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# UNIQUENESS RESULTS FOR DIFFERENTIAL POLYNOMIALS WEIGHTED SHARING A SET

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Abstract. In this paper, we explore the uniqueness problem of differential polynomials $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ of meromorphic functions $f$ and $g$, respectively with the notion of weighted sharing a set of roots of unity, where $Q$ is a polynomial of one variable. The results of the paper generalize the results due to Sultana and Sahoo [Mathematica Bohemica, 2024].

## 1. Introduction and main results

In this paper, the meromorphic function means the meromorphic function in the complex plane. We use the standard notations of Nevanlinna theory, which can be found in [2, 14, 15].

A meromorphic function $a\ (\not\equiv 0, \infty)$ is said to be small with respect to $f$ provided $T(r,a) = S(r,f)$ as $r \to \infty,\ r \notin E$ where $E$ is a set of positive real numbers of finite Lebesgue measure. We denote by $S(f)$ the collection of all small functions of $f$.

For any two non-constant meromorphic functions $f$ and $g$, and $a \in S(f) \cap S(g)$, we say that $f$ and $g$ share $a$ CM(IM) provided that $f - a$ and $g - a$ have the same zeros counting (ignoring) multiplicities.

For a meromorphic function $f$ and a set $S \subseteq \mathbb{C}$, we define $E_f(S) = \bigcup_{a\in S}\{z|f(z)-a = 0, \text{counting multiplicities}\}$, $\overline{E}_f(S) = \bigcup_{a\in S}\{z|f(z)-a = 0, \text{ignoring multiplicities}\}$. If $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$, then we say that $f$ and $g$ share S CM (IM). Evidently, if $S$ contains only one element, then it coincides with the usual definition of CM (respectively IM) shared values.

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Considering set sharing instead of value sharing some authors proved uniqueness of meromorphic functions when $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a set of roots of unity, see [3, 6, 9, 10].

Now, we recall the idea of weighted sharing, which was previously explored in the literature (see [4, 5]). This concept forms the basis for discussing the relaxation of sharing. Below, we will define this concept in more detail.

**Definition 1.1.** [4, 5] *Let $m$ be a non-negative integer or infinity and $a \in S(f)$. We denote by $E_m(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity $k$ is counted $k$ times if $k \leq m$ and $m+1$ times if $k > m$. If $E_m(a,f) = E_m(a,g)$, we say that $f$, $g$ share the function $a$ with weight $m$ and we write $f$ and $g$ share $(a,m)$. Since $E_m(a,f) = E_m(a,g)$ implies that $E_s(a,f) = E_s(a,g)$ for any integer $s\,(0 \leq s < m)$, if $f$, $g$ share $(a,m)$, then $f$, $g$ share $(a,s)$, $(0 \leq s < m)$. Moreover, we note that $f$ and $g$ share the function $a$ IM or CM if and only if $f$ and $g$ share $(a,0)$ or $(a,\infty)$ respectively.*

**Definition 1.2.** *Let $S$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $m$ be a non-negative integer. We denote by $E_f(S,m)$ the set $E_f(S,m) = \bigcup_{a \in S} E_m(a,f)$. We say that $f$ and $g$ share the set $S$ with weight $m$ if $E_f(S,m) = E_g(S,m)$ and we write $f$ and $g$ share $(S,m)$.*

Let $Q(z)$ be a polynomial of degree $q \in \mathbb{C}$ and $k$ be a positive integer. Denote the derivatives of $Q(z)$ by

$$Q'(z) = b\prod_{i=1}^{l}(z-z_i)^{m_i}, \tag{1}$$

where $b \in \mathbb{C} - \{0\}$, and denote by $\nu$ and $h$ the indexes such that $1 \leq \nu \leq h \leq l$, and

$$m_1 \geq m_2 \geq .... \geq m_\nu > k \geq m_{\nu+1} \geq ..... \geq m_l,$$
$$m_1 \geq m_2 \geq .... \geq m_h \geq k > m_{h+1} \geq ..... \geq m_l.$$

In 2017, An and Phuong [1] proved a uniqueness result on meromorphic functions when $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share a small function $\alpha$ CM. They demonstrated the following outcomes.

**Theorem 1.1.** [1] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ $(\not\equiv 0,\infty)$ be a small function with respect to $f$. Suppose that $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $\alpha$ CM. If $q > k+6+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$, then one of the following conclusions holds:*
*(1) $Q(f) = tQ(g)+c$ for a constant $c$.*
*(2) $(Q(f))^{(k)}.(Q(g))^{(k)} = \alpha^2$.*

In the same paper, An and Phuong [1], demonstrated that the conclusion (2) of Theorem 1.1 can be ruled out by imposing extra constraints on the multiple zeros of $Q'(z)$ or if $f$ and $g$ share $\infty$ IM.

**Theorem 1.2.** [1] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ $(\not\equiv 0,\infty)$ be a small function with respect to $f$. Suppose that $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $\alpha$ CM. If $q > k+6+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$ and if one of*
*(1) $h \geq 4$;*
*(2) $h = 3$ and $q \neq 2m_1 - 2k + 2$, $q \neq \frac{3m_1-2k+3}{2}$, and $q \neq 3m_i - 2k + 3$, for all $i = 1, 2, 3$; or*

*(3) h=2,*
*and $f$ and $g$ share $\infty$ IM holds, then*

$$Q(f) = tQ(g) + c \ \text{ for a constant } c.$$

The following findings were demonstrated for sharing the small function $\alpha$ IM in 2021 by Phuong [8].

