage\ across\ the\ inductor$$

$$Ri(t): the\ voltage\ acrosse\ the\ resistor$$

$Since\ v_L(t) = L\dfrac{di(t)}{dt}\ , then$

$$Ri(t) + L\frac{di(t)}{dt} = V_s(t) \qquad (3.1)$$

$i(0) = I_0$

3.1 Methods of Solution

3.1.1 Adomian decomposition method

i. Solution algorithm

From (3.1)

$$L\frac{di(t)}{dt} = v_s(t) - Ri(t) \qquad (3.2)$$

By integrating both sides of equation (3.2), we have

$$i(t) = I_0 + \frac{1}{L}\int_0^t v_s(\tau)d\tau - \frac{R}{L}\int_0^t i(\tau)d\tau \qquad (3.3)$$

Decomposing $i(t) = \sum_{n=0}^{\infty} i_n(t)$ and substitute in equation (3.3), we get the following recursive relations that represent the ADM algorithm:

$$i_0(t) = I_0 + \frac{1}{L}\int_0^t v_s(\tau)d\tau, \qquad (3.4)$$

$$i_n(t) = -\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau \qquad (3.5)$$

Finally, the ADM solution of (3.1) is

$$i(t) = \sum_{n=0}^{\infty} i_n(t). \qquad (3.6)$$

ii. Convergence analysis

- **Existence and uniqueness of the solution**

Define the mapping $F: E \to E$ where E is the Banach space, $(C[I], \|\cdot\|)$ is the space of which consists of all continuous functions defined on the interval I with the norm $\|i(t)\| = \max_{t\in I}|i(t)|$, $\forall\ 0 \leq \tau \leq t \leq T$.

Theorem 1:

The problem (1) has a unique solution whenever, $0 < \beta < 1$ *where* $\beta = \frac{R}{L}T$.

Proof:

The mapping $F: E \to E$ is defined as,

$$Fi(t) = I_0 + \frac{1}{L}\int_0^t v_s(\tau)d\tau\ - \frac{R}{L}\int_0^t i(\tau)d\tau$$

Let: $i(t), z(t) \in E$

$$\begin{aligned}
\|Fi - Fz\| &= \max_{t\in I}\left|-\frac{R}{L}\int_0^t i(\tau)d\tau\ + \frac{R}{L}\int_0^t z(\tau)d\tau\ \right| \\
&= \max_{t\in I}\left|\frac{R}{L}\int_0^t [i(\tau) - z(\tau)]d\tau\ \right| \\
&= \frac{R}{L}\max_{t\in I}|i(t) - z(t)|\left|\int_0^t d\tau\ \right| \\
&\leq \frac{R}{L}\max_{t\in I}|i(t) - z(t)|T \\
&\leq \frac{R}{L}T\|i - z\| \\
&\leq \beta\|i - z\|
\end{aligned}$$

Under the condition $0 < \beta < 1$, the mapping F is a contraction, hence, there exists a unique solution of the problem (3.1) and this completes the proof.

- **Proof of convergence**

Theorem 3.2:

The series solution (3.6) of the problem (3.1) using ADM converges if $|i_1| < \infty$ *and* $0 < \beta < 1$,

$\beta = \frac{R}{L}T$.

Proof:

Define the sequence $\{S_n\}$ such that $S_n = \sum_{k=0}^{n} i_k(t)$ is the sequence of partial sums from the series solution.

Let S_n and S_m be two arbitrary partial sums with $n > m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\|S_n - S_m\| = \max_{t\in I}|S_n - S_m| = \max_{t\in I}\left|\sum_{k=m+1}^{n} i_k(t)\right|$$

$$= \max_{t\in I}\left|\sum_{k=m+1}^{n}\frac{R}{L}\int_0^t i_k(\tau)d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t \sum_{k=m+1}^{n} i_k\,(t)d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t \sum_{k=m}^{n-1} i_k\,(t)d\tau\right|$$

$$= \max_{t\in I}\left|\frac{R}{L}\int_0^t [S_{n-1} - S_{m-1}]d\tau\right|$$

$$\le \frac{R}{L}T \max_{t\in I}|S_{n-1} - S_{m-1}|$$

$$\le \frac{R}{L}T\|S_{n-1} - S_{m-1}\|$$

$$\le \beta\|S_{n-1} - S_{m-1}\|$$

Let $n = m + 1$ then,

$\|S_{m+1} - S_m\| \le \beta\|S_m - S_{m-1}\| \le \beta^2\|S_{m-1} - S_{m-2}\| \le \cdots \le \beta^m\|S_1 - S_0\|$

