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A bivariate inverse Weibull distribution: properties, concomitant of order statistics, extropy's measure

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ABSTRACT : In this paper, some statistical properties of Cambanis bivariate inverse Weibull (CAMBIW) are studied. In addition, the distribution theory of concomitants of order statistics (OSs) for CAMBIW is investigated. Additionally, extropy and weighted extropy-two recent information measures-are provided and studied for the concomitants of OSs based on CAMBIW. Finally, a bivariate real-world data set has been examined for illustration purposes, and the performance is robust.

KEYWORDS: Inverse Weibull distribution; Cambanis bivariate; Product moments; Extropy; Weighted extropy.

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I. Introduction

The inverse Weibull (IW) distribution plays an important role in analyzing the lifetime data. It has applications in mortality and breast cancer study due to the non-monotonicity of the hazard rate. Alternatively, this probability distribution is called the Fréchet distribution or the type II extreme value distribution. It has been extensively used to analyze lifespan data with nonmonotone hazard function. There are a variety of applications for IW distribution, such as medicine, reliability engineering, and ecology, among others. The IW model was used by Keller et al. (1982) to model mechanical component failure caused by degradation. Calabria and Pulcini (1994) explained how IW distribution is interpreted in the context of load strength relationships for components. The literature is replete of works that address the characteristics of this important life distribution, such as Nigm and Abo-Eleneen (2008), in addition to several studies that demonstrate the broad applicability of the IW distribution.

The cumulative distribution function (CDF) and the probability density function (PDF) of the IW distribution are given by:

$$F(y;\theta,\eta)=e^{-\theta y^{-\eta}},\quad y\geq 0,$$

and

$$f(y;\theta,\eta)=\eta\theta y^{-\eta-1}e^{-\theta y^{-\eta}},\quad y\geq 0, \tag{1}$$

where $\theta>0$ is the scale parameter and $\eta>0$ is the shape parameter. The reliability function and the hazard rate function of the IW distribution are given respectively by:

$$R(y;\theta,\eta)=1-e^{-\theta y^{-\eta}},\quad y\geq 0,$$

and

$$H(y;\theta,\eta)=\frac{\eta\theta y^{-\eta-1}e^{-\theta y^{-\eta}}}{1-e^{-\theta y^{-\eta}}},\quad y\geq 0.$$

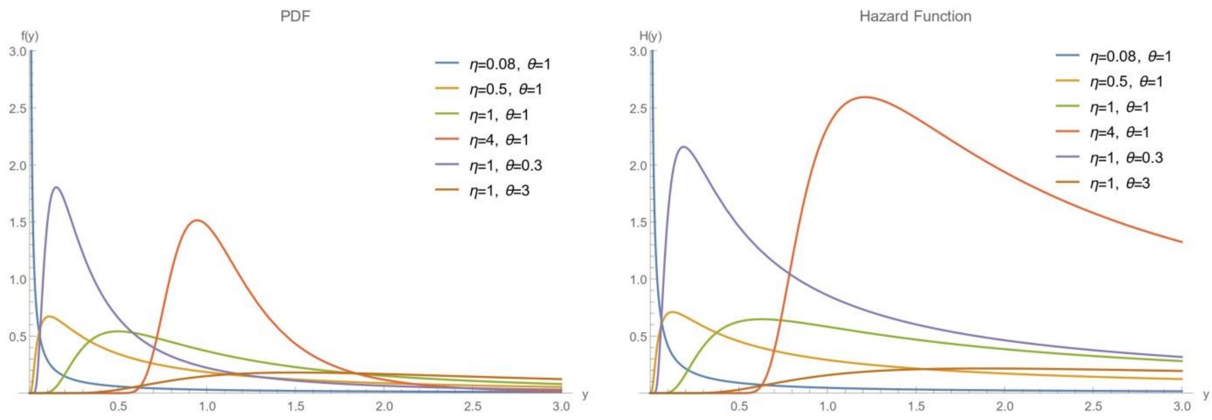


Figure 1: PDF and hazard functions of IW distribution

The IW distribution possesses notable characteristics and finds utility in several contexts. As an illustration, one notable characteristic of the IW is its ability to exhibit an unimodal failure rate function across all values of η, except for values of η that are less than or approximately equal to 0.1, where it has a decreasing trend. The finding is further supported by the evidence presented in Figure 1, which clearly demonstrates that the distribution characterized by a unimodal failure rate function has greater flexibility in its practical application.

Despite its wide application in lifetime data analysis, the univariate Weibull distribution may not be suitable for scenarios with non-monotone hazard functions, such as a unimodal shape where the risk of failure rises, peaks, and then falls. It is possible to suggest non-monotonic hazard functions based on prior knowledge in practical situations. As an example, mortality studies typically show a peak followed by a decline. A similar pattern has also been found in breast cancer studies that suggest a peak in mortality three years after surgery is followed by a gradual decline. In cases where empirical evidence suggests a possible unimodal hazard function, the IW distribution can be used to analyze the data. As an additional advantage, it is capable of modeling heavily tailed distributions.

In statistical literature, a copula is frequently used to generate bivariate distributions, as described by Nelsen (2006). Copulas can be used to characterize bivariate distributions with well-defined dependency structures. By using this function, bivariate data frames can be combined with consistent margins from 0 to 1. Based on the sort of dependency structure that is demonstrated by the two random variables (RVs), the copula function should be selected. As a result of their ability to assist in modeling and estimating the probability distribution of random vectors, copulas are commonly used in statistical applications involving high-dimensional data. A simplified procedure is achieved by calculating marginals and copula separately.

Given two marginal univariate distributions, denoted as $H_Z(z)$ representing the probability of Z being less than or equal to z, and $H_T(t)$ representing the likelihood of T being less than or equal to t, as well as a copula $C(u, v)$ and its PDF indicated as $c(u, v)$, which is the second partial derivative of $C(u, v)$ with respect to u and v, respectively. Sklar (1973) developed the concepts of the joint distribution function (JDF) and joint probability density function (JPDF) in a subsequent manner: The functions denoted as $F_{Z,T}(z, t)$ and $f_{Z,T}(z, t)$ can be expressed as $C(F_Z(z), F_T(t))$ and $f_Z(z)f_T(t)c(F_Z(z), F_T(t))$, respectively. The Farlie-Gumbel-Morgenstern (FGM) copula is widely recognized as a prominent parametric copula family that is commonly utilized in several applications. Gumbel (1960) conducted an academic investigation on the aforementioned family. The JDF and JPDF for the FGM copula are provided, respectively, as follows:

$$C(u,v) = uv[1 + \omega(1-u)(1-v)], -1 \leq \omega \leq 1,$$

and

$$c(u,v) = [1 + \omega(1-2u)(1-2v)].$$

In applications, the FGM family is adaptable and helpful as long as the correlation between the variables is minimal. For more details about this family and its generalizations see, Abd Elgawad and Alawady (2022), Alawady et al. (2021b,2022,2023), Barakat et al. (2019,2021a,b,2022a,b), Husseiny et al. (2022,2024a,b), Mansour et al. (2022), and Nagy et al. (2024).