**Theorem 1.3.** [8] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0, \infty$) be a small function with respect to $f$. Suppose that $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $\alpha$ IM. If $q > 4k + 12 + \nu(5k + 2) + 5\sum_{i=\nu+1}^{l} m_i$, then one of the following conclusions holds:*
(1) $Q(f) = tQ(g) + c$ *for a constant $c$.*
(2) $(Q(f))^{(k)}.(Q(g))^{(k)} = \alpha^2$.

**Theorem 1.4.** [8] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0, \infty$) be a small function with respect to $f$. Suppose that $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $\alpha$ CM. If $q > 4k + 12 + \nu(5k + 2) + 5\sum_{i=\nu+1}^{l} m_i$ and if one of*
*(1) $h \geq 4$;*
*(2) $h = 3$ and $q \neq 2m_1 - 2k + 2$, $q \neq \frac{3m_1-2k+3}{2}$, and $q \neq 3m_i - 2k + 3$, for all $i = 1, 2, 3$; or*
*(3) h=2,*
*and $f$ and $g$ share $\infty$ IM holds, then*

$$Q(f) = tQ(g) + c \ \text{ for a constant } c.$$

Recently, Sultana et al.[12] replaced a small function sharing by a set sharing and proved the following theorems.

**Theorem 1.5.** [12] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0, \infty$) be a small function of $f$. Let $d$ be a positive integer and $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), ......, \omega^{d-1}\alpha(z)\}$ where $\omega^d = 1$. Let $Q$ be a polynomial of degree $q$ ($> 0$) of the form (1) with $q > k + 2 + \frac{4}{d} + 2\nu(k + 1) + 2\sum_{i=\nu+1}^{l} m_i$. If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share the set $S$ CM, then one of the following holds:*
(1) $Q(f) = tQ(g) + c$ *for a constant $c$ and $t^d = 1$.*
(2) $(Q(f))^{(k)}.(Q(g))^{(k)} = t\alpha^{\frac{2}{d}}$ *with $t^d = 1$.*

**Theorem 1.6.** [12] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0, \infty$) be a small function with respect to $f$. Let $d, S$ be defined as in Theorem 1.5 and $q > k + 2 + \frac{4}{d} + 2\nu(k + 1) + 2\sum_{i=\nu+1}^{l} m_i$. If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share the set $S$ CM and if one of*
*(1) $h \geq 4$;*
*(2) $h = 3$ and $q \neq 2m_1 - 2k + 2$, $q \neq \frac{3m_1-2k+3}{2}$, and $q \neq 3m_i - 2k + 3$, for all $i = 1, 2, 3$; or*
*(3) h=2,*
*and $f$ and $g$ share $\infty$ IM holds, then*

$$Q(f) = tQ(g) + c \ \text{ for a constant } c \text{ and } t^d = 1.$$

**Theorem 1.7.** [12] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0, \infty$) be a small function of $f$. Let $d$ be a positive integer and $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), ......, \omega^{d-1}\alpha(z)\}$ where $\omega^d = 1$. Let $Q$ be a polynomial of degree $q$ ($> 0$) of the form (1) with $q > k+2+\frac{3k+10}{d}+\nu(2k+2+\frac{3k}{d})+(2+\frac{3}{d})\sum_{i=\nu+1}^{l} m_i$.*

*If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $S$ IM, then one of the conclusions of Theorem 1.5 holds.*

**Theorem 1.8.** [12] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ $(\not\equiv 0, \infty)$ be a small function with respect to $f$. Let $d, S$ be defined as in Theorem 1.5 and $q > k + 2 + \frac{3k+10}{d} + \nu(2k + 2 + \frac{3k}{d}) + (2 + \frac{3}{d})\sum_{i=\nu+1}^{l} m_i$. If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share the set $S$ IM and if one of (1), (2) and (3) of Theorem 1.6 holds, then*

$$Q(f) = tQ(g) + c \ \text{ for a constant } c \text{ and } t^d = 1.$$

In this paper, we relax the nature of sharing and prove the followings theorems.

**Theorem 1.9.** *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ $(\not\equiv 0, \infty)$ be a small function with respect to $f$. Let $d$ be a positive integer, $m$ be integer or $\infty$ and $S$ be defined as in Theorem 1.5. Let $Q$ be a polynomial of degree $q$ $(> 0)$ of the form (1). If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $(S, m)$ with one of the following conditions:*

*(i) $m \geq 2$ and*

$$q > k + 2 + \frac{4}{d} + 2\nu(k+1) + 2\sum_{i=\nu+1}^{l} m_i, \tag{2}$$

*(ii) $m = 1$ and*

$$q > k + 2 + \frac{k+10}{2d} + \nu\left(2(k+1) + \frac{k}{2d}\right) - (2 + \frac{1}{2d})\sum_{i=\nu+1}^{l} m_i, \tag{3}$$

