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NUMERICAL SOLUTION OF INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS BY SOME CONSTRUCTED ORTHOGONAL POLYNOMIALS AS THE BASES FUNCTIONS

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Abstract. In this paper, standard collocation approximation method is used as the numerical solution of integral and integro-differential equations. Two types of orthogonal polynomials are constructed and used as bases functions. The method assumed an approximate solution using the orthogonal polynomials constructed as basis function which are then substituted into the problem considered. Then, the like terms of the unknown coefficients are collected and simplified. The resulting equation is then collocated at equally spaced interior points hence leading to algebraic linear system of equations which are then solved by Guassian elimination method to obtain the unknown constants. These are substituted back into the assumed approximate solution to get the required approximate solution. Numerical solution are given to illustrate the accuracy of the method discussed in the work. It was observed that as the degree of approximant increases, approximate solution tends to the exact solution. These are evident in the graph presented.

1. Introduction

In recent years, the study of integral and integro-differential equations has sparked significant research interest. Various numerical methods, such as Variational Iteration, Homotopy Perturbation, and Adomain Decomposition, have been employed to tackle these equations. Integral and integro-differential equations find applications in diverse scientific fields. These equations, comprising both integral and differential operators, often involve unknown functions with derivatives of any order. Real-life problems are often modeled using functional equations, including partial differential equations, integral equations, integro-differential equations, and

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stochastic equations. Many physical phenomena are mathematically formulated with integro-differential equations, prevalent in fluid dynamics, biological models, and chemical kinetics.

Researchers have proposed numerous methods to solve these equations. For instance, Kythe and Puri (2002) introduced a combinatorial method for linear integral equations, while Hashim (2006) developed the Adomian decomposition method for solving boundary value problems related to fourth-order integro-differential equations. Laeli and Maalek Ghaini (2012) explored numerical solutions for Volterra-Fredholm integral equations using the moving least square method and Chebyshev polynomials. Similarly, Sweilam et al. (2013) conducted a numerical and analytical study on fourth-order integro-differential equations using a pseudospectral method.

Due to their complexity, integro-differential equations are often challenging to solve analytically, necessitating the development of efficient approximate solutions. All the above mentioned methods, existing bases functions like Power series, Chebyshev polynomials, Legendre polynomial were used.

In this present work, two new orthogonal polynomials were constructed and used as bases functions using standard collocation method for solving integral and integro-differential equations. Some numerical results were presented graphically with the exact solutions.

2. DEFINITIONS OF RELEVANT TERMS

INTEGRAL EQUATIONS:
An integral equation is the equation in which the unknown function say y(x) appears under an integral sign. Integral equation are classified into two forms which are:

(i) Volterra Integral Equation:

$$y(x) = f(x) + \lambda \int_{a}^{b(x)} K(x,t)y(t)dt$$

(ii) Fredholm Integral Equation:

$$y(x) = f(x) + \lambda \int_{a}^{b} K(x,t)y(t)dt$$

INTEGRO-DIFFERENTIAL EQUATIONS:
An integro-differential equation is a type of mathematical equation that combines both differential and integral operators. It involves functions that depends on both the values of it's function and it's derivative at a given point as well as the integral function over a specified interval. Integro-differential equation are classified into three forms:

(i) Volterra Integro-Differential equation:

$$\sum_{i=0}^{n} P_i(x)y^{(i)}(x) + \lambda \int_{a}^{b(x)} K(x,t)y(t)dt = g(x)$$

with the conditions $y^{(i)}(a_i) = \alpha_i; i = 0, 1, 2, \cdots, (n-1)$
(ii) Fredholm Integro-Differential equation:

$$\sum_{i=0}^{n} P_i(x)y^{(i)}(x) + \lambda \int_a^b K(x,t)y(t)dt = g(x)$$

with the conditions $y^{(i)}(a_i) = \alpha_i; i = 0, 1, 2, \cdots, (n-1)$
(iii) Volterra - Fredholm Integro-Differential equation:

$$\sum_{i=0}^{n} P_i(x)y^{(i)}(x) + \lambda_1 \int_a^{b(x)} K(x,t)y(t)dt + \lambda_2 \int_a^b K(x,t)y(t)dt = g(x)$$

with the conditions $y^{(i)}(a_i) = \alpha_i; i = 0, 1, 2, \cdots, (n-1)$

COLLOCATION METHOD:
A numerical method for solving differential equations that involves approximating the solution at a set of discrete points, known as collocation points, and using a set of basis functions to approximate the solution between these points.

EXACT SOLUTION:
The exact or true solution of a differential equation that satisfies the given initial and boundary conditions.

APPROXIMATE SOLUTION:
This is the expression obtained after the unknown constants have been found and substituted back into the assumed solution. It is referred to as an approximate solution or inexact solution since it is a reasonable approximation to the exact solution. it is denoted as say $y_N(x)$ where N is the degree of the approximant used in the calculation. Methods of approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be given. In this work, approximate solution used are given as

$$y_N(x) = \sum_{i=0}^{N} c_i Q_i(x)$$

where x represents the independent variables in the problem, c_i and $Q_i(x); i \geq 0$ are the unknown constants to be determined and the basis functions used respectively

ORTHOGONALITY:
Two functions, say $T_n(x)$ and $T_m(x)$ defined on the interval $a \leq x \leq b$ are said to be orthogonal if, the inner product of the functions is zero. i.e.

$$<T_n(x), T_m(x)> = \int_a^b T_n(x)T_m(x)dx = 0$$

where m, n∈N.
If, on the other hand, a third function w(x) > 0 exist ; then

$$<Tn(x), Tm(x)> = \int_a^b w(x)Tn(x)Tm(x)dx = 0$$

where, w(x) is the the weight function.

ORTHOGONAL POLYNOMIAL BASES FUNCTIONS:
A set of polynomial functions that are orthogonal to each other over a certain interval. They are commonly used as basis functions for approximating the solution of differential equation.

3. DERIVATION OF ORTHOGONAL POLYNOMIALS USED IN THIS WORK

CONSTRUCTION OF TYPE I ORTHOGONAL POLYNOMIAL BASES:
In this section , instead of normalization, the orthogonal polynomial are subjected to standardization for Legendre polynomial valid in [-1,1] is:

$$P_n(1) = 1$$

In particular $P_0(x) = 1$
Hence for n = N

$$P_n(x) = \sum_{i=1}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

The procedure is as follows:

$$P_n(1) = 1 \Rightarrow a_0 + a_1 + a_2 + a_3 + \cdots + a_N = 0$$
$$< P_n, 1 > \Rightarrow \int_{-1}^{1} w(x) \cdot P_0(x) \cdot P_N(x) dx = 0$$
$$< P_n, x > \Rightarrow \int_{-1}^{1} w(x) \cdot P_1(x) \cdot P_N(x) dx = 0$$
$$< P_n, x^2 > \Rightarrow \int_{-1}^{1} w(x) \cdot P_2(x) \cdot P_N(x) dx = 0$$
$$< P_n, x^3 > \Rightarrow \int_{-1}^{1} w(x) \cdot P_3(x) \cdot P_N(x) dx = 0$$
$$\vdots$$
$$< P_n, x^{n-1} > \Rightarrow \int_{-1}^{1} w(x) \cdot P_{N-1}(x) \cdot P_N(x) dx = 0$$