An extended family of FGM is the Cambanis family, which was recently proposed by Cambanis (1977). It is one of the more flexible extensions of the FGM family, where its JDF is given by

$$F_{Z,T}(z,t) = F_Z(z)F_T(t)[1 + \omega_1(1 - F_Z(z)) + \omega_2(1 - F_T(t)) + \omega(1 - F_Z(z))(1 - F_T(t))], \quad (2)$$

and the JPDF corresponding to the bivariate JDF (2) is

$$f_{Z,T}(z,t) = f_Z(z)f_T(t)[1 + \omega_1(1 - 2F_Z(z)) + \omega_2(1 - 2F_T(t)) + \omega(1 - 2F_Z(z))(1 - 2F_T(t))], \quad (3)$$

where f_Z and f_T are the PDFs of F_Z and F_T, respectively, and the parameters ω_1, ω_2, and ω are real constants belonging to the parameter space

$$C = \{(\omega_1, \omega_2, \omega): 1 + \omega_1 + \omega_2 + \omega \geq 0, 1 - \omega_1 - \omega_2 + \omega \geq 0, 1 - \omega_1 + \omega_2 - \omega \geq 0, 1 + \omega_1 - \omega_2 - \omega \geq 0\}.$$

We refer to the Cambanis family defined by (3), by $CAM(\omega_1, \omega_2, \omega)$. The classical FGM family can be obtained when $\omega_1 = \omega_2 = 0$. The important difference between the Cambanis family and the known extended families of FGM is that its marginals are not F_Z and F_T, but they are uniquely determined by F_Z and F_T, as $G_Z(z) = F_Z(z)[1 + \omega_1(1 - F_Z(z))]$ and $G_T(t) = F_T(t)[1 + \omega_2(1 - F_T(t))]$. This property may allow this family to fit a bivariate data set with marginal DFs (G_Z and G_T) that have a more intricate functional structure than the family's base DFs (F_Z and F_T). Cambanis (1977) has shown that the RVs corresponding to (3), i.e. Z and T, are independent if they are uncorrelated and also gave an interpretation of the parameters. For more details about this family see Abd Elgawad et al. (2021), Alawady et al. (2021a), Arun et al. (2023), Cambanis (1977), and Husseiny et al. (2024a,b).

Extropy, that was introduced by Lad et al. (2015), offers a unique perspective on information uncertainty, complementing the established concept of entropy. Both measures remain invariant under rearrangements of their mass functions, and both favor uniform distributions as their maximum. However, extropy diverges in its assessment of "sophistication" within distributions. Formally, for a non-negative RV Z with continuous PDF f_Z and CDF F_Z, extropy is defined as (Husseiny and Syam 2022): $\mathcal{J}(Z) = -1/2\int_{-\infty}^{\infty} f^2(z)dz$. Further extending this concept, Bansal and Gupta (2022) introduced weighted extropy, incorporating the variable's value into the evaluation, as

$$\mathcal{J}^w(Z) = -1/2\int_{-\infty}^{\infty} zf^2(z)dz.$$

Interestingly, while two distributions might share the same extropy value, their weighted extropy can differ significantly. This highlights a potential advantage of using weighted extropy in certain scenarios. In essence, extropy sheds light on information organization and complexity, offering valuable insights alongside traditional entropy measures. By considering both perspectives, we gain a richer understanding of information's nuances.

The remainder of the paper is structured as follows. In Section 2, we discuss the adaptability of the CAMBIW distribution and some properties such as the marginal distributions, conditional distribution, regression curve, covariance, and correlation. Furthermore, we study the concomitants of order statistics (OSs) based on CAMBIW and its moments. Moreover, in Section 3, the extropy for IW distribution and the CAMBIW distribution are derived and discussed. Furthermore, in Section 4, the weighted extropy for the IW distribution and the CAMBIW distribution are derived and discussed. In Section 5, a bivariate real-world data set has been examined for illustration purposes, and the performance is respectable.

II. Bivariate CAM-IW distribution

In this section, we investigate the CAMBIW distribution. This section examines the fundamental statistical properties of the CAMBIW distribution in order to gain a deeper understanding of it. We reveal the individual and interrelated behaviors of variables by examining the marginal and conditional distributions. Additionally, conditional expectations provide insight into the average value of one variable in light of the state of another. In the same way, product moments quantify the complex relationships among variables, whereas moment generating functions offer valuable insights into the whole distribution. We gain a comprehensive understanding of CAMBIW distribution and its behavior through this analysis of statistics. The distribution theory of concomitants of OSs for CAMBIW is also discussed. The JPDF of CAMBIW, as stated by Sklar's theorem, can be expressed as follows:

$$f_{Z,T}(z,t) = \eta_1\theta_1 z^{-\eta_1-1}e^{-\theta_1 z^{-\eta_1}}\eta_2\theta_2 t^{-\eta_2-1}e^{-\theta_2 t^{-\eta_2}}\big[1+\omega_1(1-2e^{-\theta_1 z^{-\eta_1}}) +\omega_2(1-2e^{-\theta_2 t^{-\eta_2}})+\omega(1-2e^{-\theta_1 z^{-\eta_1}})(1-2e^{-\theta_2 t^{-\eta_2}})\big],$$

and the corresponding JDF is

$$F_{Z,T}(z,t) = e^{-\theta_1 z^{-\eta_1}}e^{-\theta_2 t^{-\eta_2}}\big[1+\omega_1(1-e^{-\theta_1 z^{-\eta_1}})+\omega_2(1-e^{-\theta_2 t^{-\eta_2}}) +\omega(1-e^{-\theta_1 z^{-\eta_1}})(1-e^{-\theta_2 t^{-\eta_2}})\big].$$

Furthermore, the reliability function is given by:

$$R(z,t) = (1-e^{-\theta_1 z^{-\eta_1}})(1-e^{-\theta_2 t^{-\eta_2}})\big[1-\omega_1 e^{-\theta_1 z^{-\eta_1}}-\omega_2 e^{-\theta_2 t^{-\eta_2}}+\omega e^{-\theta_1 z^{-\eta_1}}e^{-\theta_2 t^{-\eta_2}}\big].$$

The three-dimensional JPDF of the CAMBIW distribution is illustrated in Figure 2. The figure presents subfigures featuring distinct parameter values. The parameter values in vector form, ordered in the order $(\theta_1,\eta_1,\theta_2,\eta_2,\omega_1,\omega_2,\omega)$, were placed under each subfigure from (a) to (f).