*(ii) $m = 0$ and*

$$q > k + 2 + \frac{3k+10}{d} + \nu(2k + 2 + \frac{3k}{d}) + (2 + \frac{3}{d})\sum_{i=\nu+1}^{l} m_i, \tag{4}$$

*then one of the following conclusion holds:*
(1) *$Q(f) = tQ(g) + c$ for a constant $c$ and $t^d = 1$.*
(2) *$(Q(f))^{(k)}.(Q(g))^{(k)} = t\alpha^{\frac{2}{d}}$ with $t^d = 1$.*

**Theorem 1.10.** *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ $(\not\equiv 0, \infty)$ be a small function with respect to $f$. Let $d$ be a positive integer, $m$ be integer or $\infty$ and $S$ be defined as in Theorem 1.5. Let $Q$ be a polynomial of degree $q$ $(> 0)$ of the form (1). If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $(S, m)$ with one of the following conditions $(i)$, $(ii)$ and $(iii)$ of Theorem 1.9 and if one of*
*(1) $h \geq 4$;*
*(2) $h = 3$ and $q \neq 2m_1 - 2k + 2$, $q \neq \frac{3m_1-2k+3}{2}$, and $q \neq 3m_i - 2k + 3$, for all $i = 1, 2, 3$; or*
*(3) h=2*
*and $f$ and $g$ share $\infty$ IM holds, then*

$$Q(f) = tQ(g) + c \ \text{ for a constant } c \text{ and } t^d = 1.$$

**Remark 1.** *We can easily see that Theorem 1.9 and Theorem 1.10 generalizes Theorem 1.5, Theorem 1.7 and Theorem 1.6, Theorem 1.8 respectively.*

## 2. Lemmas

In this section we present some lemmas which will needed in the sequel. Let $\tilde{F}$ and $\tilde{G}$ be non-constant meromorphic functions and $\tilde{H}$ be a defined as follows:

$$\tilde{H} := \left(\frac{\tilde{F}^{(2)}}{\tilde{F}^{(1)}} - 2\frac{\tilde{F}^{(1)}}{\tilde{F}-1}\right) - \left(\frac{\tilde{G}^{(2)}}{\tilde{G}^{(1)}} - 2\frac{\tilde{G}^{(1)}}{\tilde{G}-1}\right). \tag{5}$$

**Lemma 2.1.** [11] *Let $f$ be a non-constant meromorphic function on $\mathbb{C}$. Then*

$$m(r, \frac{f'}{f}) = S(r,f)$$

*as $r \to \infty$ outside a subset of finite measure.*

**Lemma 2.2.** [13, 15] *Let $f$ be a meromorphic function and let $c$ be a complex number. Then*

$$T(r, \frac{1}{f-c}) = T(r,f) + O(1).$$

**Lemma 2.3.** [13, 15] *(Second fundamental theorem) Let $f$ be a non-constant meromorphic function on $\mathbb{C}$. Let $a_i \in S(f)$ where $i = 1, 2, ..., q$ be distinct meromorphic functions on $\mathbb{C}$. Then*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

*holds for all $r$ outside a set $E \subset (0, \infty)$ with finite Lebesgue measure.*

**Lemma 2.4.** [7] *Let $f$ be a non-constant meromorphic function and $s$, $k$ be two positive integers. If $f^{(k)} \not\equiv 0$, then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N_{s+k}(r, \frac{1}{f}) + S(r,f).$$

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{s+k}(r, \frac{1}{f}) + S(r,f),$$

*and*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N(r, \frac{1}{f}) + S(r,f).$$

**Lemma 2.5.** [13, 15] *Let $f$ be a non-constant meromorphic function and let $a_0, a_1, ..., a_n (\not\equiv 0)$ be small functions with respect to $f$. Then*

$$T\big(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0\big) = nT(r,f) + S(r,f).$$

**Lemma 2.6.** [10] *Let $m$ be a non-negative integer or infinity. $\tilde{F}$ and $\tilde{G}$ be non-constant meromorphic functions sharing $(1, m)$ and $\tilde{H}$ as defined in (5). If $\tilde{H} \not\equiv 0$ then*

*(i) If $m \geq 2$, then*

$$T(r, \tilde{F}) \leq 2\overline{N}(r, \tilde{F}) + 2\overline{N}(r, \tilde{G}) + N_2(r, \frac{1}{\tilde{F}}) + N_2(r, \frac{1}{\tilde{G}}) + S(r, \tilde{F}) + S(r, \tilde{G}).$$

*(ii) If $m = 1$, then*

$$T(r, \tilde{F}) \leq \tfrac{5}{2}\overline{N}(r, \tilde{F}) + 2\overline{N}(r, \tilde{G}) + N_2(r, \tfrac{1}{\tilde{F}}) + N_2(r, \tfrac{1}{\tilde{G}})$$
$$+ \tfrac{1}{2}\overline{N}(r, \tfrac{1}{\tilde{F}}) + S(r, \tilde{F}) + S(r, \tilde{G}).$$

*(iii) If $m = 0$, then*

$$T(r,\tilde{F}) \le 4\overline{N}(r,\tilde{F}) + 3\overline{N}(r,\tilde{G}) + N_2(r,\tfrac{1}{\tilde{F}}) + N_2(r,\tfrac{1}{\tilde{G}})$$
$$+2\overline{N}(r,\tfrac{1}{\tilde{F}}) + \overline{N}(r,\tfrac{1}{\tilde{G}}) + S(r,\tilde{F}) + S(r,\tilde{G}).$$