Altogether, these resulted into (n+1) algebraic linear system of equations in (n+1) unknown constants.Thus the (n+1) algebraic linear system of equations are then solved to give the following polynomials:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$
$$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$
$$P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$$
$$P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$$

CONSTRUCTION OF TYPE II ORTHOGONAL POLYNOMIAL BASES:

Let

$$q_0(x) = a_{00}$$
$$q_1(x) = a_{11}x + a_{10} \cdot q_0(x)$$
$$q_2(x) = a_{22}x^2 + a_{21} \cdot q_1(x) + a_{20} \cdot q_0(x)$$
$$\vdots$$

be orthogonal polynomials.
Hence, $q_0(x)$ is computed as follows:

$$< q_0, q_0 >= 1 \Rightarrow \int_{-1}^{1} q_0(x) \cdot q_0(x) dx = 1$$
$$\Rightarrow a_{00}^2 \int_{-1}^{1} dx = 1 \Rightarrow a_{00}^2 [x]_{-1}^{1} = 1$$
$$\Rightarrow 2a_{00}^2 = 1 \Rightarrow a_{00}^2 = \frac{1}{2} \Rightarrow a_{00} = \frac{1}{\sqrt{2}}$$
$$q_0(x) = \frac{1}{\sqrt{2}}$$

Thus, $q_1(x)$ is computed as follows:
Here, we used the properties of orthogonality

$$<q_0, q_1> = 0 \ \cdots \ (i)$$
$$<q_1, q_1 >= 1 \ (Orthonormal) \ \cdots \ (ii)$$

From (i), we have

$$<q_0, q_1> = \int_{-1}^{1} q_0(x) \cdot q_1(x) dx = \int_{-1}^{1} a_{00}(a_{11}x + a_{10}q_0(x)) dx = 0$$
$$\Rightarrow \int_{-1}^{1} \frac{1}{\sqrt{2}} (a_{11}x + \frac{1}{\sqrt{2}} a_{10}) dx = 0$$
$$\Rightarrow \int_{-1}^{1} a_{11} \frac{x}{\sqrt{2}} dx + \int_{-1}^{1} \frac{a_{10}}{2} dx = 0$$
$$\Rightarrow \frac{a_{11}}{\sqrt{2}} \int_{-1}^{1} x dx + \frac{a_{10}}{2} \int_{-1}^{1} dx = 0$$
$$\Rightarrow a_{10} = 0$$

From (ii), we have

$$<q_1, q_1 >= \int_{-1}^{1} q_1(x) \cdot q_1(x) dx = 1$$
$$\Rightarrow \int_{-1}^{1} [a_{11}x + a_{10}q_0(x)]^2 dx = 1$$
$$\Rightarrow \int_{-1}^{1} [a_{11}x]^2 dx = 1$$

$$\Rightarrow a_{11} = \sqrt{\frac{3}{2}}$$

$$\Rightarrow q_1(x) = \sqrt{\frac{3}{2}}x$$

Thus, in general $q_n(x) = \sqrt{\frac{2n+1}{2}}P_n(x)$, where $P_n(x)$, a polynomial of degree n and is called Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This Process is continued to give

$$q_2(x) = \frac{\sqrt{10}}{4}(3x^2-1)$$

$$q_3(x) = \sqrt{\frac{7}{4}}(\frac{5x^3-3x}{2})$$

$$q_4(x) = \frac{3}{\sqrt{2}}(\frac{35x^4-30x^2+3}{8})$$

$$q_5(x) = \frac{(63x^5-70x^3+15x)\sqrt{22}}{8}$$

$$q_6(x) = \frac{\sqrt{26}(231x^6-315x^4+105x^2-5)}{32}$$

4. THEOREMS THAT GUIDES THE DERIVATION OF THE ORTHOGONALS POLYNOMIAL USED IN THIS WORK

THEOREM 1:

Given a continuous weight function w(x) defined on a closed interval [a, b], there exists a sequence of orthogonal polynomials $P_n(x)$ such that,
1. Each polynomial $P_n(x)$ has a degree n.
2. $P_n(x)$ is orthogonal to all lower-degree polynomials $P_m(x)$ for m ¡n.
3. The polynomials $P_n(x)$ form a complete basis for the space of all polynomials defined over [a, b].

Proof:

We start by constructing a sequence of monic polynomials $\{Q_n(x)\}$ where each polynomial $Q_n(x)$ has a degree n. These monic polynomials are constructed using the Gram-Schmidt orthogonalization process.
Let $Q_0(x) = 1$ be the first polynomial in the sequence. Using the monic polynomials $\{Qn(x)\}$, we construct orthogonal polynomials $\{Pn(x)\}$ as follows:

$$P_0(x) = Q_0(x)$$

$$P_n(x) = Q_n(x) - \sum_{k=0}^{n-1}\frac{< Q_n, P_k >}{||P_k||^2}P_k(x)$$

Where ⟨f, g⟩ represents the inner product of two functions f(x) and g(x) over the interval [a, b] and $||f||^2$ is the norm of a function f(x) with respect to the inner product and weight function w(x).
We need to show that $P_n(x)$ is orthogonal to all lower-degree polynomials $P_m(x)$ for m ¡ n. This can be established using the properties of the inner product:

For $n > m, <P_n, P_m> = 0$
This orthogonality property is verified by expanding the inner product expression and applying the orthogonality of Q_n to lower-degree polynomials.
Assume that $P_n(x)$ is an orthogonal to all polynomials of degree less than n.
Now, consider $P_{n+1}(x)$:

$$[P_{n+1}(x), P_m(x)] = [Q_{n+1}(x) - \frac{\sum[Q_k(x), P_n]P_n(x)}{[P_n, P_n]} P_m(x)]$$

By the orthogonality of $Q_{n+1}(x)$ with respect to all lower-degree polynomials and the inductive assumption, all the cross terms in the summation vanish, leaving only:

$$[P_{n+1}(x), P_m(x)] = [Q_{n+1}(x), P_m(x)] - \sum[P_n(x), P_m(x)][P_{n-1}, P_n]$$

Using the orthogonality properties of the monic polynomials $Q_n(x)$ and the fact that $[P_n(x), P_m(x)] = 0$for m ¡ n, we can simplify this to:

$$[P_{n+1}(x), P_m(x)] = [Q_{n+1}(x), P_m(x)]$$

Since $Q_{n+1}(x)$ is orthogonal to all lower-degree polynomials, it follows that $P_{n+1}(x)$ is orthogonal to all lower-degree polynomials as well. The completeness of the basis follows from the fact that the orthogonal polynomials $P_n(x)$ form a complete basis for the space of all polynomials, as each polynomial is orthogonal to all lower-degree polynomials. This completes the proof of the existence of an orthogonal polynomial basis.

THEOREM 2:

For a function f(x) in the space of continuous functions $C^{(n)}$ [a, b] and a fractional derivative order $0 < \alpha \leq 1$, there exists a polynomial $P_m(x)$ defined as follows:

$$P_m(x) = x^m - \frac{<x^m, P_0>}{<P_0, P_0>} P_0(x) - \cdots - \frac{<x_m, P_{m-1}>}{<P_{m-1}, P_{m-1}>} P_{m-1}(x)$$

This theorem provides a construction for polynomials$P_m(x)$ that can be used to approximate fractional derivatives of f(x) when f(x) is n times continuously differentiable.