A. *Conditional distribution*

The following shows the conditional PDF for T given Z

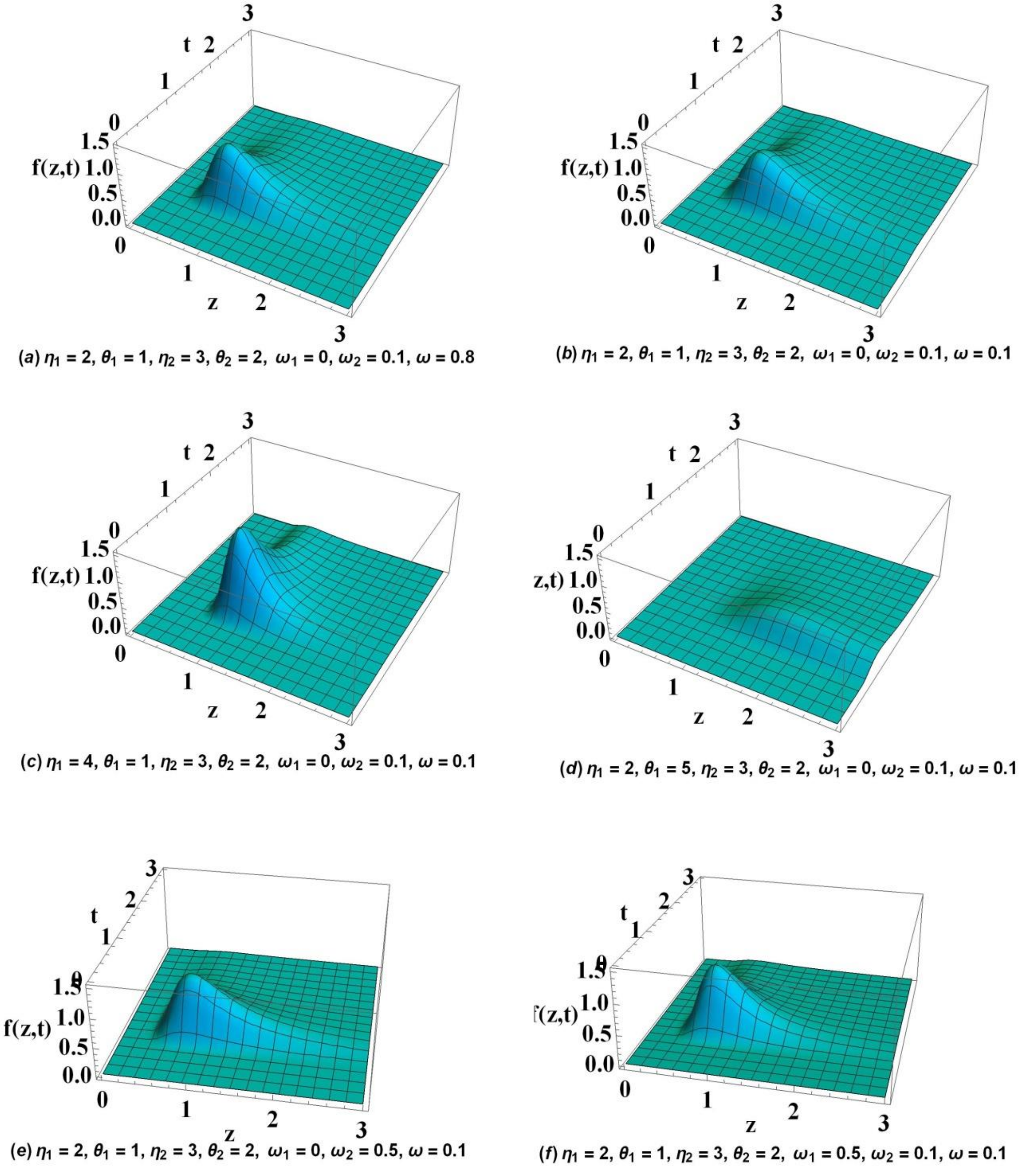
$$f_{T|Z}(t|z) = \eta_2\theta_2 t^{-\eta_2-1}e^{-\theta_2 t^{-\eta_2}}\left[1+\frac{\omega_2+\omega\left(1-2e^{-\theta_1 z^{-\eta_1}}\right)}{1+\omega_1\left(1-2e^{-\theta_1 z^{-\eta_1}}\right)}\left(1-2e^{-\theta_2 t^{-\eta_2}}\right)\right].$$

Therefore, the regression curve (conditional expectation) of T given $Z=z$ for CAMBIW is

$$E(T|Z=z) = \theta_2^{1/\eta_2}\Gamma\left(1-\frac{1}{\eta_2}\right)\left[1+\frac{\omega_2+\omega\left(1-2e^{-\theta_1 z^{-\eta_1}}\right)}{1+\omega_1\left(1-2e^{-\theta_1 z^{-\eta_1}}\right)}\left(1-2^{\frac{1}{\eta_2}}\right)\right], \eta_2>1.$$

The regression curve for several values of the parameters are illustrated in Figure 3. From Figure 3, when ω is positive (negative) increases (decreases) the overall value of the regression curve, resulting in an increasing (decreasing) trend as Z increases.

It is evident that the previously presented regression curve is non-linear in Z. Using an analogous analysis, we can demonstrate that the regression curve of Z conditioned on a fixed value of T also demonstrates non-linear behavior with respect to t.



(a) $\eta_1 = 2, \theta_1 = 1, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0, \omega_2 = 0.1, \omega = 0.8$

(b) $\eta_1 = 2, \theta_1 = 1, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0, \omega_2 = 0.1, \omega = 0.1$

(c) $\eta_1 = 4, \theta_1 = 1, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0, \omega_2 = 0.1, \omega = 0.1$

(d) $\eta_1 = 2, \theta_1 = 5, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0, \omega_2 = 0.1, \omega = 0.1$

(e) $\eta_1 = 2, \theta_1 = 1, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0, \omega_2 = 0.5, \omega = 0.1$

(f) $\eta_1 = 2, \theta_1 = 1, \eta_2 = 3, \theta_2 = 2, \omega_1 = 0.5, \omega_2 = 0.1, \omega = 0.1$

Figure 2: JPDF for CAMBIW distribution

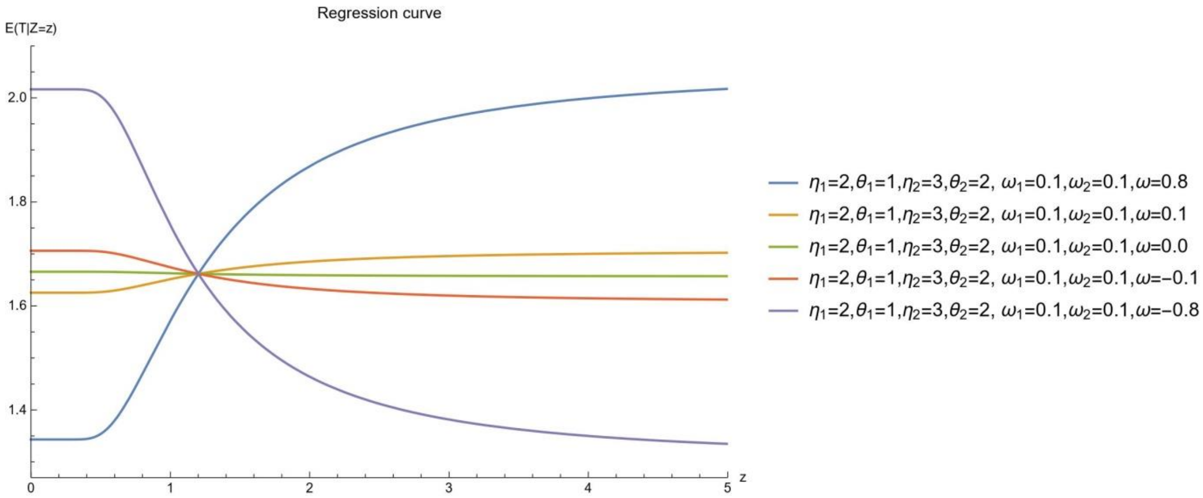


Figure 3: Regression curve for CAMBIW distribution

B. *Product moments*

The joint moments of order l_1 and l_2 around zero, denoted by $\mu_{Z,T}^{(l_1,l_2)}$ can be expressed as:

$$\mu_{Z,T}^{(l_1,l_2)} = \theta_1^{l_1/\eta_1}\Gamma\left(1-\frac{l_1}{\eta_1}\right)\theta_2^{l_2/\eta_2}\Gamma\left(1-\frac{l_2}{\eta_2}\right)\Big[1+\omega_1(1-2^{\frac{l_1}{\eta_1}})$$
$$+\omega_2(1-2^{\frac{l_2}{\eta_2}})+\omega(1-2^{\frac{l_1}{\eta_1}})(1-2^{\frac{l_2}{\eta_2}})\Big], \eta_1 > l_1, \eta_2 > l_2.$$