*The same inequality holds for $T(r,\tilde{G})$.*

**Lemma 2.7.** [12] *Let $Q$ be a polynomial of degree $q$ in $\mathbb{C}$, and let $k$ be a positive integer. Let*

$$Q'(z) = b\prod_{i=1}^{l}(z-z_i)^{m_i}$$

*where $b \in \mathbb{C}^\star$. Let $f$ and $g$ be two non-constant meromorphic functions. Assume that $((Q(f)^{(k)})^d = (Q(g)^{(k)})^d$. If $q - 2l - 2k - 4 > 0$, then $Q(f) = tQ(g) + c$, for a constant $c$ and $t^d = 1$.*

**Lemma 2.8.** [12] *Let $f$ and $g$ be two non-constant meromorphic functions, and $\alpha$ ($\not\equiv 0,\infty$) be a small function with respect to $f$ and $g$. If*

$$((Q(f)^{(k)})^d.(Q(g)^{(k)})^d = \alpha^2,$$

*then $h \le 2$ or $h = 3$ and $q = 2m_1 - 2k + 2$, $q = \frac{3m_1-2k+3}{2}$, or $q = 3m_i - 2k + 3$, for $i = 1,2,3$. If further assume that $f$ and $g$ share $\infty$ IM, then also $h = 1$.*

**Lemma 2.9.** *Let $f$ and $g$ be two non-constant meromorphic functions, and let $\alpha$ be a small function with respect to $f$. Let $d, S$ be defined as in Theorem 1.5 and $q > 5 + \frac{1}{d} + \nu(k+1) + \sum_{i=\nu+1}^{l} m_i$. If $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share $(S,m)$, then $T(r,f) = O(T(r,g))$, $T(r,g) = O(T(r,f))$ and $\alpha$ is a small function with respect to $g$.*

*Proof.* Proof follows from the proof of Lemma 2.6 of [12]. □

## 3. Proof of the Main Theorems

Proof of **Theorem 1.9 :**

*Proof.* Let

$$F := (Q(f)^{(k)},\ F_1 := Q(f),\ \tilde{F} := \frac{F^d}{\alpha} \text{ and } G := (Q(g))^{(k)},\ G_1 := Q(g).\ \tilde{G} := \frac{G^d}{\alpha}.$$

It is easy to prove that $S(r,\tilde{F}) = S(r,F) = S(r,f)$ and $S(r,\tilde{G}) = S(r,G) = S(r,g)$. By Lemma 2.9, $\alpha$ is a small function with respect to $g$ also. Since $F$ and $G$ share $(S,m)$, it follows that $\tilde{F}$ and $\tilde{G}$ share $(1,m)$. We note that

$$N_2(r,\tilde{F}) = 2\overline{N}(r,f) \le 2T(r,f) + S(r,f)$$

and

$$N_2(r,\tfrac{1}{F}) = N_2\left(r,\tfrac{1}{(F_1')^{(k-1)}}\right) \le (k-1)\overline{N}(r,F_1') + N_{k+1}\left(r,\tfrac{1}{F_1'}\right) + S(r,f), \quad (6)$$

and

$$\overline{N}(r,\frac{1}{F}) \le (k-1)\overline{N}(r,F_1') + N_k\left(r,\frac{1}{F_1'}\right) + S(r,f). \quad (7)$$

Similarly,

$$N_2(r,\tfrac{1}{G}) = N_2\left(r,\tfrac{1}{(G_1')^{(k-1)}}\right) \le (k-1)\overline{N}(r,G_1') + N_{k+1}\left(r,\tfrac{1}{G_1'}\right) + S(r,g), \quad (8)$$

and

$$\overline{N}(r, \frac{1}{G}) \leq (k-1)\overline{N}(r, G_1') + N_k\left(r, \frac{1}{G_1'}\right) + S(r, g). \tag{9}$$

Again we write

$$Q(z) - R(z) = a(z-\beta)Q'(z),$$

where $a \neq 0$ and $\beta$ are constants and $R(z)$ is a polynomial of degree atmost $q-2$. Applying Lemmas 2.1, 2.2, we have

$$\begin{aligned} m\left(r, \tfrac{1}{Q(f)-R(f)}\right) &= m\left(r, \tfrac{(Q(f))'}{Q(f)-R(f)}. \tfrac{1}{(Q(f))'}\right) \\ \leq m\left(r, \tfrac{f'}{a(f-\beta)}\right) &+ m\left(r, \tfrac{1}{F_1'}\right) + O(1) \leq m\left(r, \tfrac{1}{F_1'}\right) + S(r, f). \end{aligned}$$

From this we get

$$\begin{aligned} T(r, F_1') &= m(r, \tfrac{1}{F_1'}) + N(r, \tfrac{1}{F_1'}) + O(1) \\ &\geq T\left(r, \tfrac{1}{Q(f)-R(f)}\right) - N(r, \tfrac{1}{Q(f)-R(f)}) + N(r, \tfrac{1}{F_1'}) + O(1) \\ &\geq qT(r, f) - N(r, \tfrac{1}{(Q(f))'}) - N(r, \tfrac{1}{f-\beta}) + N(r, \tfrac{1}{F_1'}) + O(1). \end{aligned}$$