from the triangle inequality we have,

$$\|S_n - S_m\| \le \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\|$$

$$\le [\beta^m + \beta^{m+1} + \cdots + \beta^{n-1}]\|S_1 - S_0\|$$

$$\le \beta^m[1 + \beta + \cdots + \beta^{n-m-1}]\|S_1 - S_0\|$$

$$\le \beta^m\left[\frac{1-\beta^{n-m}}{1-\beta}\right]\|i(t)\|$$

Since $0 < \beta < 1$, and $n > m$, then $(1 - \beta^{n-m}) \le 1$. Consequently,

$$\|S_n - S_m\| \le \frac{\beta^m}{1-\beta}\|i_1(t)\|$$

$$\leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

but $|i_1(t)| < \infty$ and as $m \to \infty$, $\|S_n - S_m\| \to 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space, so the series $\sum_{n=0}^{\infty} i_n(t)$ converges, and This statement concludes the proof.

- **Error analysis**

For the ADM, we can assess the maximum absolute truncation error of the series solution as outlined in the subsequent theorem

Theorem 3.3:

The maximum absolute truncation error of the series solution (3.6) to the problem (3.1) is estimated to be

$$\max_{t \in I} |y(t) - \sum_{k=0}^{m} i_k(t)| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|.$$

Proof: From theorem 3.2 we have,

$$\|S_n - S_m\| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

But, $S_n = \sum_{i=0}^{n} i_k(t)$ as $n \to \infty$, then $S_n \to i(t)$, so

$$\|i(t) - S_m\| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

Therefore, the maximum absolute truncation error in the interval I is

$$\max_{t \in I} |i(t) - \sum_{i=0}^{m} i_k(t)| \leq \frac{\beta^m}{1-\beta} \max_{t \in I} |i_1(t)|$$

Moreover, this completes the proof.

3.1.2 Successive approximation method (PM)

i. Solution algorithm

Applying PM to IE (3.3), the solution is

$$i_0(t) = I_0 + \frac{1}{L} \int_0^t v_s(t) d\tau \quad (3.7)$$

$$i_n(t) = i_0(t) - \frac{R}{L} \int_0^t i(\tau) d\tau . \quad (3.8)$$

All the functions $i_n(t)$ are continuous functions, and $i_n(t)$ is the sum of successive differences.

$$i_n(t) = i_0(t) + \sum_{k=1}^{n} i_n(t) - i_{n-1}(t)$$

This means that the sequence $i_n(t)$ convergence is equivalent to the infinite series convergence.

The final PM solution takes the form

$$i(t) = \lim_{n\to\infty} i_n(t).$$

ii. Convergence analysis

We can deduce that if the series $\sum_{k=1}^{n} i_k(t) - i_{k-1}(t)$ is convergent, then the sequence $\{i_n(t)\}$ will converge to $i(t)$.

To prove that the sequence $\{i_n(t)\}$ is convergent, consider the associated series,

$$\sum_{k=0}^{\infty} i_k(t) - i_{k-1}(t)$$

For k=1, we get

$$i_1(t) - i_0(t) = -\frac{R}{L}\int_0^t i_0(\tau)d\tau$$

$$|i_1(t) - i_0(t)| = \left|-\frac{R}{L}\int_0^t i_0(\tau)d\tau\right|$$

$$\leq |i_0(t)|\left[\frac{R}{L}T\right]$$

$$\leq \frac{R}{L}T\eta \leq \varphi_1$$

Where $|i_0(t)| \leq \eta \text{ and } \varphi_1 = \frac{R}{L}T\eta.$

Now, we will get an estimate for $i_n(t) - i_{n-1}(t)$, $n \geq 2$

$$i_n(t) - i_{n-1}(t) = -\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau + \frac{R}{L}\int_0^t i_{n-2}(\tau)d\tau$$

$$|i_n(t) - i_{n-1}(t)| = \left|-\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau + \frac{R}{L}\int_0^t i_{n-2}(\tau)d\tau\right|$$

$$\leq \left[\frac{R}{L}\int_0^t d\tau\right]|i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \left[\frac{R}{L}T\right]|i_{n-1}(t) - i_{n-2}(t)|$$

$$\leq \beta|i_{n-1}(t) - i_{n-2}(t)|$$

In the above equation, if we put n=2

$$|i_2(t) - i_1(t)| \leq \left[\frac{R}{L}T\right]|i_1(t) - i_0(t)|$$

$$|i_2(t) - i_1(t)| \le \beta\varphi_1$$

Doing the same for n=3, 4, …

$$|i_3(t) - i_2(t)| \le \beta|i_2(t) - i_1(t)| \le \beta^2\varphi_1,$$

$$|i_4(t) - i_3(t)| \le \beta|i_3(t) - i_2(t)| \le \beta^3\varphi_1,$$

$$\vdots$$

Then the general solution will be,

$$|i_n(t) - i_{n-1}(t)| \le \beta^{n-1}\varphi_1$$

Since $\beta < 1$, so the sequence $\{i_n(t)\}$ will be convergent.