The covariance and correlation (ρ) between Z and T are calculated as follows:

$$COV(Z,T) = \theta_1^{1/\eta_1}\Gamma\left(1-\frac{1}{\eta_1}\right)\theta_2^{1/\eta_2}\Gamma\left(1-\frac{1}{\eta_2}\right)(1-2^{\frac{1}{\eta_1}})(1-2^{\frac{1}{\eta_2}})[\omega-\omega_1\omega_2], \eta_1 > 1, \eta_2 > 1,$$

and

$$\rho = \frac{\Gamma\left(1-\frac{1}{\eta_1}\right)\Gamma\left(1-\frac{1}{\eta_2}\right)(1-2^{\frac{1}{\eta_1}})(1-2^{\frac{1}{\eta_2}})}{\sqrt{\left(\Gamma\left(1-\frac{2}{\eta_1}\right)[1+\omega_1(1-2^{\frac{2}{\eta_1}})]-\Gamma\left(1-\frac{1}{\eta_1}\right)^2[1+\omega_1(1-2^{\frac{1}{\eta_1}})]^2\right)}}$$
$$\times \frac{[\omega-\omega_1\omega_2]}{\sqrt{\left(\Gamma\left(1-\frac{2}{\eta_2}\right)[1+\omega_2(1-2^{\frac{2}{\eta_2}})]-\Gamma\left(1-\frac{1}{\eta_2}\right)^2[1+\omega_2(1-2^{\frac{1}{\eta_2}})]^2\right)}}, \tag{4}$$

where $\eta_1 > 2, \eta_2 > 2$.

Table 1 displays the coefficient of correlation for CAMBIW by using (4). The result of this table shows that the maximum value of ρ from CAMBIW is 0.30024, the correlation is negative only when $\omega < 0$, and the following two properties are also clear from (4) and from Table 1, $\rho(\omega_1,-\omega_2,-\omega) = \rho(-\omega_1,\omega_2,-\omega)$, $\rho(-\omega_1,\omega_2,\omega) = \rho(\omega_1,-\omega_2,\omega)$.

Table 1: The coefficient of correlation, ρ, in CAMBIW

ρ	ω_1	ω_2	ω	η_1	η_2	ρ	ω_1	ω_2	ω	η_1	η_2
-0.12831	0.1	0.1	-0.8	3	3	-0.04623	0.1	-0.5	-0.4	3	3
-0.06495	0.1	0.1	-0.4	3	3	-0.05272	0.1	-0.25	-0.4	3	3
-0.00158	0.1	0.1	0.0	3	3	-0.06095	0.1	0.0	-0.4	3	3
0.06178	0.1	0.1	0.4	3	3	-0.07207	0.1	0.25	-0.4	3	3
0.12514	0.1	0.1	0.8	3	3	-0.08878	0.1	0.5	-0.4	3	3
-0.04624	-0.5	0.1	-0.4	3	3	0.05944	0.1	-0.5	0.4	3	3
-0.05273	-0.25	0.1	-0.4	3	3	0.05975	0.1	-0.25	0.4	3	3
-0.06095	0.0	0.1	-0.4	3	3	0.06095	0.1	0.0	0.4	3	3
-0.07207	0.25	0.1	-0.4	3	3	0.06359	0.1	0.25	0.4	3	3
-0.08878	0.5	0.1	-0.4	3	3	0.06905	0.1	0.5	0.4	3	3
0.05944	-0.5	0.1	0.4	3	3	0.16821	0.1	0.1	0.8	3	15
0.05975	-0.25	0.1	0.4	3	3	0.22610	0.1	0.1	0.8	15	15
0.06095	0.0	0.1	0.4	3	3	0.28224	0.	0.	0.99	30	40
0.06359	0.25	0.1	0.4	3	3	0.29432	0.3	0.3	0.99	50	60
0.06905	0.5	0.1	0.4	3	3	0.30024	0.22	0.22	0.999	150	150

C. Concomitant of rth OS

Abd Elgawad et al. (2021) and Alawady et al. (2021a) derived the PDF and CDF of the concomitant of rth OS $T_{[r:n]}$, $1 \leq r \leq n$, denoted by $g_{[r,n]}(t)$ and $G_{[r,n]}(t)$, respectively, as follows:

$$g_{[r,n]}(t) = f(t)(1 + C_{r,n}(1 - 2F(t))),$$
$$G_{[r,n]}(t) = F(t)(1 + C_{r,n}(1 - F(t))), \tag{5}$$

where

$$C_{r,n} = \begin{cases} \omega_2 + \omega(\frac{m-2r+1}{m+1}), & \omega_1 = 0, \\ \frac{\omega}{\omega_1} + \left(\omega_2 - \frac{\omega}{\omega_1}\right)\sum_{j=0}^{\infty} \frac{(-4\omega_1)^j \beta(m-r+j+1,r)}{(1-\omega_1)^{2j+1}\beta(m-r+1,r)} (\prod_{h=1}^{j} \frac{(2h-1)}{2h}), & \omega_1 \neq 0. \end{cases} \tag{6}$$

For more details see Abd Elgawad et al. (2021) and Alawady et al. (2021a). By using (1) and (5), we get the PDF and CDF of the concomitant of rth OS for CAMBIW as follows:

$$g_{[r,n]}(t) = \eta_2\theta_2 t^{-\eta_2-1} e^{-\theta_2 t^{-\eta_2}}\left[1 + C_{r,n}\left(2e^{-\theta_2 t^{-\eta_2}} - 1\right)\right],$$
$$G_{[r,n]}(t) = e^{-\theta_2 t^{-\eta_2}}\left[1 + C_{r,n}\left(e^{-\theta_2 t^{-\eta_2}} - 1\right)\right]. \tag{7}$$

From (7), the l th moment of $T_{[r:n]}$ is

$$\mu_{[r:n]}^{(l)} = E\left(T_{[r:n]}^l\right) = \int_{-\infty}^{\infty} t^l g_{[r,n]}(t)\mathrm{d}t$$
$$= \left(1 - C_{r,n}\right)\mu_T^{(l)} + 2C_{r,n}(\mu_T^{(l)} - I^*),$$

where

$$\mu_T^{(l)} = \mathbb{E}\left(T^l\right) = \int_0^{\infty} t^l \theta_2\eta_2 t^{-\eta_2-1} e^{-\theta_2 t^{-\eta_2}} dt = \theta_2^{l/\eta_2}\Gamma\left(1 - \frac{l}{\eta_2}\right), \eta_2 > l,$$

and

$$I^{*}=\mathbb{E}\left(T^{l}e^{-\theta_{2}T^{-\eta_{2}}}\right)=2^{\frac{l}{\eta_{2}}-1}\theta_{2}^{l/\eta_{2}}\Gamma\left(1-\frac{l}{\eta_{2}}\right),\eta_{2}>l.$$

III. Extropy

The extropy for the IW distribution with PDF given in (1) is given by

$$\mathcal{J}(Z)=-\frac{1}{2}\int_{-\infty}^{\infty}f^{2}(z)dz.$$

$$=-\frac{1}{2}\int_{0}^{\infty}\left(\eta\theta z^{-\eta-1}e^{-\theta z^{-\eta}}\right)^{2}dz$$

$$=-2^{-\frac{1}{\eta}-3}\eta\theta^{-1/\eta}\Gamma\left(2+\frac{1}{\eta}\right).$$

