Electronic Journal of Mathematical Analysis and Applications
Vol. 13(1) Jan. 2025, No.9.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY DIFFERENTIAL OPERATOR

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Abstract. The study of operators plays an essential role in Mathematics, especially in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. The class of analytic functions, which has an essential place in the theory of geometric functions, has been studied by many researchers before. This topic still maintains its popularity today. In this work, we introduce and investigate a new subclass of analytic functions in the open unit disc E with negative coefficients. The object of the present paper is to determine the coefficient estimates, extreme points, integral means inequalities and subordination results for this class.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of functions $u(z)$ which are all univalent in E. A function $u \in \mathcal{A}$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

We denote this class with $S^*(\upsilon)$.

2020 *Mathematics Subject Classification.* 30C45, 30C50 .

Key words and phrases. analytic function, uniformly starlike function, coefficient estimate, subordination.

Submitted August, 12, 2024. Revised Sep., 18, 2024. Accepted Jan. 28, 2025.

A function $u \in \mathcal{A}$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{3}$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ (a_n \geq 0,\ z \in E) \tag{4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon),\ C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [12] and Orhan [9].

Let u be a function in the class $\mathcal{A}$. We define the following differential operator introduced by Deniz and Ozkan [3] and others (see [4, 5, 6]).

$$\begin{aligned}
&\mathfrak{D}_\lambda^0 u(z) = u(z)\\
&\mathfrak{D}_\lambda^1 u(z) = \mathfrak{D}_\lambda u(z) = \lambda z^3 u'''(z) + (2\lambda + 1)z^2 u''(z) + zu'(z)\\
&\mathfrak{D}_\lambda^2 u(z) = \mathfrak{D}_\lambda(\mathfrak{D}_\lambda^1 u(z))\\
&\quad\quad .\\
&\quad\quad .\\
&\quad\quad .\\
&\mathfrak{D}_\lambda^m u(z) = \mathfrak{D}_\lambda(\mathfrak{D}_\lambda^{m-1} u(z)),
\end{aligned}$$

where $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If u is given by (1), then from the definition of the operator $\mathfrak{D}_\lambda^m u(z)$, it is to see that

$$\mathfrak{D}_\lambda^m u(z) = z + \sum_{n=2}^{\infty} n^{2m}[\lambda(n-1) + 1]^m a_n z^n. \tag{5}$$

Many differential operators studied by various authors can be seen in the literature (see [1, 2, **?**, 13]).
Denote by

$$\mathcal{R}^\delta := \frac{z}{(1-z)^{\delta+1}} * u(z), \quad (\delta \in \mathbb{N}_0)$$

then implies that

$$\mathcal{R}^\delta u(z) := \frac{z(z^{\delta-1}u(z))^\delta}{\delta!}, \quad (\delta \in \mathbb{N}_0).$$

The operator $\mathcal{R}^\delta u$ is called Ruschweyh derivative operator [10]. Noor [8] defined and investigated an integral operator $\mathcal{N}^\delta : \mathcal{A} \longrightarrow \mathcal{A}$ analogous to $\mathcal{R}^\delta u$ as follows:

Let $u_\delta(z) := \frac{z}{(1-z)^{\delta+1}}, \quad \delta \in \mathbb{N}_0$ and $u_\delta^{(-1)}$ be defined such that

$$u_\delta(z) * u_\delta^{(-1)}(z) = \frac{z}{1-z}.$$

Then

$$\mathcal{N}^\delta u(z) = u_\delta^{(-1)}(z) * u(z) = \left[\frac{z}{(1-z)^{\delta+1}}\right]^{(-1)} * u(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\delta+1)n!}{\Gamma(\delta+n)} a_n z^n := \zeta(z). \tag{6}$$

In [7], Kaziltepe et al. defined the following convolution operator:

$$\begin{aligned}
\mathbb{D}_\lambda^0 u(z) &= u(z)\\
\mathbb{D}_\lambda^1 u(z) &= \mathbb{D}_\lambda u(z)\\
&= \lambda z^3 \zeta'''(z) + (2\lambda+1) z^2 \zeta''(z) + z\zeta'(z)\\
&= z + \sum_{n=2}^{\infty} n^2[\lambda(n-1)+1]\frac{\Gamma(\delta+1)n!}{\Gamma(\delta+n)} a_n z^n\\
&\vdots\\
\mathbb{D}_\lambda^m u(z) &= \mathbb{D}_\lambda(\mathbb{D}_\lambda^{m-1} u(z)), \quad (m \in \mathbb{N}).
\end{aligned}$$

It can be easly seen that

$$\begin{aligned}
\mathbb{D}_\lambda^m u(z) &= z + \sum_{n=2}^{\infty} n^{2m}[\lambda(n-1)+1]\frac{\Gamma(\delta+1)n!}{\Gamma(\delta+n)} a_n z^n\\
&= z + \sum_{n=2}^{\infty} \Theta(n,m,\lambda,\delta) a_n z^n, \qquad (7)
\end{aligned}$$

where

$$\Theta(n,m,\lambda,\delta) = n^{2m}[\lambda(n-1)+1]\frac{\Gamma(\delta+1)n!}{\Gamma(\delta+n)}, \qquad (8)$$

where $m, z \in \mathbb{N}_0$ and $\lambda \geq 0$. Now, by making use of the linear operator $\mathbb{D}_\lambda^m u(z)$, we define a new subclass of functions belonging to the class $\mathcal{A}$.