$$i(t) = \lim_{n\to\infty}\left(-\frac{R}{L}\int_0^t i_{n-1}(\tau)d\tau\right) = -\frac{R}{L}\int_0^t i(\tau)d\tau$$

3.2 Numerical Examples

3.2.1 RL Current Growth

For the circuit of Figure 3.2, i(0) = 0, and the 50 Ω resistor represents the resistance of the inductor. Compute and sketch i(t) for t > 0.

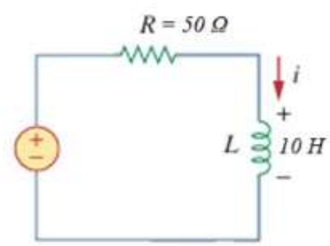

Figure 3.2

- **Solution:**

i. We can get the exact solution of (3.1) by applying Laplace transform.

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) = 0.126(1 - e^{-5t})$$

ii. From (3.4) and (3.5) we get

$$i_0(t) = 0.1\int_0^t 6.3\, d\tau,$$

$$i_n(t) = -5\int_0^t i_{n-1}(\tau)d\tau, \quad n \ge 1.$$

hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t)$$

iii. From (3.7) and (3.8) we get

$$i_0(t) = 0.1\int_0^t 6.3\, d\tau\,,$$

$$i_n(t) = 0.1\int_0^t 6.3\, d\tau - 5\int_0^t i_{n-1}(\tau)d\tau\,, \quad n \geq 1.$$

hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures illustrate a comparison among the exact solution, the ADM, and the PM. These visuals demonstrate that as the number of terms n increases, the accuracy of the solution improves, ultimately converging to the exact solution.

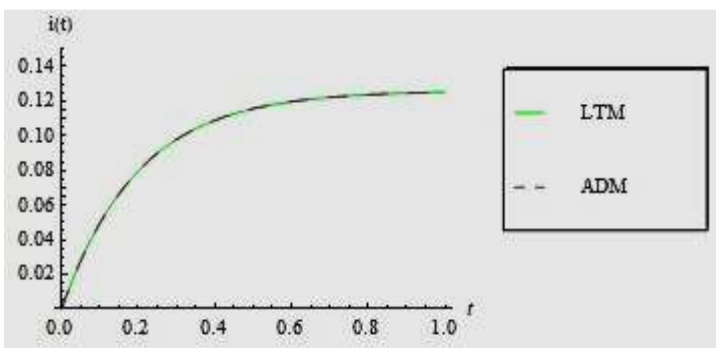

Figure 3.4: ADM and LTM Solutions

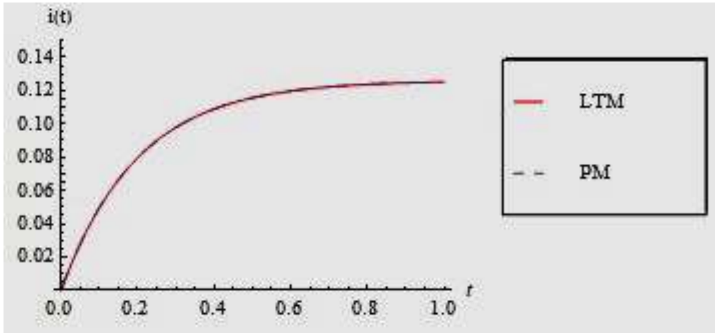

Figure 3.5: PM and LTM Solutions

Table 3.1 shows the absolute relative error (ARE) of ADM and PM solutions. The time comparison between them is given in Table 3.2.