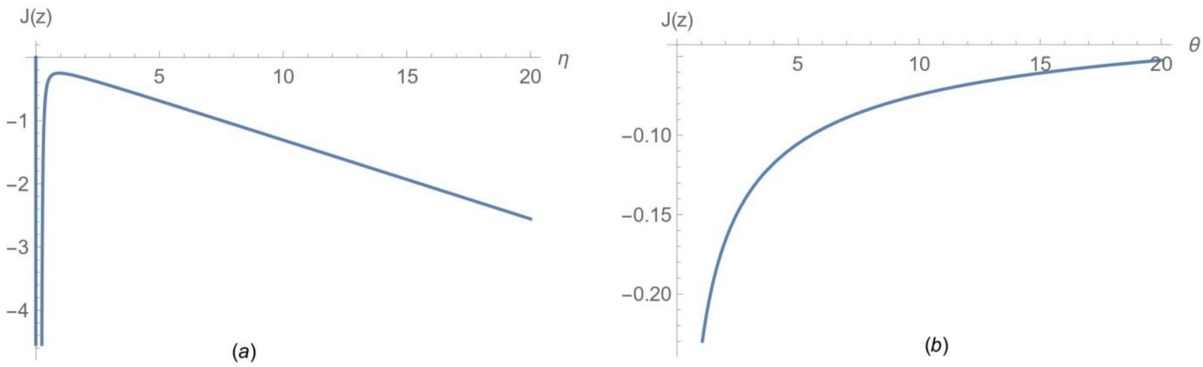


Figure 4: Extropy of IW distribution

Figure 4 (a,b) shows the extropy of IW when the value of $\theta=2$ (see, a) for various values of η, and the value of $\eta=2$ for various values of θ (see, b). It is easy to see from Figure 4 (a) that the value of extropy decreases with an increase in η, except in a very short interval it increases. While it increases with an increase in θ, see Figure 4 (b).

Proposition 1 *If $T_{[r:n]}$ is the rth concomitant of OS based on the CAMBIW, then the extropy of $T_{[r:n]}$, is given by*

$$\mathcal{J}(T_{[r:n]})=(1+C_{r,n})^{2}\mathcal{J}(T)+2(1+C_{r,n})C_{r,n}I_{1}-2(C_{r,n})^{2}I_{2}, \tag{8}$$

where $\mathcal{J}(T)$ is the extropy of the RV T,

$$I_{1}=3^{-\frac{1}{\eta_{2}}-2}\eta_{2}\theta_{2}^{-1/\eta_{2}}\Gamma\left(2+\frac{1}{\eta_{2}}\right),$$

and

$$I_{2}=4^{-\frac{1}{\eta_{2}}-2}\eta_{2}\theta_{2}^{-1/\eta_{2}}\Gamma\left(2+\frac{1}{\eta_{2}}\right).$$

Proof. By using (5), the extropy of $T_{[r:n]}$ is

$$\mathcal{J}(T_{[r:n]})=\frac{-1}{2}\int_{0}^{\infty}g_{[r,n]}^{2}(t)dt$$

$$=\frac{-1}{2}\int_{0}^{\infty}[f(t)(1-C_{r,n}(2F(t)-1))]^{2}dt$$

$$=\frac{-1}{2}[(1+C_{r,n})^{2}\int_{0}^{\infty}f^{2}(t)dt-4(1+C_{r,n})C_{r,n}\int_{0}^{\infty}f^{2}(t)F(t)dt$$

$$+4(C_{r,n})^{2}\int_{0}^{\infty}f^{2}(t)F^{2}(t)dt]$$

$$= [(1 + C_{r,n})^2 \mathcal{J}(T) + 2(1 + C_{r,n})C_{r,n}I_1 - 2(C_{r,n})^2 I_2,$$

where

$$I_1 = \int_0^\infty (\eta_2\theta_2 t^{-\eta_2-1})^2 e^{-3\theta_2 t^{-\eta_2}}\, dt$$
$$= 3^{-\frac{1}{\eta_2}-2}\eta_2\theta_2^{-1/\eta_2}\Gamma\left(2 + \frac{1}{\eta_2}\right)$$

and

$$I_2 = \int_0^\infty (\eta_2\theta_2 t^{-\eta_2-1})^2 e^{-4\theta_2 t^{-\eta_2}}\, dt$$
$$= 4^{-\frac{1}{\eta_2}-2}\eta_2\theta_2^{-1/\eta_2}\Gamma\left(2 + \frac{1}{\eta_2}\right).$$

This completes the proof.

The following features can be extracted from Table 2:

- In general, when r remains constant, the value of $\mathcal{J}(T_{[r:n]})$ increases (decreases) if $\omega < 0 (\omega > 0)$ as n increases.
- In general, the value of $\mathcal{J}(T_{[r:n]})$ decreases (increases) if $\omega < 0 (\omega > 0)$ as r increases.
- In almost cases, the value of $\mathcal{J}(T_{[r:n]})$ decreases (increases) if $r < \frac{n+1}{2} (r > \frac{n+1}{2})$ as ω increases.
- In general, the value of $\mathcal{J}(T_{[r:n]})$ decreases as ω_2 increases.

IV. Weighted extropy

The weighted extropy for the IW distribution with PDF given in (1) is given by

$$\mathcal{J}^w(Z) = -\frac{1}{2}\int_{-\infty}^{\infty} z f^2(z) dz.$$

Thus, we get $$\mathcal{J}^w(Z) = -\frac{1}{2}\int_0^\infty z\left(\eta\theta z^{-\eta-1} e^{-\theta z^{-\eta}}\right)^2 dz = -\frac{\eta}{8}.$$

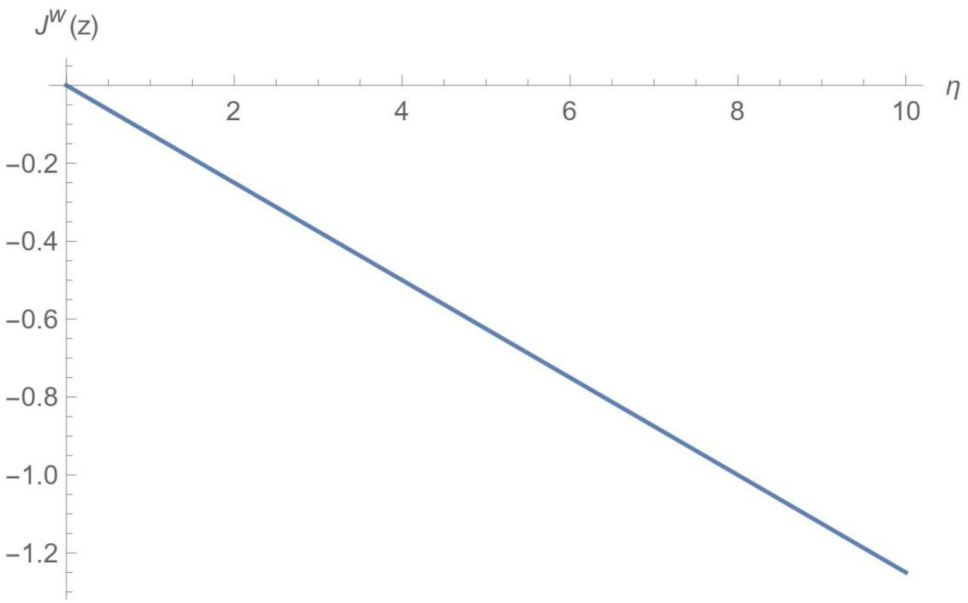


Figure 5: Weighted extropy of IW distribution

It is easy to notice that the value of weighted extropy does not depend on θ. So for various values of η, Figure 5 is plotted. It is easy to see from the graph that the value of weighted extropy decreases with an increase in η.