Proof:

Start with a polynomial $P_0(x) = 1$. This is a polynomial of degree 0. For each m ≥ 1, define $P_m(x)$ as follows:

$$P_m(x) = x^m - \sum_{k=0}^{m-1} \frac{<x^m, P_k>}{<P_k, P_k} P_k(x)$$

This definition involves subtracting certain linear combinations of lower-degree polynomials from x_m. Normalize $P_m(x)$ by dividing it by its norm;

$$P_m(x) = \frac{P_m(x)}{\sqrt{<P_m, P_m>}}$$

This step ensures that each $P_m(x)$ has a unit norm. increasing values of m to generate the sequence of polynomials $P_m(x)$. Now, let's summarize the key properties of the resulting sequence of polynomials: $P_m(x)$: Each $P_m(x)$ has a degree m, as it starts with x_m and subtracts linear combinations of lower-degree polynomials. The

construction ensures that each $P_m(x)$ is orthogonal to all lower-degree polynomials $P_k(x)$ for k ¡ m.
The normalization step ensures that each $P_m(x)$ has a unit norm. This theorem and its proof show that you can construct a sequence of polynomials $P_m(x)$ with the specified properties, which can be useful for approximating fractional derivatives of functions in the space $C^{(n)}$ [a, b].
Inner Product $<x^m, P_k>$: This represents the inner product between the polynomials x_m and $P_k(x)$. It measures the similarity or overlap between these polynomials over the interval [a, b]. Mathematically, it is calculated as:

$$< x^m, P_k >= \int_a^b x^m P_k(x)dx$$

Inner Product $< P_m, P_m>$ This represents the inner product of $P_m(x)$ with itself, measuring its norm or magnitude over [a, b]. Mathematically, it is calculated as:

$$< P_m, P_m >= \int_a^b P_m(x)^2 dx$$

5. DESCRIPTION OF METHOD USED IN THIS WORK

In this section, the method used in this work is referred to as standard collocation method. The method is demonstrated on the two types of equations mentioned in the work.

STANDARD COLLOCATION METHOD ON INTEGRO-DIFFERENTIAL EQUATION:

We consider the n^{th} order integro-diferential equation of the form

$$\sum_{i=0}^{n} P_i(x)y^{(i)}(x) + \lambda \int_a^{b(x)} K(x,t)y(t)dt = g(x) \tag{1}$$

The conditions are given as

$$y^{(i)}(a_i) = \alpha_i; i = 0, 1, 2, \cdots, (n-1) \tag{2}$$

Let $y_N(x) = \sum_{k=0}^{N} c_i Q_i(x)$ to be the approximate solution of (1) and (2)
That is,

$$y_N(x) = c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x) \tag{3}$$

Thus (1) is expanded as

$$P_0y(x) + P_1y^{'}(x) + P_2y^{''}(x) + \cdots + P_ny^{(n)}(x) + \lambda \int_a^{b(x)} K(x,t)y(t)dt = g(x) \tag{4}$$

substituting (3) into (4) to get

$$P_0y_N(x)+P_1y_N^{'}(x)+P_2y_N^{''}(x)+\cdots+P_ny_N^{(n)}(x)+\lambda \int_a^{b(x)} K(x,t)y_N(t)dt = g(x) \tag{5}$$

Putting (3) into (5) to get

$$P_0\{c_0Q_0(x)+c_1Q_1(x)+c_2Q_2(x)+\cdots+c_NQ_N(x)\}+P_1\{c_0Q_0^{'}(x)+c_1Q_1^{'}(x)+c_2Q_2^{'}(x)+\cdots+c_NQ_N^{'}(x)\}$$

$$+P_2\{c_0Q_0^{''}(x)+c_1Q_1^{''}(x)+c_2Q_2^{''}(x)+\cdots+c_NQ_N^{''}(x)\}+\cdots+P_n\{c_0Q_0^{(n)}(x)+c_1Q_1^{(n)}(x)+c_2Q_2^{(n)}(x)+\cdots+c_NQ_N^{(n)}(x)\}$$

$$+\lambda\int_{a}^{b(x)}K(x,t)\{c_0Q_0(t)+c_1Q_1(t)+c_2Q_2(t)+\cdots+c_NQ_N(t)\}dt=g(x) \quad (6)$$

Hence, collecting like term in (6) to get

$$\{P_0Q_0(x)+P_1Q_0^{'}(x)+P_2Q_0^{''}(x)+\cdots+P_nQ_0^{(n)}(x)+\lambda\int_{a}^{b(x)}K(x,t)Q_0(t)dt\}c_0$$

$$+\{P_0Q_1(x)+P_1Q_1^{'}(x)+P_2Q_1^{''}(x)+\cdots+P_nQ_1^{(n)}(x)+\lambda\int_{a}^{b(x)}K(x,t)Q_1(t)dt\}c_1$$

$$\vdots$$

$$+\{P_0Q_N(x)+P_1Q_N^{'}(x)+P_2Q_N^{''}(x)+\cdots+P_nQ_N^{(n)}(x)+\lambda\int_{a}^{b(x)}K(x,t)Q_N(t)dt\}c_N=g(x) \quad (7)$$

Thus (7) gives (N+1) unknown constants to be determined.
In (2), that is,

$$y_N(a_1)=\alpha_0 \Rightarrow c_0Q_0(a_1)+c_1Q_1(a_1)+c_2Q_2(a_1)+\cdots+c_NQ_N(a_1)=\alpha_0 \quad (8)$$

$$y_N^{'}(a_1)=\alpha_1 \Rightarrow c_0Q_0^{'}(a_1)+c_1Q_1^{'}(a_1)+c_2Q_2^{'}(a_1)+\cdots+c_NQ_N^{'}(a_1)=\alpha_1 \quad (9)$$

$$\vdots$$

$$y_N^{(n-1)}(a_1)=\alpha_{n-1} \Rightarrow c_0Q_0^{(n-1)}(a_1)+c_1Q_1^{(n-1)}(a_1)+c_2Q_2^{(n-1)}(a_1)+\cdots+c_NQ_N^{(n-1)}(a_1)=\alpha_{n-1} \quad (10)$$

Thus, (7) is collocated at point $x=x_k$ to get

$$\{P_0Q_0(x_k)+P_1Q_0^{'}(x_k)+P_2Q_0^{''}(x_k)+\cdots+P_nQ_0^{(n)}(x)+\lambda\int_{a}^{b(x_k)}K(x_k,t)Q_0(t)dt\}c_0$$

$$+\{P_0Q_1(x_k)+P_1Q_2^{'}(x_k)+P_2Q_1^{''}(x_k)+\cdots+P_nQ_1^{(n)}(x_k)+\lambda\int_{a}^{b(x_k)}K(x_k,t)Q_1(t)dt\}c_1$$

$$\vdots$$

$$+\{P_0Q_N(x_k)+P_1Q_N^{'}(x_k)+P_2Q_N^{''}(x_k)+\cdots+P_nQ_N^{(n)}(x_k)+\lambda\int_{a}^{b(x)}K(x_k,t)Q_N(t)dt\}c_N=g(x_k) \quad (11)$$

Where,

$$x_k=a+\frac{(b-a)k}{(N-n+2)}; K=1,2,3,\cdots,(N-n+1) \quad (12)$$

Hence (11) gives(N-n+1) algebraic linear system of equations in (N+1) unknown constants. Extra (n) equation are obtained from (8) to (10). Altogether, we have (N+1) algebraic equations in (N+1) unknown constants which are then solved using Guassian elimination method to obtain the unknown constants. These are substituted into (3) to get the required approximate solution.