Thus using the Lemma 2.4 to the function $F_1'$ we get

$$\begin{aligned} T(r, F) &\geq T(r, F_1') + N_2(r, \tfrac{1}{F}) - N_{k+1}(r, \tfrac{1}{F_1'}) + S(r, f) \\ &\geq qT(r, f) - N(r, \tfrac{1}{(Q(f))'}) - N(r, \tfrac{1}{f-\beta}) + N(r, \tfrac{1}{F_1'}) \\ &\quad + N_2(r, \tfrac{1}{F}) - N_{k+1}(r, \tfrac{1}{F_1'}) + S(r, f). \end{aligned} \tag{10}$$

Case 1: $\tilde{H} \not\equiv 0$. Then by Lemma 2.6, we get following subcases:
Subcase 1.1: If $m \geq 2$, then using Lemma 2.5 we have

$$\begin{aligned} dT(r, F) = T(r, \tilde{F}) &\leq dN_2\left(r, \tfrac{1}{F}\right) + dN_2\left(r, \tfrac{1}{G}\right) + 2\overline{N}\left(r, F\right) \\ &\quad + 2\overline{N}\left(r, G\right) + S(r, f) + S(r, g). \end{aligned} \tag{11}$$

Using (6)–(10) and (11) we get

$$\begin{aligned} dqT(r, f) &\leq d(k-1)\overline{N}(r, G_1') + dN_{k+1}\left(r, \tfrac{1}{G_1'}\right) + 2\overline{N}(r, G) + 2\overline{N}(r, F) \\ &+ dN(r, \tfrac{1}{(Q(f))'}) + dN(r, \tfrac{1}{f-\beta}) - dN(r, \tfrac{1}{F_1'}) + dN_{k+1}(r, \tfrac{1}{F_1'}) + S(r, f) + S(r, g) \\ &\leq (d(k-1)+2)\overline{N}(r, g) + d(k+1)\textstyle\sum_{i=1}^{\nu} N(r, \tfrac{1}{g-z_i}) + dN(r, \tfrac{1}{g'}) \\ &+ d\textstyle\sum_{i=\nu+1}^{l} m_i N(r, \tfrac{1}{g-z_i}) + 2\overline{N}(r, f) + d(k+1)\textstyle\sum_{i=1}^{\nu} N(r, \tfrac{1}{f-z_i}) \\ &+ d\textstyle\sum_{i=\nu+1}^{l} m_i N(r, \tfrac{1}{g-z_i}) + dN(r, \tfrac{1}{f-\beta}) + +S(r, f) + S(r, g) \\ &\leq \left(d(k+1) + 2 + d\nu(k+1) + d\textstyle\sum_{i=\nu+1}^{l} m_i\right) T(r, g) \\ &+ \left(2 + d + d\nu(k+1) + d\textstyle\sum_{i=\nu+1}^{l} m_i\right) T(r, f) + +S(r, f) + S(r, g). \end{aligned}$$

This implies

$$\begin{aligned} &\left(dq - 2 - d - d\nu(k+1) - d\textstyle\sum_{i=\nu+1}^{l} m_i\right) T(r, f) \\ &\leq \left(d(k+1) + 2 + d\nu(k+1) + d\textstyle\sum_{i=\nu+1}^{l} m_i\right) T(r, g) + S(r, f) + S(r, g). \end{aligned} \tag{12}$$

Similarly, we get

$$\begin{aligned}&\left(dq-2-d-d\nu(k+1)-d\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,g)\\ &\leq\left(d(k+1)+2+d\nu(k+1)+d\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,f)+S(r,f)+S(r,g).\quad(13)\end{aligned}$$

Combining (12) and (13), we get

$$\left(dq-4-d(k+2)-2d\nu(k+1)-2d\sum_{i=\nu+1}^{l}m_i\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g).$$

Therefore, we have $q\leq k+2+\frac{4}{d}+2\nu(k+1)+2\sum_{i=\nu+1}^{l}m_i$, which contradicts (2).

Subcase 1.2: $m=1$, then

$$\begin{aligned}dT(r,f)=T(r,\tilde{F})&\leq dN_2\left(r,\tfrac{1}{F}\right)+dN_2\left(r,\tfrac{1}{G}\right)+\tfrac{5}{2}\overline{N}\left(r,F\right)\\ &+2\overline{N}\left(r,G\right)+\tfrac{1}{2}\overline{N}\left(r,\tfrac{1}{F}\right)+S\left(r,f\right)+S\left(r,g\right).\quad(14)\end{aligned}$$