Proposition 2 *If $T_{[r:n]}$ is the concomitant of the rth OS based on the CAMBIW, then the weighted extropy of $T_{[r:n]}$, is given by*

$$\mathcal{J}^W(T_{[r:n]}) = (1 + C_{r,n})^2 \mathcal{J}^W(T) + \frac{2\eta_2}{9}(1 + C_{r,n})C_{r,n} - \frac{\eta_2}{8}C_{r,n}^2, \tag{9}$$

where $\mathcal{J}^W(T)$ is the weighted extropy of the RV T.

Proof. By using (5), the weighted extropy of $T_{[r:n]}$ is

$$\mathcal{J}^W(T_{[r:n]}) = \frac{-1}{2}\int_0^\infty t g_{[r,n]}^2(t)dt$$

$$= \frac{-1}{2}\int_0^\infty t[f(t)(1 - C_{r,n}(2F(t) - 1))]^2 dt$$

$$= \frac{-1}{2}[(1 + C_{r,n})^2 \int_0^\infty t f^2(t)dt - 4(1 + C_{r,n})C_{r,n}\int_0^\infty t f^2(t)F(t)dt$$

$$+4(C_{r,n})^2 \int_0^\infty t f^2(t)F^2(t)dt]$$

$$= [(1 + C_{r,n})^2 \mathcal{J}^W(T) + 2(1 + C_{r,n})C_{r,n}J_1 - 2(C_{r,n})^2 J_2,$$

where

$$J_1 = \int_0^\infty t(\eta_2\theta_2 t^{-\eta_2-1})^2 e^{-3\theta_2 t^{-\eta_2}}\, dt = \frac{\eta_2}{9},$$

and

$$J_2 = \int_0^\infty t(\eta_2\theta_2 t^{-\eta_2-1})^2 e^{-4\theta_2 t^{-\eta_2}} dt = \frac{\eta_2}{16}.$$

This completes the proof.

The following features can be extracted from Table 3:

- In almost cases, the value of $\mathcal{J}^W(T_{[r:n]})$ decreases (increases) if $\omega < 0(\omega > 0)$ as r increases.
- In almost cases, the value of $\mathcal{J}^W(T_{[r:n]})$ decreases (increases) if $r < \frac{n+1}{2}(r > \frac{n+1}{2})$ as ω increases.
- In almost cases, the value of $\mathcal{J}^W(T_{[r:n]})$ decreases as ω_2 increases.

Table 2: $\mathcal{J}(T_{[r:n]})$ at $\eta_2 = 2, \theta_2 = 1$

n	r	ω											
		-0.6	-0.3	0.3	0.6	-0.6	-0.3	0.3	0.6	-0.6	0.6	0.3	-0.3
		$\omega_1 = -0.1, \omega_2 = -0.2$				$\omega_1 = -0.1, \omega_2 = 0.2$				$\omega_1 = 0.1, \omega_2 = -0.2$		$\omega_1 = 0.1, \omega_2 = 0.2$	
5	1	-0.1776	-0.1912	-0.2343	-0.2637	-0.2112	-0.2357	-0.3002	-0.3403	-0.181	-0.2626	-0.2344	-0.2924
	2	-0.1933	-0.2016	-0.2214	-0.233	-0.237	-0.2501	-0.2793	-0.2956	-0.193	-0.2369	-0.247	-0.2786
	3	-0.2135	-0.2126	-0.2108	-0.2099	-0.2657	-0.2644	-0.2618	-0.2605	-0.2099	-0.2135	-0.2618	-0.2644
	4	-0.2369	-0.224	-0.202	-0.193	-0.2963	-0.2786	-0.247	-0.2331	-0.233	-0.1933	-0.2793	-0.2501
	5	-0.2626	-0.2356	-0.1948	-0.181	-0.3281	-0.2924	-0.2344	-0.212	-0.2637	-0.1776	-0.3002	-0.2357
15	1	-0.1708	-0.1854	-0.2436	-0.2873	-0.1973	-0.2269	-0.315	-0.3736	-0.1755	-0.2797	-0.2275	-0.3011
	2	-0.1747	-0.189	-0.2377	-0.2722	-0.2058	-0.2323	-0.3056	-0.3523	-0.1785	-0.2697	-0.2314	-0.2961
	3	-0.1796	-0.1927	-0.2322	-0.2586	-0.2149	-0.2378	-0.2968	-0.3329	-0.1821	-0.2598	-0.2356	-0.291
	4	-0.1853	-0.1966	-0.2272	-0.2464	-0.2245	-0.2433	-0.2887	-0.3153	-0.1861	-0.2502	-0.2401	-0.2859
	5	-0.1917	-0.2006	-0.2225	-0.2354	-0.2346	-0.2487	-0.2811	-0.2993	-0.1908	-0.2408	-0.2449	-0.2807
	6	-0.1987	-0.2047	-0.2182	-0.2257	-0.245	-0.2542	-0.274	-0.2847	-0.1962	-0.2316	-0.25	-0.2755
	7	-0.2063	-0.2088	-0.2141	-0.2169	-0.2558	-0.2596	-0.2674	-0.2714	-0.2023	-0.2228	-0.2554	-0.2703
	8	-0.2143	-0.213	-0.2104	-0.2092	-0.2668	-0.2649	-0.2612	-0.2593	-0.2092	-0.2143	-0.2612	-0.2649
	9	-0.2228	-0.2172	-0.207	-0.2023	-0.2781	-0.2703	-0.2554	-0.2484	-0.2169	-0.2063	-0.2674	-0.2596
	10	-0.2316	-0.2215	-0.2038	-0.1962	-0.2896	-0.2755	-0.25	-0.2384	-0.2257	-0.1987	-0.274	-0.2542
	11	-0.2408	-0.2258	-0.2008	-0.1908	-0.3012	-0.2807	-0.2449	-0.2295	-0.2354	-0.1917	-0.2811	-0.2487
	12	-0.2502	-0.2301	-0.198	-0.1861	-0.3129	-0.2859	-0.2401	-0.2213	-0.2464	-0.1853	-0.2887	-0.2433
	13	-0.2598	-0.2344	-0.1955	-0.1821	-0.3247	-0.291	-0.2356	-0.214	-0.2586	-0.1796	-0.2968	-0.2378
	14	-0.2697	-0.2387	-0.1931	-0.1785	-0.3366	-0.2961	-0.2314	-0.2073	-0.2722	-0.1747	-0.3056	-0.2323
	15	-0.2797	-0.2429	-0.1909	-0.1755	-0.3485	-0.3011	-0.2275	-0.2014	-0.2873	-0.1708	-0.315	-0.2269
		$\omega_1 = 0, \omega_2 = -0.2$				$\omega_1 = 0, \omega_2 = -0.1$				$\omega_1 = 0, \omega_2 = 0.1$		$\omega_1 = 0, \omega_2 = 0.2$	
5	1	-0.1793	-0.1931	-0.235	-0.2632	-0.1856	-0.2017	-0.2485	-0.2791	-0.2017	-0.3145	-0.235	-0.2962
	2	-0.1931	-0.2017	-0.2227	-0.235	-0.2017	-0.2116	-0.235	-0.2485	-0.2227	-0.2791	-0.2485	-0.2791
	3	-0.2116	-0.2116	-0.2116	-0.2116	-0.2227	-0.2227	-0.2227	-0.2227	-0.2485	-0.2485	-0.2632	-0.2632
	4	-0.235	-0.2227	-0.2017	-0.1931	-0.2485	-0.235	-0.2116	-0.2017	-0.2791	-0.2227	-0.2791	-0.2485
	5	-0.2632	-0.235	-0.1931	-0.1793	-0.2791	-0.2485	-0.2017	-0.1856	-0.3145	-0.2017	-0.2962	-0.235
15	1	-0.1731	-0.1882	-0.2433	-0.2832	-0.1779	-0.1962	-0.2575	-0.3006	-0.1911	-0.339	-0.2272	-0.3075
	2	-0.1766	-0.1911	-0.2383	-0.271	-0.1823	-0.1995	-0.252	-0.2875	-0.1972	-0.324	-0.2318	-0.3006
	3	-0.1807	-0.1941	-0.2334	-0.2594	-0.1873	-0.2029	-0.2467	-0.275	-0.2041	-0.3098	-0.2366	-0.294
	4	-0.1856	-0.1972	-0.2287	-0.2485	-0.1931	-0.2065	-0.2416	-0.2632	-0.2116	-0.2962	-0.2416	-0.2875
	5	-0.1911	-0.2006	-0.2242	-0.2383	-0.1995	-0.2103	-0.2366	-0.252	-0.2198	-0.2832	-0.2467	-0.2811
	6	-0.1972	-0.2041	-0.2198	-0.2287	-0.2065	-0.2143	-0.2318	-0.2416	-0.2287	-0.271	-0.252	-0.275
	7	-0.2041	-0.2078	-0.2156	-0.2198	-0.2143	-0.2184	-0.2272	-0.2318	-0.2383	-0.2594	-0.2575	-0.269
	8	-0.2116	-0.2116	-0.2116	-0.2116	-0.2227	-0.2227	-0.2227	-0.2227	-0.2485	-0.2485	-0.2632	-0.2632
	9	-0.2198	-0.2156	-0.2078	-0.2041	-0.2318	-0.2272	-0.2184	-0.2143	-0.2594	-0.2383	-0.269	-0.2575
	10	-0.2287	-0.2198	-0.2041	-0.1972	-0.2416	-0.2318	-0.2143	-0.2065	-0.271	-0.2287	-0.275	-0.252
	11	-0.2383	-0.2242	-0.2006	-0.1911	-0.252	-0.2366	-0.2103	-0.1995	-0.2832	-0.2198	-0.2811	-0.2467
	12	-0.2485	-0.2287	-0.1972	-0.1856	-0.2632	-0.2416	-0.2065	-0.1931	-0.2962	-0.2116	-0.2875	-0.2416
	13	-0.2594	-0.2334	-0.1941	-0.1807	-0.275	-0.2467	-0.2029	-0.1873	-0.3098	-0.2041	-0.294	-0.2366
	14	-0.271	-0.2383	-0.1911	-0.1766	-0.2875	-0.252	-0.1995	-0.1823	-0.324	-0.1972	-0.3006	-0.2318
	15	-0.2832	-0.2433	-0.1882	-0.1731	-0.3006	-0.2575	-0.1962	-0.1779	-0.339	-0.1911	-0.3075	-0.2272