STANDARD COLLOCATION METHOD ON INTEGRAL EQUATION:
Consider the integral equation of the form

$$y(x)+\lambda\int_{a}^{b(x)}K(x,t)y(t)dt=f(x) \quad (13)$$

Let $y_N(x) = \sum_{i=o}^{N} c_iQ_i(x)$ be the approximate solution of (13)
That is,

$$y_N(x) = c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x) \tag{14}$$

Substituting (14) into (13) to get,

$$y_N(x) + \lambda \int_a^{b(x)} K(x,t)y_N(t)dt = f(x) \tag{15}$$

Putting (14) into (15) to get

$$\{c_0Q_0(x)+c_1Q_1(x)+c_2Q_2(x)+\cdots+c_NQ_N(x)\}+\lambda \int_a^{b(x)} K(x,t)\{c_0Q_0(x)+c_1Q_1(x)+c_2Q_2(x)+\cdots+c_NQ_N(x)\}dt = f(x) \tag{16}$$

Collecting like terms in (16) to get

$$\{Q_0(x)+\lambda \int_a^{b(x)} K(x,t)Q_0(t)dt\}c_0+\{Q_1(x)+\lambda \int_a^{b(x)} K(x,t)Q_1(t)dt\}c_1+\{Q_2(x)+\lambda \int_a^{b(x)} K(x,t)Q_2(t)dt\}c_2 + \cdots + \{Q_N(x) + \lambda \int_a^{b(x)} K(x,t)Q_N(t)dt\}c_N = f(x) \tag{17}$$

Thus,(17) gives (N+1) unknown constant to be determined. Hence, (17) is collocated at point $x = x_k$ to get

$$\{Q_0(x_k)+\lambda \int_a^{b(x_k)} K(x_k,t)Q_0(t)dt\}c_0+\{Q_1(x_k)+\lambda \int_a^{b(x_k)} K(x_k,t)Q_1(t)dt\}c_1+\{Q_2(x_k) +\lambda \int_a^{b(x)} K(x_k,t)Q_2(t)dt\}c_2+\cdots+\{Q_N(x_k) + \lambda \int_a^{b(x_k)} K(x_k,t)Q_N(t)dt\}c_N = f(x_k) \tag{18}$$

Where,

$$x_k = a + \frac{(b-a)k}{(N+2)}; K = 1, 2, 3, \cdots, (N+1) \tag{19}$$

Hence, (18) gives(N+1) algebraic linear system of equation in (N+1) unknown constants which are then solved using Guassian elimination method to obtain the unknown constants these are then substituted into (14) to get the required approximate solution.

6. NUMERICAL EXPERIMENT ON EXAMPLES:

In this section,we have demonstrated standard collocation method for case N=4 for example 1 and other examples listed, the same procedure for case N=4 are followed.

EXAMPLE 1:

Consider the fourth-order integro-differential equation of the form:

$$y^{(iv)} + (x^2 - 1)y(x) + \int_0^x e^{t-2x}y(t)dt = xe^{-x}(x + e^{-x}) \tag{20}$$

subject to the conditions

$$y(0)=1, y^{'}(0)=-1, y^{''}(0)=1, y^{'''}(0)=-1$$

and with the exact solution

$$y(x)=e^{-x}$$

Method of Solution

Example 1 is solved using Standard collocation method and the constructed Type I orthogonal polynomial as the basis function.

Let the approximate solution be given as :

$$y_N(x)=\sum_{i=0}^{N} c_i Q_i(x) \tag{21}$$

For N=4, (21) becomes

$$y_4 = c_0Q_0 + c_1Q_1 + c_2Q_2 + c_3Q_3 + c_4Q_4 \tag{22}$$

The Type I orthogonal polynomial is shifted to the interval of [0,1] to have:

$$Q_0 = 1$$
$$Q_1 = 2x-1$$
$$Q_2 = 6x^2-6x+1$$
$$Q_3 = 20x^3-30x^2+12x-1$$
$$Q_4 = 70x^4-140x^3+90x^2-20x+1$$

Substituting the orthogonal polynomial into (22) to get

$$y_4 = c_0+c_1(2x-1)+c_2(6x^2-6x+1)+c_3(20x^3-30x^2+12x-1)+c_4(70x^4-140x^3+90x^2-20x+1) \tag{23}$$

Hence, (23) is differentiated four times to get

$$y_4^{(iv)} = 1680c_4$$

substituting $y_4^{(iv)}$, $y_4(x)$,$y_4(t)$ into (21) to get

$$1680c_4+(x^2-1)\{c_0+c_1(2x-1)+c_2(6x^2-6x+1)+c_3(20x^3-30x^2+12x-1)+c_4(70x^4-140x^3+90x^2-20x+1)\}$$

$$+\int_0^x e^{t-2x}\{c_0+c_1(2t-1)+c_2(6t^2-6t+1)+c_3(20t^3-30t^2+12t-1)+c_4(70t^4-140t^3+90t^2-20t+1)\}dt$$
$$= xe^{-x}(x+e^{-x}) \tag{24}$$

Evaluating the integral in (24), to get

$$\{x^2+e^{-x}-e^{-2x}-1\}c_0+\{(x^2-1)(2x-1)+2xe^{-x}-3e^{-x}+3e^{-2x}\}c_1$$
$$+\{(x^2-1)(6x^2-6x+1)+6e^{-x}x^2-18xe^{-x}+19e^{-x}-19e^{-2x}\}c_2$$
$$\{(x^2-1)(20x^3-30x^2+12x-1)+20e^{-x}x^3-90e^{-x}x^2+192xe^{-x}-193e^{-x}+193e^{-2x}\}c_3$$
$$\{1680+(x^2-1)(70x^4-140x^3+90x^2-20x+1)+70e^{-x}x^4-420e^{-x}x^3+1350e^{-x}x^2-2720xe^{-x}$$
$$+2721e^{-x}-2721e^{-2x}\}c_4 = xe^{-x}(x+e^{-x}) \tag{25}$$

From the boundary conditions given,

That is,

$$y_4(0)=1 \Leftrightarrow c_0-c_1+c_2-c_3+c_4=1 \tag{26}$$

$$y_4^{'}(0) = -1 \Leftrightarrow 2c_1 - 6c_2 + 12c_3 - 20c_4 = -1 \quad (27)$$

$$y_4^{''}(0) = 1 \Leftrightarrow 12c_2 - 60c_3 + 180c_4 = 1 \quad (28)$$

$$y_4^{'''}(0) = -1 \Leftrightarrow 120c_3 - 840c_4 = -1 \quad (29)$$

Collocating (25) at point $x = x_k$ to get

$$\{x_k^2 + e^{-x_k} - e^{-2x_k} - 1\}c_0 + \{(x_k^2 - 1)(2x_k - 1) + 2x_k e^{-x_k} - 3e^{-x_k} + 3e^{-2x_k}\}c_1$$
$$+\{(x_k^2 - 1)(6x_k^2 - 6x_k + 1) + 6e^{-x_k}x_k^2 - 18x_k e^{-x_k} + 19e^{-x_k} - 19e^{-2x_K}\}c_2$$
$$\{(x_k^2-1)(20x_k^3-30x_k^2+12x_k-1)+20e^{-x_k}x_k^3-90e^{-x_k}x_k^2+192x_k e^{-x_k}-193e^{-x_k}+193e^{-2x_k}\}c_3$$
$$\{1680+(x_k^2-1)(70x_k^4-140x_k^3+90x_k^2-20x_k+1)+70e^{-x_k}x_k^4-420e^{-x_k}x_k^3+1350e^{-x_k}x_k^2-2720x_k e^{-x_k}$$
$$+2721e^{-x_k} - 2721e^{-2x_k}\}c_4 = x_k e^{-x_k}(x_k + e^{-x_k}) \quad (30)$$