Using (6)–(10) and (14) we get

$$\begin{aligned}dqT(r,f)&\leq d(k-1)\overline{N}(r,G_1')+dN_{k+1}\left(r,\tfrac{1}{G_1'}\right)+2\overline{N}(r,G)+\tfrac{5}{2}\overline{N}(r,F)\\ &\tfrac{(k-1)}{2}\overline{N}(r,F_1')+\tfrac{1}{2}N_k\left(r,\tfrac{1}{F_1'}\right)+dN(r,\tfrac{1}{(Q(f))'})+dN(r,\tfrac{1}{f-\beta})-dN(r,\tfrac{1}{F_1'})\\ &+dN_{k+1}(r,\tfrac{1}{F_1'})+S(r,f)+S(r,g)\leq(d(k-1)+2)\overline{N}(r,g)\\ &+d(k+1)\textstyle\sum_{i=1}^{\nu}N(r,\tfrac{1}{g-z_i})+dN(r,\tfrac{1}{g'})+d\textstyle\sum_{i=\nu+1}^{l}m_iN(r,\tfrac{1}{g-z_i})\\ &+\tfrac{k+4}{2}\overline{N}(r,f)+(d(k+1)+\tfrac{k}{2})\textstyle\sum_{i=1}^{\nu}N(r,\tfrac{1}{f-z_i})+(d+\tfrac{1}{2})\textstyle\sum_{i=\nu+1}^{l}m_iN(r,\tfrac{1}{f-z_i})\\ &+\tfrac{1}{2}N(r,\tfrac{1}{f'})+dN(r,\tfrac{1}{f-\beta})+S(r,f)+S(r,g)\\ &\leq\left(d(k+1)+2+d\nu(k+1)+d\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,g)\\ &+\left(3+\tfrac{k}{2}+d+\nu(d(k+1)+\tfrac{k}{2})+(d+\tfrac{1}{2})\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,f)++S(r,f)+S(r,g).\end{aligned}$$

This implies

$$\begin{aligned}&\left(dq-3-\tfrac{k}{2}-d-\nu(d(k+1)+\tfrac{k}{2})-(d+\tfrac{1}{2})\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,f)\\ &\leq\left(d(k+1)+2+d\nu(k+1)+d\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,g)+S(r,f)+S(r,g).\quad(15)\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\left(dq-3-\tfrac{k}{2}-d-\nu(d(k+1)+\tfrac{k}{2})-(d+\tfrac{1}{2})\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,g)\\ &\leq\left(d(k+1)+2+d\nu(k+1)+d\textstyle\sum_{i=\nu+1}^{l}m_i\right)T(r,f)+S(r,f)+S(r,g).\quad(16)\end{aligned}$$

Combining (15) and (16), we get

$$\begin{aligned}&\left(dq-5-d(k+2)-\tfrac{k}{2}-\nu\left((2d(k+1)+\tfrac{k}{2}\right)-(2d+\tfrac{1}{2})\textstyle\sum_{i=\nu+1}^{l}m_i\right)(T(r,f)+T(r,g))\\ &\leq S(r,f)+S(r,g).\end{aligned}$$

Therefore, we have $q\leq k+2+\frac{k+10}{2d}+\nu\left(2(k+1)+\frac{k}{2d}\right)-(2+\frac{1}{2d})\sum_{i=\nu+1}^{l}m_i$, which contradicts (3).

Subcase 1.3: $m = 0$, then

$$dT(r,F) = T(r,\tilde{F}) \leq 4\overline{N}(r,F) + 3\overline{N}(r,G) + dN_2(r,\tfrac{1}{F}) + dN_2(r,\tfrac{1}{G}) + 2\overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + S(r,f) + S(r,g). \quad (17)$$

Using (6)–(10) and (17) we get

$$\begin{aligned} dqT(r,f) &\leq d(k-1)\overline{N}(r,G_1') + dN_{k+1}\left(r,\tfrac{1}{G_1'}\right) + 3\overline{N}(r,G) + 4\overline{N}(r,F) \\ &+2(k-1)\overline{N}(r,F_1') + 2N_k\left(r,\tfrac{1}{F_1'}\right) + (k-1)\overline{N}(r,G_1') + N_k\left(r,\tfrac{1}{G_1'}\right) \\ &+dN(r,\tfrac{1}{(Q(f))'}) + dN(r,\tfrac{1}{f-\beta}) - dN(r,\tfrac{1}{F_1'}) + dN_{k+1}(r,\tfrac{1}{F_1'}) + S(r,f) + S(r,g) \\ &\leq (d(k-1)+k+2)\overline{N}(r,g) + (d(k+1)+k)\textstyle\sum_{i=1}^{\nu} N(r,\tfrac{1}{g-z_i}) + (d+1)N(r,\tfrac{1}{g'}) \\ &+(d+1)\textstyle\sum_{i=\nu+1}^{l} m_i N(r,\tfrac{1}{g-z_i}) + (2k+2)\overline{N}(r,f) + (d(k+1)+2k)\sum_{i=1}^{\nu} N(r,\tfrac{1}{f-z_i}) \\ &+(d+2)\textstyle\sum_{i=\nu+1}^{l} m_i N(r,\tfrac{1}{g-z_i}) + dN(r,\tfrac{1}{f-\beta}) + 2N(r,\tfrac{1}{f'}) + S(r,f) + S(r,g) \\ &\leq \left(d(k+1)+k+4+\nu(d(k+1)+k) + (d+1)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,g) \\ &+\left(2k+6+d+\nu(d(k+1)+2k) + (d+2)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,f) + +S(r,f) + S(r,g). \end{aligned}$$

This implies

$$\begin{aligned} &\left(dq - 2k - d - \nu((k+1)+2k) - (d+2)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,f) \\ &\leq \left(d(k+1)+k+4+\nu(d(k+1)+k) + (d+1)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,g) \\ &\quad +S(r,f) + S(r,g). \end{aligned} \quad (18)$$