Table 3: $\mathcal{J}^{W}(T_{[r:n]})$ at $\eta_2 = 2$

N	r	ω											
		-0.6	-0.3	0.3	0.6	-0.6	-0.3	0.3	0.6	-0.6	0.6	0.3	-0.3
		$\omega_1 = -0.1, \omega_2 = -0.2$				$\omega_1 = -0.1, \omega_2 = 0.2$				$\omega_1 = 0.1, \omega_2 = -0.2$		$\omega_1 = 0.1, \omega_2 = 0.2$	
5	1	-0.2371	-0.2364	-0.2497	-0.2636	-0.241	-0.2503	-0.2834	-0.3072	-0.2364	-0.263	-0.2498	-0.279
	2	-0.2367	-0.2383	-0.2445	-0.2491	-0.2509	-0.2569	-0.2718	-0.2808	-0.2367	-0.2508	-0.2554	-0.2714
	3	-0.2417	-0.2414	-0.2408	-0.2406	-0.2646	-0.264	-0.2626	-0.262	-0.2406	-0.2417	-0.2626	-0.264
	4	-0.2508	-0.2455	-0.2384	-0.2367	-0.2812	-0.2714	-0.2554	-0.2492	-0.2491	-0.2367	-0.2718	-0.2569
	5	-0.263	-0.2502	-0.2369	-0.2364	-0.2998	-0.279	-0.2498	-0.2412	-0.2636	-0.2371	-0.2834	-0.2503
15	1	-0.2407	-0.2361	-0.2538	-0.2762	-0.2374	-0.2466	-0.292	-0.3281	-0.2377	-0.272	-0.2469	-0.2839
	2	-0.2381	-0.2362	-0.2512	-0.268	-0.2394	-0.2489	-0.2865	-0.3147	-0.2368	-0.2667	-0.2485	-0.2811
	3	-0.2366	-0.2366	-0.2488	-0.261	-0.2422	-0.2512	-0.2815	-0.3027	-0.2363	-0.2616	-0.2503	-0.2782
	4	-0.2361	-0.2372	-0.2467	-0.2551	-0.2457	-0.2537	-0.2769	-0.2922	-0.2361	-0.2569	-0.2522	-0.2754
	5	-0.2365	-0.2381	-0.2449	-0.2502	-0.2498	-0.2562	-0.2728	-0.2829	-0.2364	-0.2525	-0.2544	-0.2726
	6	-0.2377	-0.2391	-0.2433	-0.2461	-0.2545	-0.2588	-0.269	-0.2747	-0.2372	-0.2486	-0.2568	-0.2698
	7	-0.2395	-0.2402	-0.2419	-0.2429	-0.2596	-0.2615	-0.2655	-0.2676	-0.2385	-0.245	-0.2594	-0.267
	8	-0.242	-0.2416	-0.2407	-0.2404	-0.2652	-0.2642	-0.2623	-0.2614	-0.2404	-0.242	-0.2623	-0.2642
	9	-0.245	-0.243	-0.2397	-0.2385	-0.2712	-0.267	-0.2594	-0.2561	-0.2429	-0.2395	-0.2655	-0.2615
	10	-0.2486	-0.2445	-0.2388	-0.2372	-0.2774	-0.2698	-0.2568	-0.2515	-0.2461	-0.2377	-0.269	-0.2588
	11	-0.2525	-0.2462	-0.2381	-0.2364	-0.284	-0.2726	-0.2544	-0.2477	-0.2502	-0.2365	-0.2728	-0.2562
	12	-0.2569	-0.2479	-0.2375	-0.2361	-0.2908	-0.2754	-0.2522	-0.2445	-0.2551	-0.2361	-0.2769	-0.2537
	13	-0.2616	-0.2497	-0.237	-0.2363	-0.2978	-0.2782	-0.2503	-0.2419	-0.261	-0.2366	-0.2815	-0.2512
	14	-0.2667	-0.2516	-0.2367	-0.2368	-0.305	-0.2811	-0.2485	-0.2398	-0.268	-0.2381	-0.2865	-0.2489
	15	-0.272	-0.2535	-0.2364	-0.2377	-0.3123	-0.2839	-0.2469	-0.2383	-0.2762	-0.2407	-0.292	-0.2466
		$\omega_1 = 0, \omega_2 = -0.2$				$\omega_1 = 0, \omega_2 = -0.1$				$\omega_1 = 0, \omega_2 = 0.1$		$\omega_1 = 0, \omega_2 = 0.2$	
5	1	-0.2367	-0.2367	-0.25	-0.2633	-0.2361	-0.2383	-0.2561	-0.2717	-0.2383	-0.2917	-0.25	-0.2811
	2	-0.2367	-0.2383	-0.245	-0.25	-0.2383	-0.2411	-0.25	-0.2561	-0.245	-0.2717	-0.2561	-0.2717
	3	-0.2411	-0.2411	-0.2411	-0.2411	-0.245	-0.245	-0.245	-0.245	-0.2561	-0.2561	-0.2633	-0.2633
	4	-0.25	-0.245	-0.2383	-0.2367	-0.2561	-0.25	-0.2411	-0.2383	-0.2717	-0.245	-0.2717	-0.2561
	5	-0.2633	-0.25	-0.2367	-0.2367	-0.2717	-0.2561	-0.2383	-0.2361	-0.2917	-0.2383	-0.2811	-0.25
15	1	-0.2389	-0.2362	-0.2537	-0.2739	-0.237	-0.2372	-0.2605	-0.2836	-0.2364	-0.3064	-0.2467	-0.2876
	2	-0.2374	-0.2364	-0.2514	-0.2674	-0.2362	-0.2378	-0.2578	-0.2763	-0.2374	-0.2974	-0.2486	-0.2836
	3	-0.2364	-0.2368	-0.2493	-0.2614	-0.2361	-0.2386	-0.2553	-0.2695	-0.2389	-0.2889	-0.2507	-0.2799
	4	-0.2361	-0.2374	-0.2474	-0.2561	-0.2367	-0.2396	-0.2529	-0.2633	-0.2411	-0.2811	-0.2529	-0.2762
	5	-0.2364	-0.2381	-0.2456	-0.2514	-0.2378	-0.2407	-0.2507	-0.2578	-0.2439	-0.2739	-0.2553	-0.2728
	6	-0.2374	-0.2389	-0.2439	-0.2474	-0.2396	-0.242	-0.2486	-0.2529	-0.2474	-0.2674	-0.2578	-0.2695
	7	-0.2389	-0.2399	-0.2424	-0.2439	-0.242	-0.2434	-0.2467	-0.2486	-0.2514	-0.2614	-0.2605	-0.2663
	8	-0.2411	-0.2411	-0.2411	-0.2411	-0.245	-0.245	-0.245	-0.245	-0.2561	-0.2561	-0.2633	-0.2633
	9	-0.2439	-0.2424	-0.2399	-0.2389	-0.2486	-0.2467	-0.2434	-0.242	-0.2614	-0.2514	-0.2663	-0.2605
	10	-0.2474	-0.2439	-0.2389	-0.2374	-0.2529	-0.2486	-0.242	-0.2396	-0.2674	-0.2474	-0.2695	-0.2578
	11	-0.2514	-0.2456	-0.2381	-0.2364	-0.2578	-0.2507	-0.2407	-0.2378	-0.2739	-0.2439	-0.2728	-0.2553
	12	-0.2561	-0.2474	-0.2374	-0.2361	-0.2633	-0.2529	-0.2396	-0.2367	-0.2811	-0.2411	-0.2762	-0.2529
	13	-0.2614	-0.2493	-0.2368	-0.2364	-0.2695	-0.2553	-0.2386	-0.2361	-0.2889	-0.2389	-0.2799	-0.2507
	14	-0.2674	-0.2514	-0.2364	-0.2374	-0.2763	-0.2578	-0.2378	-0.2362	-0.2974	-0.2374	-0.2836	-0.2486
	15	-0.2739	-0.2537	-0.2362	-0.2389	-0.2836	-0.2605	-0.2372	-0.237	-0.3064	-0.2364	-0.2876	-0.2467