Where,

$$x_k = \frac{k}{2}; k = 1$$

$When\, k = 1; x_1 = \frac{1}{2}$ hence (30) becomes

$$1679.721828c_4+0.03664497c_3+0.360393204c_2-0.109422995c_1-0.5113487815c_0 = 0.3355723856 \quad (31)$$

Solving equations (26),(27),(28),(29) and(31) by Guassian elimination method to get

$c_0 = 0.6300498932, c_1 = -0.3149002136, c_2 = 0.04888079982, c_3 = -0.005808386730, c_4 = 0.0003607066577$

The values $c_i (i = 0(1)4)$ are then substituted into (23) and simplified to get the required approximate solution for case N=4 that is

$$y_4(x) = 1.000000000-1.000000000x+0.5000000000x^2-0.1666666667x^3+0.02524946604x^4 \quad (32)$$

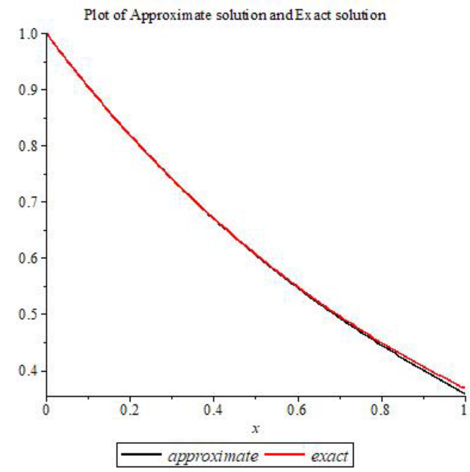
The same procedure for case N=4 has been followed and the required approximate solutions for case N=5 and N=6 are:

$y_5(x) = 1.000000000-1.000000000x+0.5000000000x^2-0.1666666666x^3+0.03832345290x^4-0.005081413748x^5$

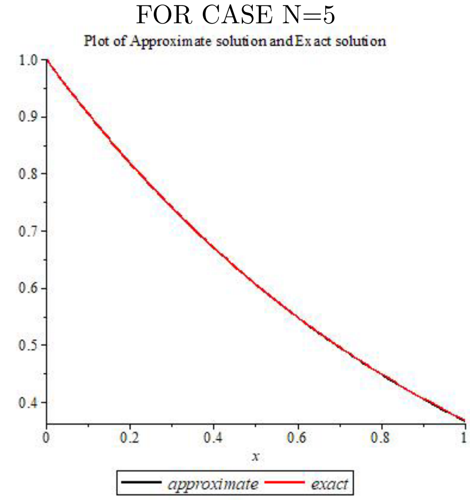
$y_6(x) = 0.9999999996-1.000000000x+0.5000000001x^2-0.1666666667x^3+0.04121595640x^4-0.007647854693x^5 + 0.0008467725135x^6$

GRAPHICAL REPRESENTATION OF EXAMPLE 1

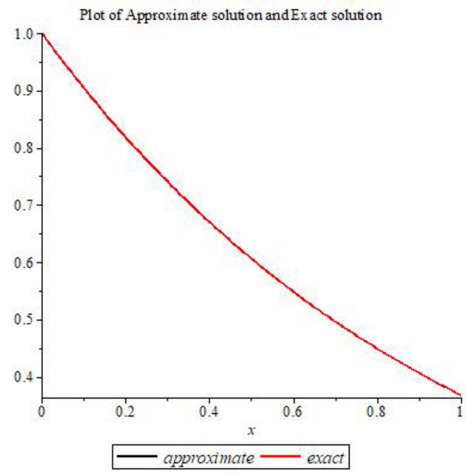
FOR CASE N=4



FOR CASE N=5



FOR CASE N=6



EXAMPLE 2:

Consider the fourth-order integro-differential equation of the form:

$$y^{(iv)} + (x^2 - 1)y(x) + \int_0^x e^{t-2x} y(t)dt = xe^{-x}(x + e^{-x}) \tag{33}$$

subject to the conditons

$$y(0) = 1, y^{'}(0) = -1, y^{''}(0) = 1, y^{'''}(0) = -1$$

and with the exact solution given as

$$y(x) = e^{-x}$$

Method of Solution

Using Standard collocation method and the constructed Type II orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=4

$y_4(x) = 1.000000000 - 1.000000000x + 0.5000000000x^2 - 0.1666666666x^3 + 0.02524946601x^4$

For N=5

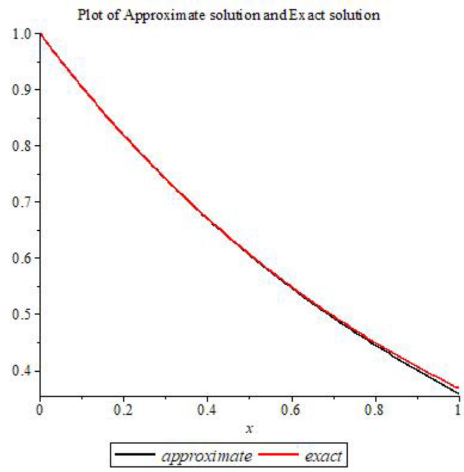
$y_5(x) = 0.9999999999 - 1.000000000x + 0.5000000000x^2 - 0.1666666666x^3 + 0.03832345299x^4 - 0.005081413780x^5$

For N=6

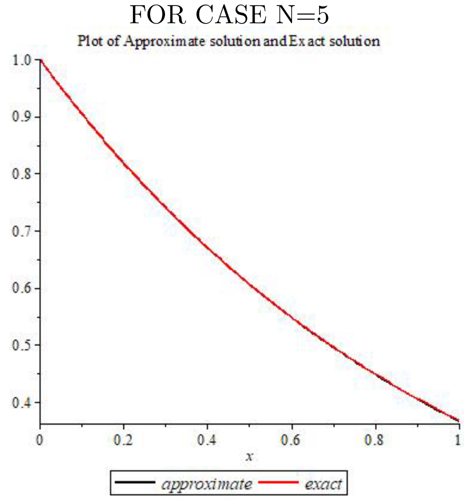
$y_6(x) = 0.9999999999 - 1.000000000x + 0.5000000000x^2 - 0.1666666667x^3 + 0.04121595644x^4 - 0.007647854744x^5 + 0.0008467725310x^6$

GRAPHICAL REPRESENTATION OF EXAMPLE 2

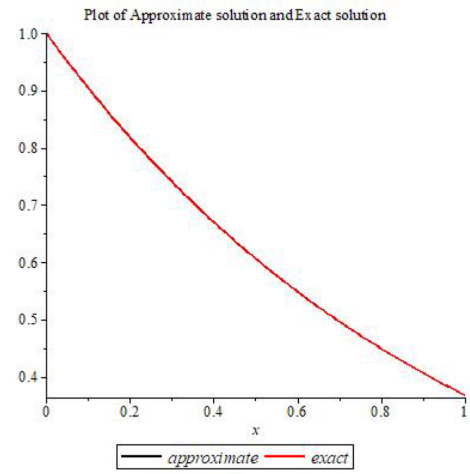
FOR CASE N=4



FOR CASE N=5



FOR CASE N=6



EXAMPLE 3:

Consider the second-order integro-differential equation of the form:

$$y^{''}(x) - xy^{'}(x) + \int_0^1 xty(t)dt = x + e^x - xe^x$$

Subject to the conditions

$$y(0) = 1, y^{'}(0) = 1$$

and with the exact solution given as

$$y(x) = e^x$$

Method of Solution

Using Standard collocation method and the constructed Type I orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=2

$$y_2(x) = 0.9999999997 + 0.9999999998x + 0.8662732122x^2$$

For N=3

$$y_3(x) = 1.000000000 + 1.000000000x + 0.4208961966x^2 + 0.2740717760x^3$$

For N=4

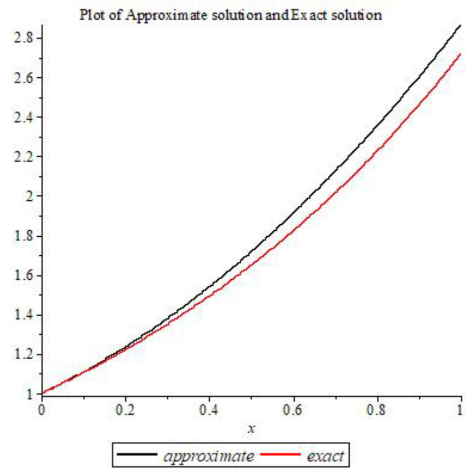
$$y_4(x) = 1.000000000 + 0.9999999998x + 0.5117108075x^2 + 0.1393269541x^3 + 0.06935222037x^4$$

For N=5

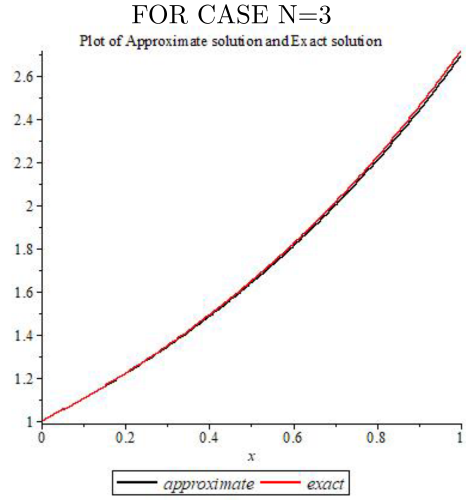
$$y_5(x) = 1.000000001 + 1.000000000x + 0.4987812630x^2 + 0.1707906361x^3 + 0.03469837907x^4 + 0.01382710217x^5$$

GRAPHICAL REPRESENTATION OF EXAMPLE 3

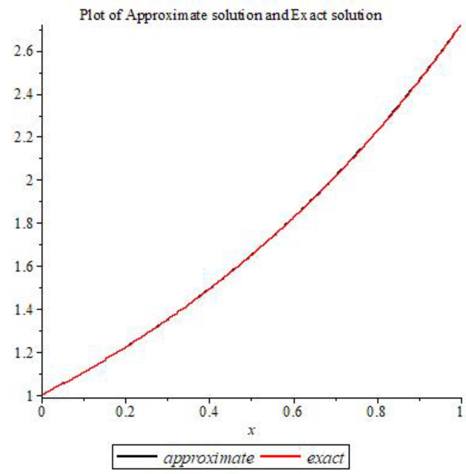
FOR CASE N=2



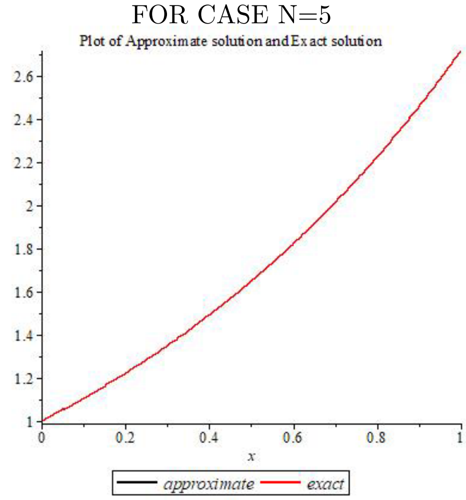
FOR CASE N=3



FOR CASE N=4



FOR CASE N=5



EXAMPLE 4:

Consider the second-order integro-differential equation of the form:

$$y^{''}(x) - xy^{'}(x) + \int_0^1 xty(t)dt = x + e^x - xe^x$$

Subject to the conditions

$$y(0) = 1, y^{'}(0) = 1$$

and with the exact solution given as

$$y(x) = e^x$$

Method of Solution

Using Standard collocation method and the constructed Type II orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get

the following required approximate solutions:
For N=2

$$y_2(x) = 0.9999999992 + 1.000000000x + 0.8662732126x^2$$

For N=3

$$y_3(x) = 0.9999999992 + 1.000000000x + 0.4208961964x^2 + 0.2740717763x^3$$

For N=4

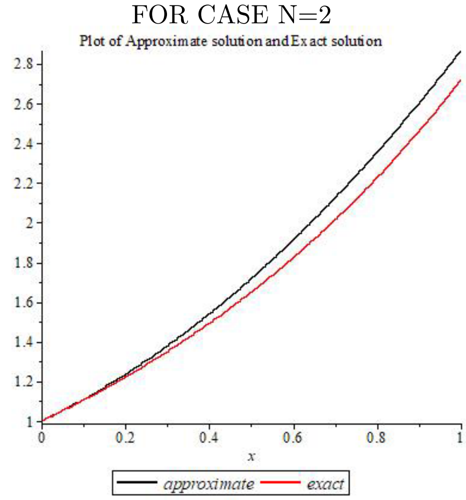
$$y_4(x) = 1.000000000+1.000000000x+0.5117108073x^2+0.1393269551x^3+0.06935221985x^4$$

For N=5

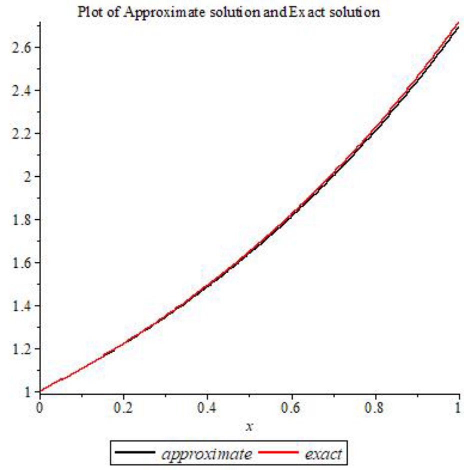
$$y_5(x) = 0.9999999985+1.000000000x+0.4987812612x^2+0.1707906413x^3+0.03469837291x^4+0.01382710479x^5$$