Definition 1.1. *For $0 \leq \hbar < 1, 0 \leq \sigma < 1$ and $0 < \varsigma < 1$, we let $TS_\lambda^m(\hbar,\sigma,\varsigma)$ be the subclass of u consisting of functions of the form (4) and its geometrical condition satisfy*

$$\left|\frac{\hbar\left((\mathbb{D}_\lambda^m u(z))' - \frac{\mathbb{D}_\lambda^m u(z)}{z}\right)}{\sigma(\mathbb{D}_\lambda^m u(z))' + (1-\hbar)\frac{\mathbb{D}_\lambda^m u(z)}{z}}\right| < \varsigma,\ z \in E,$$

where $\mathbb{D}_\lambda^m u(z)$ is given by (7).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS_\lambda^m(\hbar,\sigma,\varsigma)$.

Theorem 2.1. *Let the function u be defined by (4). Then $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$ if and only if*

$$\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)a_n \leq \varsigma(\sigma+(1-\hbar)), \qquad (9)$$

where $0 < \varsigma < 1, 0 \leq \hbar < 1$ and $0 \leq \sigma < 1$. The result (9) is sharp for the function

$$u(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)} z^n,\ n \geq 2.$$

Proof. Suppose that the inequality (9) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\hbar\left((\mathbb{D}_\lambda^m u(z))' - \frac{\mathbb{D}_\lambda^m u(z)}{z}\right)\right| - \varsigma\left|\sigma\left((\mathbb{D}_\lambda^m u(z))' + (1-\hbar)\frac{\mathbb{D}_\lambda^m u(z)}{z}\right)\right| \\
=&\left|-\hbar\sum_{n=2}^{\infty}(n-1)\Theta(n,m,\lambda,\delta)a_n z^{n-1}\right| \\
&-\varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\Theta(n,m,\lambda,\delta)a_n z^{n-1}\right| \\
\leq&\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)a_n - \varsigma(\sigma+(1-\hbar)) \\
\leq&\ 0.
\end{aligned}$$

Hence, by maximum modulus principle, $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$. Now assume that $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$ so that

$$\left|\frac{\hbar\left((\mathbb{D}_\lambda^m u(z))' - \frac{\mathbb{D}_\lambda^m u(z)}{z}\right)}{\sigma(\mathbb{D}_\lambda^m u(z))' + (1-\hbar)\frac{\mathbb{D}_\lambda^m u(z)}{z}}\right| < \varsigma, \ \ z \in E.$$

Hence,

$$\left|\hbar\left((\mathbb{D}_\lambda^m u(z))' - \frac{\mathbb{D}_\lambda^m u(z)}{z}\right)\right| < \varsigma\left|\sigma\left((\mathbb{D}_\lambda^m u(z))' + (1-\hbar)\frac{\mathbb{D}_\lambda^m u(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\hbar(n-1)\Theta(n,m,\lambda,\delta)a_n z^{n-1}\right| \\
&< \varsigma\left|\sigma+(1-\hbar)-\sum_{n=2}^{\infty}(n\sigma+1-\hbar)\Theta(n,m,\lambda,\delta)a_n z^{n-1}\right|.
\end{aligned}$$

Thus,

$$\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)a_n \leq \varsigma(\sigma+(1-\hbar))$$

and this completes the proof. □

Corollary 2.0. *Let the function* $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$. *Then*

$$a_n \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}z^n, \ \ n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS_\lambda^m(\hbar,\sigma,\varsigma)$.

Theorem 3.2. *Let the function* $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$. *Then*

$$\begin{aligned}
|z| - \frac{\varsigma(\sigma+(1-\hbar))}{\Theta(2,m,\lambda,\delta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2 &\leq |u(z)| \\
&\leq |z| + \frac{\varsigma(\sigma+(1-\hbar))}{\Theta(2,m,\lambda,\delta)[\hbar+\varsigma(2\sigma+1-\hbar)]}|z|^2.
\end{aligned}$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{\Theta(2,m,\lambda,\delta)[\hbar+\varsigma(2\sigma+1-\hbar)]}z^2.$$

Proof.