Similarly, we get

$$\begin{aligned} &\left(dq - 2k - d - \nu((k+1)+2k) - (d+2)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,g) \\ &\leq \left(d(k+1)+k+4+\nu(d(k+1)+k) + (d+1)\textstyle\sum_{i=\nu+1}^{l} m_i\right)T(r,f) \\ &\quad +S(r,f) + S(r,g). \end{aligned} \quad (19)$$

Combining (18) and (19), we get

$$\left(dq - d(k+2) - 3k - 10 - \nu(2d(k+1)+3k) - (2d+3)\textstyle\sum_{i=\nu+1}^{l} m_i\right)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g).$$

Therefore, we have $q \leq k + 2 + \frac{3k+10}{d} + \nu(2k + 2 + \frac{3k}{d}) + (2 + \frac{3}{d})\sum_{i=\nu+1}^{l} m_i$, which contradicts (4).

Case 2: $\tilde{H} \equiv 0$. Thus, when we integrate twice, we get

$$\frac{1}{\tilde{G}-1} = \frac{A}{\tilde{F}-1} + B,$$

where $A\ (\neq 0)$ and $B$ are constants.
Thus

$$\tilde{G} = \frac{(B+1)\tilde{F} + (A-B-1)}{B\tilde{F} + (A-B)}, \quad (20)$$

and

$$\tilde{F}=\frac{(B-A)\tilde{G}+(A-B-1)}{B\tilde{G}-(B+1)}. \tag{21}$$

Now we consider the following three subcases:
Subcase 2.1: $B\neq 0,-1$. Then from (21) we have

$$\overline{N}\left(r,\frac{1}{\tilde{G}-\frac{B+1}{B}}\right)=\overline{N}(r,\tilde{F}).$$

Using Lemma 2.3 we get

$$\begin{aligned}dT(r,g)=T(r,\tilde{G})&\leq \overline{N}(r,\tilde{G})+\overline{N}(r,\tfrac{1}{\tilde{G}})+\overline{N}\left(r,\tfrac{1}{\tilde{G}-\frac{B+1}{B}}\right)+S(r,\tilde{G})\\ &\leq \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}(r,F)+S(r,g). \end{aligned}\tag{22}$$

Using (7), (9), (22) and (10) for $G$, we get

$$\begin{aligned}&dqT(r,g)\leq dT(r,G)+dN(r,\tfrac{1}{Q'(g)})+dN(r,\tfrac{1}{g-\beta})-dN(r,\tfrac{1}{G_1'})\\ &-dN_2(r,\tfrac{1}{G})+dN_{k+1}(r,\tfrac{1}{G_1'})+S(r,g)\leq \overline{N}(r,g)+\overline{N}(r,\tfrac{1}{G})\\ &+\overline{N}(r,f)+dN(r,\tfrac{1}{Q'(g)})+dN(r,\tfrac{1}{g-\beta})+S(r,g)\\ &\leq (k+2+d+(d+1)k\nu+(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,g)+T(r,f)\\ &+S(r,f)+S(r,g)\end{aligned}$$

Therefore we get

$$\begin{aligned}&(dq-d-k-2-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,g)\leq T(r,f)\\ &+S(r,f)+S(r,g).\end{aligned}\tag{23}$$

If $A-B-1\neq 0$, then it follows from (20) that

$$\overline{N}\left(r,\frac{1}{\tilde{F}-\frac{-A+B+1}{B+1}}\right)=\overline{N}(r,\frac{1}{\tilde{G}}).$$

Again by Nevanlinna second fundamental theorem we have

$$\begin{aligned}dT(r,f)=T(r,\tilde{F})&\leq \overline{N}(r,\tilde{F})+\overline{N}(r,\tfrac{1}{\tilde{F}})+\overline{N}\left(r,\tfrac{1}{\tilde{F}-\frac{-A+B+1}{B+1}}\right)+S(r,\tilde{F})\\ &\leq \overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,\tfrac{1}{G})+S(r,f)+S(r,g).\end{aligned}\tag{24}$$

Using (7), (9), (24) and (10), we get

$$\begin{aligned}&dqT(r,f)\leq dT(r,F)+dN(r,\tfrac{1}{Q'(f)})+dN(r,\tfrac{1}{f-\beta})-dN(r,\tfrac{1}{F_1'})\\ &-dN_2(r,\tfrac{1}{F})+dN_{k+1}(r,\tfrac{1}{F_1'})+S(r,f)\leq \overline{N}(r,f)+\overline{N}(r,\tfrac{1}{F})\\ &+\overline{N}(r,\tfrac{1}{G})+dN(r,\tfrac{1}{Q'(f)})+dN(r,\tfrac{1}{f-\beta})+S(r,f)+S(r,g)\\ &\leq (k+2+d+(d+1)k\nu+(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,f)\\ &+(k+1+k\nu+\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Therefore we get

$$\begin{aligned}&(dq-d-k-2-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,f)\\ &\leq (k+1+k\nu+\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,g)+S(r,f)+S(r,g).\end{aligned}\tag{25}$$

Combining (23) and (25), we get

$$\begin{aligned}&\Big(dq-d-k-2-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i\Big)\left(T(r,f)+T(r,g)\right)\\&\leq T(r,f)+(k+1+k\nu+\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,g)+S(r,f)+S(r,g),\\ \Rightarrow&\Big(dq-d-2k-3-(d+2)k\nu-(d+2)\textstyle\sum_{i=\nu+1}^{l}m_i\Big)\left(T(r,f)+T(r,g)\right)\\&\leq S(r,f)+S(r,g).\end{aligned}$$

Therefore, we have $q\leq 1+\frac{2k+3}{d}+\nu k(1+\frac{2}{d})+(1+\frac{2}{d})\sum_{i=\nu+1}^{l}m_i$, which contradicts assumptions (2)–(4).