GRAPHICAL REPRESENTATION OF EXAMPLE 4

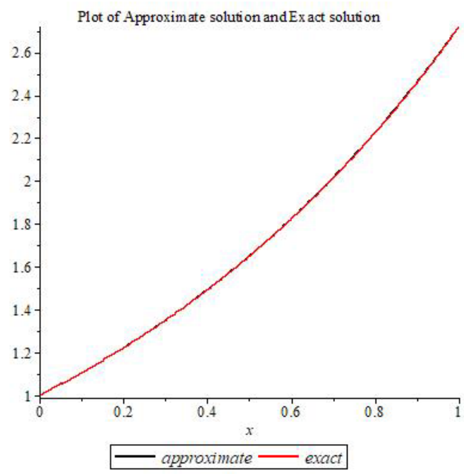
FOR CASE N=2



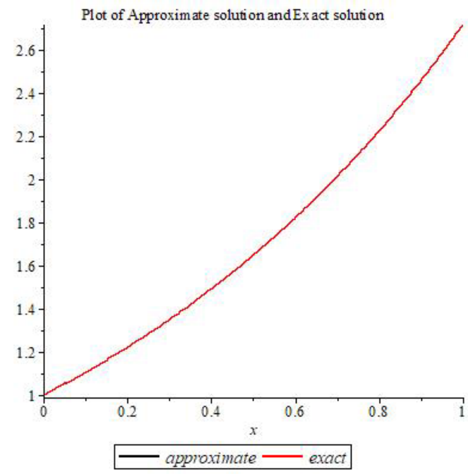
FOR CASE N=3



FOR CASE N=4



FOR CASE N=5



EXAMPLE 5:

Consider the integral equation of the form:

$$y(x) = 2x + sinx + x^2 - cosx + 1 - \int_0^x y(t)dt$$

and with the exact solution given as

$$y(x) = 2x + sinx$$

Method of Solution:

Using Standard collocation method and the constructed Type I orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=4

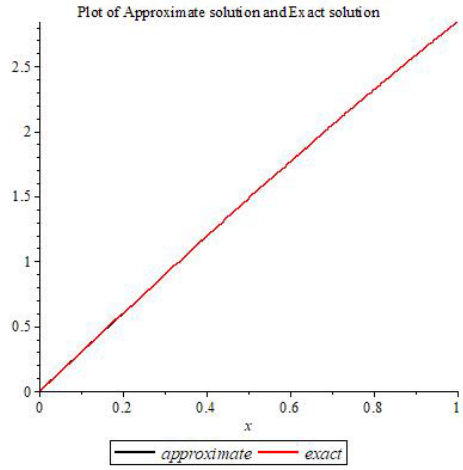
$y_4(x) = 0.0001075540509 + 2.998431466x + 0.0078689370x^2 - 0.1848485336x^3 + 0.01979924134x^4$

For N=6

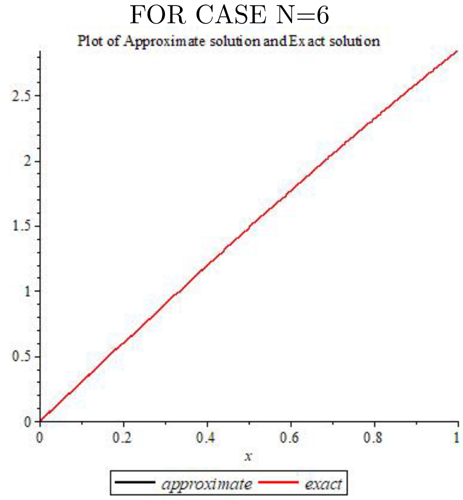
$y_6(x) = -0.0000004084111935 + 3.000008823x - 0.0000714525x^2 - 0.1663698696x^3 - 0.00069454462x^4 + 0.009260408080x^5 - 0.0006615446621x^6$

GRAPHICAL REPRESENTATION OF EXAMPLE 5

FOR CASE N=4



FOR CASE N=6



EXAMPLE 6:
Consider the integral equation of the form:

$$y(x) = 2x + sinx + x^2 - cosx + 1 - \int_0^x y(t)dt$$

and with the exact solution

$$y(x) = 2x + sinx$$

Method of Solution:
Using Standard collocation method and the constructed Type II orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=4

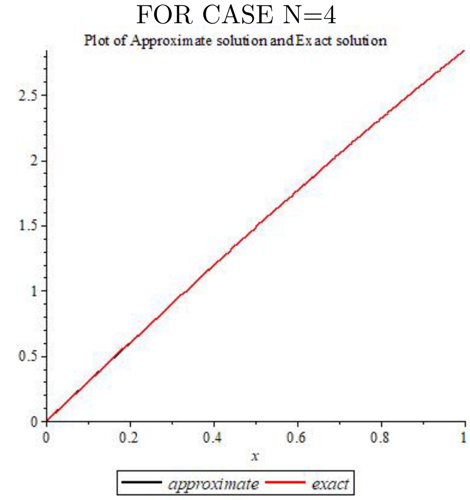
$y_4(x) = 0.000107552255 + 2.998431475x + 0.0078689159x^2 - 0.1848485068x^3 + 0.01979922873x^4$

For N=6

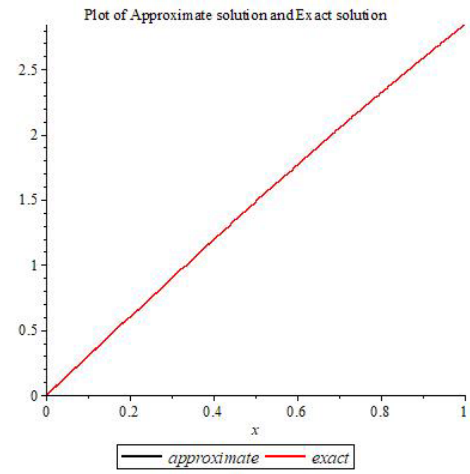
$y_6(x) = -0.000000408411193510 + 3.000008823x - 0.0000714525x^2 - 0.1663698696x^3 - 0.00069454462x^4 + 0.009260408080x^5 - 0.0006615446621x^6$

GRAPHICAL REPRESENTATION OF EXAMPLE 6

FOR CASE N=4



FOR CASE N=6



EXAMPLE 7:

Consider the integral equation of the form:

$$y(x) = 1 + x^2 + cosx - x - \frac{1}{3}x^3 - sinx + \int_0^x y(t)dt$$

and with the exact solution

$$y(x) = 1 + x^2 + cosx$$

Method of Solution:

Using Standard collocation method and the constructed Type I orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=2

$$y_2(x) = 2.006516878 - 0.387140054e - 1x + 0.5651984106x^2$$

For N=4

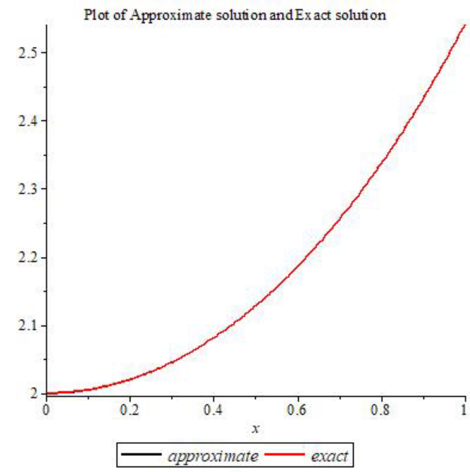
$$y_4(x) = 1.999943526+0.0007046883x+0.4967157472x^2+0.00664837042x^3+0.03635060313x^4$$