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \le |z| + \sum_{n=2}^{\infty} a_n |z|^n$$
$$\le |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}. \tag{10}$$

Thus,

$$|u(z)| \le |z| + \frac{\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} |z|^2.$$

Also,

$$\begin{aligned} |u(z)| &\ge |z| - \sum_{n=2}^{\infty} a_n |z|^n \\ &\ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\ge |z| - \frac{\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} |z|^2. \end{aligned}$$

□

Theorem 3.3. *Let $u \in TS_\lambda^m(\hbar, \sigma, \varsigma)$. Then*

$$1 - \frac{2\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} |z| \le |u'(z)| \le 1 + \frac{2\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} |z|$$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]} z^2.$$

Proof. Notice that

$$\begin{aligned} &\Theta(2, \delta, \alpha, \beta)[\hbar + \varsigma(2\sigma + 1 - \hbar)] \sum_{n=2}^{\infty} n a_n \\ \le &\sum_{n=2}^{\infty} n[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta) a_n \\ \le &\varsigma(\sigma + (1-\hbar)), \end{aligned} \tag{11}$$

from Theorem 2.1. Thus,

$$\begin{aligned} |u'(z)| &= \left| 1 - \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \\ &\le 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\le 1 + |z| \sum_{n=2}^{\infty} n a_n \\ &\le 1 + |z| \frac{2\varsigma(\sigma + (1-\hbar))}{\Theta(2, m, \lambda, \delta)[\hbar + \varsigma(2\sigma + 1 - \hbar)]}. \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}
|u'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \\
&\geq 1-\sum_{n=2}^{\infty} na_n|z|^{n-1} \\
&\geq 1-|z|\sum_{n=2}^{\infty} na_n \\
&\geq 1-|z|\frac{2\varsigma(\sigma+(1-\hbar))}{\Theta(2,m,\lambda,\delta)[\hbar+\varsigma(2\sigma+1-\hbar)]}. \qquad (13)
\end{aligned}$$

Combining (12) and (13), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS_{\lambda}^{m}(\hbar,\sigma,\varsigma)$.

Theorem 4.4. *Let $u \in TS_{\lambda}^{m}(\hbar,\sigma,\varsigma)$. Then u is starlike in $|z| < R_1$ of order ρ, $0 \leq \rho < 1$, where*

$$R_1 = \inf_n \left\{\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Theta(n,m,\lambda,\delta)}{(n-\rho)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \qquad (14)$$

Proof. u is starlike of order ρ, $0 \leq \rho < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \rho.$$

Thus, it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus,

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\rho \ if \ \sum_{n=2}^{\infty}\frac{(n-\rho)}{(1-\rho)}a_n|z|^{n-1} \leq 1. \qquad (15)$$

Hence, by Theorem 2.1, (15) will be true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Theta(n,m,\lambda,\delta)}{(n-\rho)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, \ n \geq 2. \qquad (16)$$

The theorem follows easily from (16). □

Theorem 4.5. *Let $u \in TS_{\lambda}^{m}(\hbar,\sigma,\varsigma)$. Then u is convex in $|z| < R_2$ of order ρ, $0 \leq \rho < 1$, where*

$$R_2 = \inf_n \left\{\frac{(1-\rho)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Theta(n,m,\lambda,\delta)}{n(n-\rho)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \qquad (17)$$

Proof. u is convex of order $\rho,\ 0 \leq \rho < 1$ if

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \rho.$$

Thus, it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}}.$$

Thus,

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1 - \rho \ if \ \sum_{n=2}^{\infty} \frac{n(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \leq 1. \tag{18}$$

Hence, by Theorem 2.1, (18) will be true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{n(n-\rho)\varsigma(\sigma + (1-\hbar))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{19}$$

The theorem follows easily from (19). □

Theorem 4.6. *Let $u \in TS_{\lambda}^{m}(\hbar, \sigma, \varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order $\rho,\ 0 \leq \rho < 1$, where*

$$R_3 = \inf_n \left\{\frac{(1-\rho)(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{n\varsigma(\sigma + (1-\hbar))}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{20}$$

Proof. u is close-to-convex of order $\rho, 0 \leq \rho < 1$ if

$$\Re\left\{u'(z)\right\} > \rho.$$

Thus, it is enough to show that

$$|u'(z) - 1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Thus,

$$|u'(z) - 1| \leq 1 - \rho \ if \ \sum_{n=2}^{\infty} \frac{n}{(1-\rho)} a_n |z|^{n-1} \leq 1. \tag{21}$$

Hence, by Theorem 2.1, (21) will be true if

$$\frac{n}{1-\rho}|z|^{n-1} \leq \frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{n\varsigma(\sigma + (1-\hbar))}\right]^{\frac{1}{n-1}}, n \geq 2. \tag{22}$$

The theorem follows easily from (22). □

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS_\lambda^m(\hbar,\sigma,\varsigma)$.