Therefore $A-B-1=0$. Then by (20)

$$\overline{N}\left(r,\frac{1}{\tilde{F}+\frac{1}{B}}\right)=\overline{N}(r,\tilde{G}).$$

Again using Lemma 2.3 we get

$$\begin{aligned}dT(r,f)=T(r,\tilde{F})&\leq\overline{N}(r,\tilde{F})+\overline{N}(r,\tfrac{1}{\tilde{F}})+\overline{N}\left(r,\tfrac{1}{\tilde{F}+\frac{1}{B}}\right)+S(r,\tilde{F})\\&\leq\overline{N}(r,F)+\overline{N}(r,\tfrac{1}{F})+\overline{N}(r,G)+S(r,F).\end{aligned}\tag{26}$$

Using (7), (9), (26) and (10), we get

$$\begin{aligned}&dqT(r,f)\leq dT(r,F)+dN(r,\tfrac{1}{Q'(f)})+dN(r,\tfrac{1}{f-\beta})-dN(r,\tfrac{1}{F_1'})\\&-dN_2(r,\tfrac{1}{F})+dN_{k+1}(r,\tfrac{1}{F_1'})+S(r,f)\leq\overline{N}(r,g)+\overline{N}(r,\tfrac{1}{F})\\&+\overline{N}(r,f)+dN(r,\tfrac{1}{Q'(f)})+dN(r,\tfrac{1}{f-\beta})+S(r,f)+S(r,g)\\&\leq(k+2+d+(d+1)k\nu+(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i)T(r,f)+T(r,g)\\&+S(r,f)+S(r,g).\end{aligned}$$

Therefore we get

$$\begin{aligned}\Big(dq-d-k-2-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)&\leq T(r,g)\\&+S(r,f)+S(r,g).\end{aligned}\tag{27}$$

Combining (23) and (27), we get

$$\begin{aligned}&\Big(dq-d-k-2-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i\Big)\left(T(r,f)+T(r,g)\right)\\&\leq T(r,f)+T(r,g)+S(r,f)+S(r,g),\\ \Rightarrow&\Big(dq-d-k-3-(d+1)k\nu-(d+1)\textstyle\sum_{i=\nu+1}^{l}m_i\Big)\left(T(r,f)+T(r,g)\right)\\&\leq S(r,f)+S(r,g).\end{aligned}$$

Therefore, we have $q\leq 1+\frac{k+3}{d}+\nu k(1+\frac{1}{d})+(1+\frac{1}{d})\sum_{i=\nu+1}^{l}m_i$, which violates assumptions (2)–(4).

Subcase 2.2: $B=-1$. Then

$$\tilde{G}=\frac{A}{A+1-\tilde{F}},$$

and

$$\tilde{F} = \frac{(1+A)\tilde{G} - A}{\tilde{G}}.$$

If $A + 1 \neq 0$,

$$\overline{N}\left(r, \frac{1}{\tilde{F} - (A+1)}\right) = \overline{N}(r, \tilde{G}),$$

$$\overline{N}\left(r, \frac{1}{\tilde{G} - \frac{A}{A+1}}\right) = \overline{N}(r, \frac{1}{\tilde{F}}).$$

By similar argument as Subcase 2.1 we have a contradiction.
Therefore $A + 1 = 0$ then $\tilde{F}\tilde{G} = 1$. and so we get $(Q(f))^{(k)}.(Q(g))^{(k)} = t\alpha^{\frac{2}{d}}$, with $t^d = 1$.

Subcase 2.3: $B = 0$. Then (20) and (21) gives $\tilde{G} = \frac{\tilde{F}+A-1}{A}$ and $\tilde{F} = A\tilde{G} + 1 - A$. If $A - 1 \neq 0$, $\overline{N}\left(r, \frac{1}{A-1+\tilde{F}}\right) = \overline{N}(r, \frac{1}{\tilde{G}})$ and $\overline{N}\left(r, \frac{1}{\tilde{G} - \frac{A-1}{A}}\right) = \overline{N}(r, \frac{1}{\tilde{F}})$. Proceeding similarly as in Subcase 2.1 we get a contradiction.
Therefore, $A - 1 = 0$ then $\tilde{F} \equiv \tilde{G}$ i.e., $(Q(f))^{(k)} = t(Q(g))^{(k)}$, with $t^d = 1$. Then by Lemma 2.7, we get $Q(f) = tQ(g) + c$ for a constant $c$ and $t^d = 1$. □

Proof of **Theorem 1.10 :**

*Proof.* The proof of the theorem follows from Theorem 1.9 and Lemma 2.8. □

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