For N=6

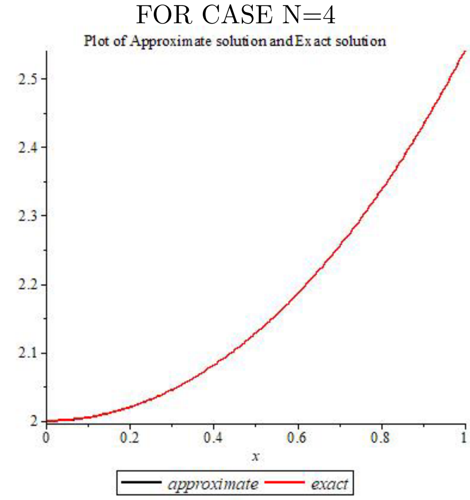
$$y_6(x) = 2.000000254 - 0.0000048983x + 0.5000377458x^2 - 0.00014656859 * x^3 + 0.04197854212x^4 - 0.000357183616x^5 - 0.001205772368x^6$$

GRAPHICAL REPRESENTATION OF EXAMPLE 7

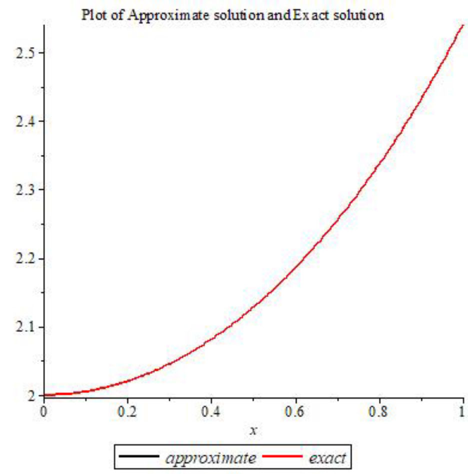
FOR CASE N=2



FOR CASE N=4



FOR CASE N=6



EXAMPLE 8:

Consider the integral equation of the form:

$$y(x) = 1 + x^2 + cosx - x - \frac{1}{3}x^3 - sinx + \int_0^x y(t)dt$$

and with the exact solution

$$y(x) = 1 + x^2 + cosx$$

Method of Solution:

Using Standard collocation method and the constructed Type II orthogonal polynomial as the bases, Then following the procedure of example 1 case N=4 we get the following required approximate solutions:

For N=2

$$y_2(x) = 2.006516877 - 0.0387140056x + 0.5651984114x^2$$

For N=4

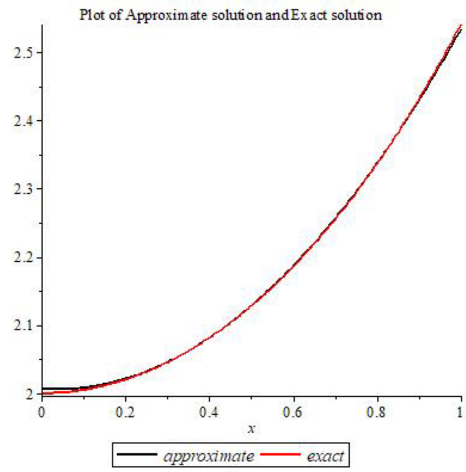
$y_4(x) = 1.999990022+0.0001687630x+0.4989522438x^2+0.00283005321x^3+0.03865132080x^4$

For N=6

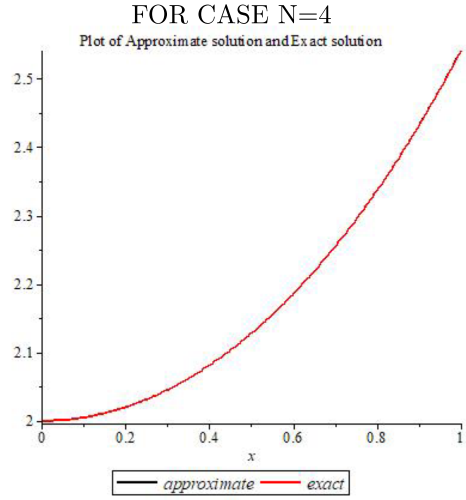
$y_6(x) = 2.000000216-0.0000042352x+0.5000330932x^2-0.00013087131x^3+0.04195123710x^4-0.000333649080 * x^5 - 0.001213716257x^6$

GRAPHICAL REPRESENTATION OF EXAMPLE 8

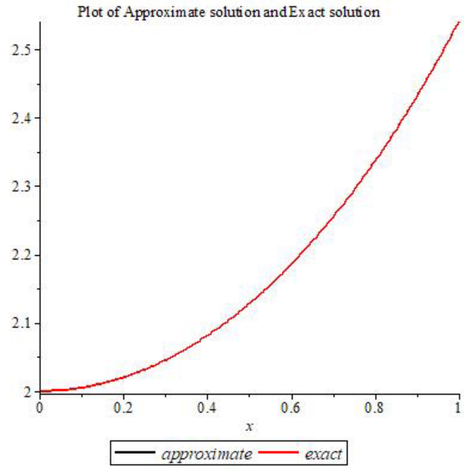
FOR CASE N=2



FOR CASE N=4



FOR CASE N=6



7. CONCLUSION

In this work, the standard collocation method is used to solve integral and integro-differential equations using the constructed type I and type II orthogonal polynomials as the bases functions. From the results of the graph presented, we observed that as N is small in each examples, the graph diverge from the exact solution and as N increases the approximate solutions converge faster to the exact solutions for both polynomials. This shows that the standard collocation method is suitable for solving integral and integro-differential equations and the type I and II orthogonal polynomials constructed and used as bases functions served as additional polynomials to the existing ones in the literature.

References

[1] Abbasbandy, S., Allahviranloo, T., & Cattani, C. (2020). Spectral collocation method based on hybrid of block-pulse functions and Chebyshev polynomials for solving Volterra integro-differential equations. *Mathematics, 8*(5), 694.

[2] Dehghan, M., & Shokri, A. (2008). A numerical method for solving integro-differential equations of Fredholm type by using Legendre polynomials. *Applied Mathematics and Computation, 198*(1), 94-102.

[3] Gautschi, W. (2004). *Orthogonal polynomials: Computation and approximation.* Oxford University Press.

[4] Gong, Y., Gao, L., & Wei, X. (2019). The construction of generalized Jacobi polynomials and the spectral methods for linear integro-differential equations. *Journal of Scientific Computing, 78*(3), 1749-1775.

[5] Hashim, I. (2006). Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Journal of Computational and Applied Mathematics, 193*(2), 658-664.

[6] Kythe, P. K., & Puri, P. (2002). *Computational methods for linear integral equations.* Birkhäuser.

[7] Laeli, D. H., & Maalek-Ghaini, F. M. (2012). Numerical solution of Volterra-Fredholm integral equations by moving least square method and Chebyshev polynomials. *Applied Mathematics Modelling, 36*(6), 3283-3288.

[8] Maleknejad, K., Shateyi, S., & Kiani, S. L. (2015). Numerical solution of nonlinear Fredholm integral equations by using the Chebyshev spectral collocation method. *Numerical Algorithms, 68*(2), 263-280.

[9] Shampine, L. F. (1974). Numerical solution of ordinary differential equations using orthogonal polynomials. *Journal of Computational Physics, 14*(1), 114-122.

[10] Shateyi, S. (2010). Chebyshev spectral collocation method for solving integro-differential equations of Fredholm type. *Numerical Algorithms, 55*(3), 309-327.

[11] Sweilam, N. H., Khader, M. M., & Kota, Y. W. (2013). Numerical and analytical study for fourth-order integro-differential equations using a pseudospectral method. *Mathematical Problems in Engineering, 2013*, 434753.

[12] Wazwaz, A. M. (n.d.). *Linear and nonlinear integral equations*. Springer.

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