Table 3.1: ARE of ADM and PM solutions

t	$\left\lvert\frac{i_{ADM} - i_{Exact}}{i_{Exact}}\right\rvert$	$\left\lvert\frac{i_{PM} - i_{exact}}{i_{Exact}}\right\rvert$
0.1	9.302×10^{-17}	1.4×10^{-16}
0.2	9.07×10^{-17}	8.712×10^{-17}
0.3	1.168×10^{-16}	0
0.4	4.302×10^{-16}	2.548×10^{-16}
0.5	5.475×10^{-17}	2.4×10^{-16}
0.6	4.045×10^{-16}	9.273×10^{-16}
0.7	3.789×10^{-16}	4.542×10^{-16}
0.8	2.372×10^{-15}	0
0.9	2.886×10^{-15}	4.455×10^{-16}
1	1.066×10^{-15}	1.774×10^{-15}

From table 3.1 we can see that the two methods are close to each other, but PM is more accurate.

Table 3.2: time comparison

ADM time	**PM time**
0.14	1.531

From table 3.2 we deduce that the ADM gives results faster than PM.

3.2.2 RL Circuit with An Initial Current

For the circuit of Figure 3.5, i(0) = 0.72. Compute and sketch i(t) for t > 0.

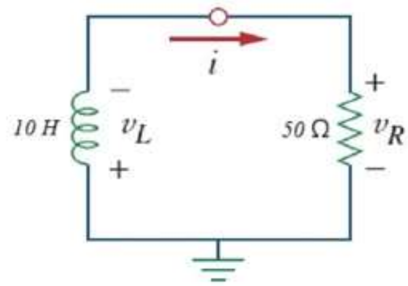

Figure 3.5

- **Solution:**

i. We can get the exact solution of (3.1) by applying the Laplace transform:

$$i(t) = I_0 e^{-\frac{R}{L}t} = 0.72e^{-5t}$$

ii. From (3.4) and (3.5) we get,

$$i_0(t) = 0.72,$$

$$i_n(t) = -5 \int_0^t i_{n-1}(\tau)d\tau, \quad n \geq 1.$$

hence,

$$i(t) = \sum_{n=0}^{\infty} i_n(t)$$

iii. From (3.7) and (3.8) we get,

$$i_0(t) = 0.72,$$

$$i_n(t) = 0.72 + 5 \int_0^t i_{n-1}(\tau)d\tau, \quad n \geq 1.$$

hence,

$$i(t) = \lim_{n\to\infty} i_n(t).$$

Figures illustrate a comparison among the exact solution, the Adomian Decomposition Method (ADM), and the Picard Method (PM). These visuals demonstrate that as the number of terms n increases, the accuracy of the solution improves, ultimately converging to the exact solution.

Notice: All calculations and graphical representations in the paper were performed using MATHEMATICA software for the examples presented.

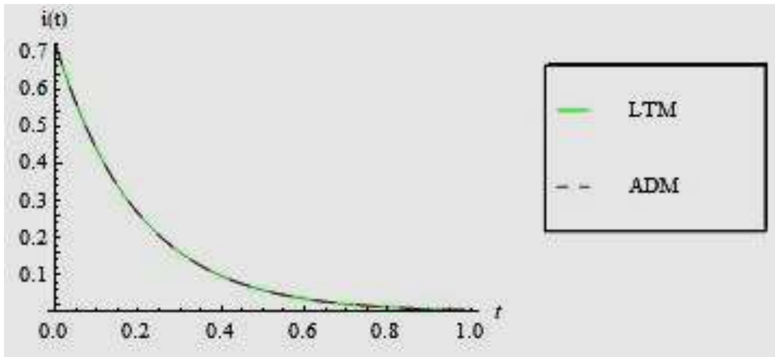

Figure 3.6: ADM and LTM solutions

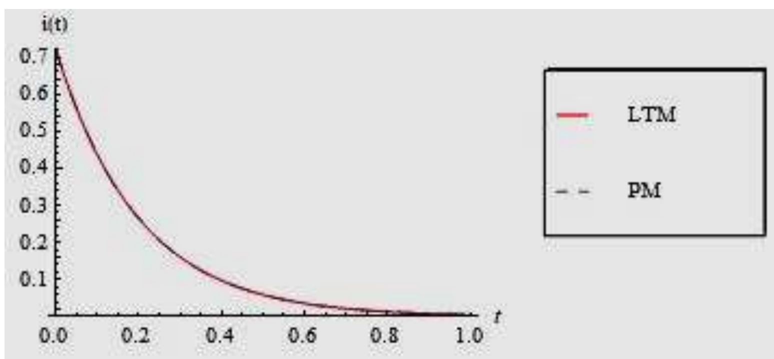

Figure 3.7: PM and LTM Solutions

Table 3.3 shows the absolute relative error (ARE) of ADM and PM solutions. The time comparison between them is given in Table 3.4.