V. Real data set

In this section, we present an analysis of a bivariate real data set to demonstrate that the distribution CAMBIW can be used as a good lifetime model. We compare it with the Cambanis bivariate exponential (CAMBE) distribution where Z, T have exponential distributions (i.e., $f_Y(y) =$

$f(y;\theta) = \theta e^{-\theta y}, \quad y \geq 0$).

This section examined the relationship between how long people have had diabetes and their serum creatinine levels (a marker of kidney function). Patients were divided into two groups based on their creatinine levels: those with diabetic nephropathy (high creatinine) and those without. The data came from pathology reports of 132 patients with type 2 diabetes, collected between 2012 and 2013. The average duration of diabetes and corresponding average creatinine levels were reported for various time intervals. For example, the average duration of diabetes for one group was 7.4 years, and their average creatinine level was 1.925 mg/dl. The data are represented in El-Sherpieny et al. (2022).

Table 4: $-\ln L$, AIC, AICc, BIC, HQIC, CAIC

	$-\ln L$	AIC	AICc	BIC	HQIC	CAIC
CAMBE	223.738	457.476	462.091	462.198	458.275	462.091
CAMBIW	187.834	389.667	399.849	396.279	390.786	399.849

Table 4 discusses the negative log-likelihood values $-\ln L$, and different measures, namely Akaike information criterion (AIC), corrected AIC (AICc), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent AIC (CAIC). The Maximum likelihood estimates of parameters for the CAMBIW distribution are calculated, $\hat{\theta}_1 = 1.4847, \hat{\eta}_1 = 0.0246, \hat{\theta}_2 = 1.0344, \hat{\eta}_2 = 0.4125, \widehat{\omega}_1 = -0.953853, \widehat{\omega}_2 = 0.350663, \widehat{\omega} = -0.311891$. It is evident from Table 4 that the CAMBIW model outperforms the CAMBE model regarding data fit. Figure 6 examines the extropy and weighted extropy for the model CAMBIW for the rth concomitant of OSs $T_{[r:19]}, r = 1, \ldots, 19$. Figure 6 shows that when r increases, the value of extropy and weighted extropy increases.

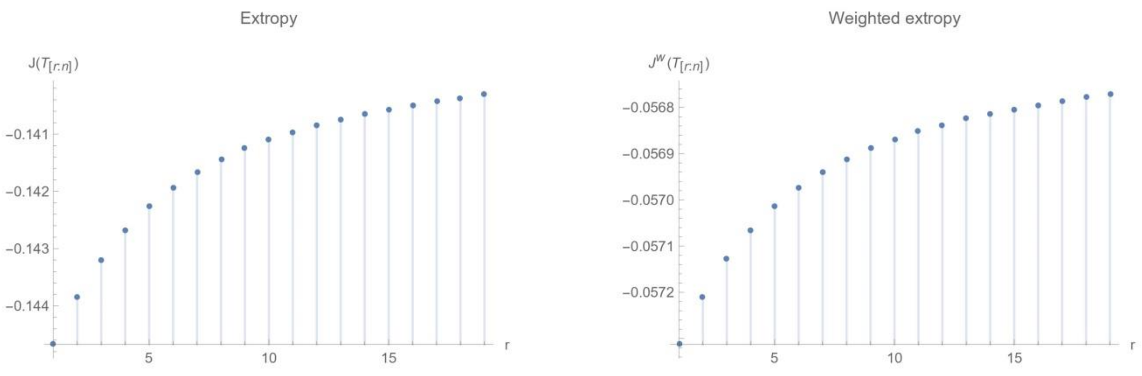


Figure 6: Extropy and weighted extropy of real data set

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