Theorem 5.7. *Let* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)} z^n, \text{ for } n = 2,3,\cdots.$$

Then $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$ *if and only if it can be expressed in the form*

$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \text{ where } \theta_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$, hence, we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma(\sigma + (1-\hbar))\theta_n}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)} z^n.$$

Now, $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$, since

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma + (1-\hbar))} \\ &\quad \times \frac{\varsigma(\sigma + (1-\hbar))\theta_n}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)} \\ &= \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1.\end{aligned}$$

Conversely, suppose $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.

Now $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$ implies from Theorem 2.1

$$a_n \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)}.$$

Setting $\theta_n = \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma+(1-\hbar))} a_n, n = 2,3,\cdots$

and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS_\lambda^m(\hbar,\sigma,\varsigma)$.

Theorem 6.8. *Let* $u, g \in TS(\hbar,\sigma,\varsigma,\vartheta)$. *Then* $u * g \in TS(\hbar,\sigma,\zeta,\vartheta)$ *for*

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } (u*g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\hbar))\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2\Theta(n,m,\lambda,\delta) - \varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. $u \in TS_\lambda^m(\hbar,\sigma,\varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma + (1-\hbar))} a_n \leq 1 \tag{23}$$

and

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma+(1-\hbar))} b_n \leq 1. \tag{24}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\zeta(\sigma+(1-\hbar))} a_n b_n \leq 1. \tag{25}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma+(1-\hbar))} \sqrt{a_n b_n} \leq 1. \tag{26}$$

Therefore, it is enough to show that

$$\begin{aligned} &\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\zeta(\sigma+(1-\hbar))} a_n b_n \\ \leq\ &\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}{\varsigma(\sigma+(1-\hbar))} \sqrt{a_n b_n}. \end{aligned}$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma}. \tag{27}$$

From (26),

$$\sqrt{a_n b_n} \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)}.$$

Thus, it is enough to show that

$$\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Theta(n,m,\lambda,\delta)} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma}$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\hbar))\hbar(n-1)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\Theta(n,m,\lambda,\delta)-\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

□

7. Closure Theorems

We shall prove the following closure theorems for the class $TS_\lambda^m(\hbar,\sigma,\varsigma)$.

Theorem 7.9. *Let $u_j \in TS_\lambda^m(\hbar,\sigma,\varsigma), j=1,2,\ldots,s$. Then*

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS_\lambda^m(\hbar,\sigma,\varsigma).$$

For $u_j(z)=z-\sum_{n=2}^{\infty} a_{n,j} z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j u_j(z) \\ &= z-\sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n \\ &= z-\sum_{n=2}^{\infty} e_n z^n, \end{aligned}$$

where $e_n = \sum\limits_{j=1}^{s} c_j a_{n,j}$. Thus, $g(z) \in TS_\lambda^m(\hbar, \sigma, \varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} e_n \leq 1$$

that is, if

$$\begin{aligned} &\sum_{n=2}^{\infty} \sum_{j=1}^{s} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} c_j a_{n,j} \\ =&\sum_{j=1}^{s} c_j \sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} a_{n,j} \\ \leq&\sum_{j=1}^{s} c_j = 1. \end{aligned}$$

□

Theorem 7.10. *Let* $u, g \in TS_\lambda^m(\hbar, \sigma, \varsigma)$. *Then*

$$h(z) = z - \sum_{n=2}^{\infty} (a_n^2 + b_n^2) z^n \in TS_\lambda^m(\hbar, \sigma, \varsigma), where$$

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \Theta(n, m, \lambda, \delta) - 2\varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. Since $u, g \in TS_\lambda^m(\hbar, \sigma, \varsigma)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty} \left[\frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} a_n \right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} b_n \right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty} \frac{1}{2} \left[\frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} \right]^2 (a_n^2 + b_n^2) \leq 1. \tag{28}$$

But $h(z) \in TS(\hbar, \sigma, \zeta, q, m)$, if and only if

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}{\zeta(\sigma + (1-\hbar))} (a_n^2 + b_n^2) \leq 1, \tag{29}$$

where $0 < \zeta < 1$, however (28) implies (29) if

$$\begin{aligned} &\frac{[\hbar(n-1) + \zeta(n\sigma + 1 - \hbar)]\Theta(n, m, \lambda, \delta)}{\zeta(\sigma + (1-\hbar))} \\ \leq&\frac{1}{2} \left[\frac{(\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar))\Theta(n, m, \lambda, \delta)}{\varsigma(\sigma + (1-\hbar))} \right]^2. \end{aligned}$$

Simplifying, we get

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2 \Theta(n, m, \lambda, \delta) - 2\varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

□

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