Table 3.3: ARE of ADM and PM solutions

t	$\left\|\frac{i_{ADM} - i_{Exact}}{i_{Exact}}\right\|$	$\left\|\frac{i_{PM} - i_{exact}}{i_{Exact}}\right\|$
0.1	6.501×10^{-17}	1.271×10^{-16}
0.2	3.082×10^{-16}	4.192×10^{-16}
0.3	8.318×10^{-16}	6.911×10^{-16}
0.4	3.283×10^{-15}	0
0.5	1.608×10^{-15}	7.514×10^{-15}

0.6	$\mathbf{1.286 \times 10^{-14}}$	$\mathbf{0}$
0.7	$\mathbf{2.99 \times 10^{-14}}$	$\mathbf{1.021 \times 10^{-14}}$
0.8	$\mathbf{4.939 \times 10^{-14}}$	$\mathbf{7.577 \times 10^{-14}}$
0.9	$\mathbf{1.602 \times 10^{-14}}$	$\mathbf{4.164 \times 10^{-14}}$
1	$\mathbf{2.013 \times 10^{-13}}$	$\mathbf{2.288 \times 10^{-14}}$

From table 3.3 we can see that the two methods are close to each other, but PM is more accurate.

Table 3.4: time comparison

ADM time	PM time
$0.094\ sec.$	$1.422\ sec.$

From table 3.4 we deduce that the ADM gives results faster than PM.

4. Conclusion

In this study, we compared the Adomian decomposition method and the Picard method for solving ordinary differential equations in electric circuits. While both methods are effective, ADM is faster, making it preferable for time-sensitive applications. Future research could explore hybrid approaches that leverage the strengths of both methods.

References

1. K. Abbaoui and Y. Cherruault, (1994), Convergence of Adomian's method applied to differential equations, Computers Math. Applic., 28, 103-109.
2. Steven T. Karris, Circuit Analysis II with MATLAB® Applications (2003)
3. Charles K. Alexander and Matthew N. O. Sadiku, Fundamentals of Electric Circuits (fifth edition) (2013)
4. O. Heaviside, Electrical Papers, Macmillan, London, 1892.
5. T. J. Bromwich, Examples of operational methods in mathematical physics, Philos. Mag., 37, 1919, pp. 407-419.
6. H. D. Davis, The Theory of Linear Operators, Principia Press, Bloomington, Indiana, 1936.
7. M. A. Al-Bassam, Some existence theorems on differential equations of generalized order, Journal

für Reine und Angewandte Mathematik, vol. 218, 1965, pp. 70-78.

8. I. M. Gelfand and G. E. Shilov, Generalized Functions, vol. 1. Nauka, Moscow, 1959 (in Russian).
9. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1995.
10. M. Saleh, and I. L. El-kalla, On the generality and convergence of Adomian's method for solving ordinary differential equations, International J. of Differential Equations and Applications, vol. 7, no. 1, (2003), pp. 15-28.
11. Y. Cherruault, Convergence of Adomian method, Kybernetes, 18, 1989, pp. 31-38.
12. M. Benabidallah, and Y. Cherruault, Application of the Adomian method for solving a class of boundary problems, Kybernetes vol. 33, no. 1, (2004), pp. 118-132
13. J. Biazar, E. Babolian, and R. Islam, Solution of the system of ordinary differential equations by Adomian decomposition method, J. Appl. Math. and Comput., 147, (2004), pp. 713-719.
14. R. Goren o, and S. Vessella, Abel Integral Equations: Analysis and Applications, Lecture Notes in Mathematics # 1461, Springer-Verlag, Berlin 1991.
15. J. D. Craig, and J. C. Brown, Inverse Problems in Astronomy, Adam Hilger Ltd, Bristol, 1986
16. G. F. D. Duff, and D. Naylor, Differential Equations of Applied Mathematics, Wiley & Son, New York 1966.
17. L. M. Delves, and I. J. Mohammed, Computational Methods for Integral Equations, Cambridge University Press, 1985.
18. J. Stoer, and R. Bulirsch, Introduction to Numerical Analysis, Springer-Verlag, 2002.
19. Debnath, L., Bhatta, D., Integral Transforms and Their Applications, Chapman & Hall, CRC Press, Boca Raton, 2007.
20. Gomez, F., Rosales, J., Gula, M., RLC electrical circuit of non-integer order, Cent. Eur. J. Phys., 2013, 1361-1365.