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LYAPUNOV-TYPE INEQUALITIES OF MULTI-LAYER FRACTIONAL HALF-LINEAR ∇-DIFFERENCE BOUNDARY VALUE PROBLEMS

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Abstract. The main problem of interest in this paper is to investigate the nontrivial solutions of 3-leyer nested half-linear higher-order fractional ∇-difference equations subject to the fully left-sided boundary conditions. In our investigation the discrete Holder inequality and integration by parts play fundamental roles. Thanks to these discrete mathematical tools, Lyapunov-type inequalities of the related boundary value problems are obtained. Relying on the obtained Lyapunov-type inequalities, we can study three classes of important qualitative dynamics of related fractional-order difference boundary value problems. The first class is investigating on the nontrivial solutions of the corresponding multi-layer half-linear fractional ∇-difference problems and the second class is devoted to study on the eigenvalue regions of related eigenvalue problems. At the third step, making use of the obtained Lyapunov-type inequalities some nonexistence results are presented.

1. Introduction

Nowadays fractional calculus has gained a prominant place within the advanced dynamical systems. As it is known, the main aim of the dynamical systems is to design and study models inspired by the real life phenomena and propose relevant practical methods to engineerize them. In this way the most precious models are those who cantain more and more details of the phenomenon under investigation. Keeping this point in mind and make concentration on this fact that since fractional-order operators are memory preserving mathematical tools, this is reasonable that why we are interested to consider these operators to make accurate dynamical systems models involving mush more details in comparison with the classic integer-order models.

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On the other hand, in order to reach an engineering chracteristic of a continuous fractional-order dynamical system, it can be proposed to study its corresponding discrete model, and consequently it is justified that why we have to organize continued investigation on discrete fractional-order dynamical systems. An interested follower can find valuable details on the discrete fractional calculus and its applications in dynamical systems in the monographs and research papers [2]-[6], [11]-[13], [18], [19], [21], [27], [28], [29] and cited bibliography therein.
Here is worthy place to have a little review on the discrete Lyapunov-type inequalities prior to finalize this section. The history of the Lyapunov-type inequalities has begun at the late ninteenth century, when the stability of motion was being investigated by the A. M. Lyapunov. In 1892, Lyapunov considered the second-order Hill type differential equation

$$x^{''} + q(t)x = 0, \quad -\infty < t < \infty,$$

for which $q(t)$ stands for a T-periodic coefficient. As an instability criterion, it has proven that if $q(t) \not\equiv 0$ and $q(t) \geq 0$ for all $t \in \mathbb{R}$, then the following inequality is satisfied:

$$\int_0^T q(t)dt > \frac{4}{T}.$$

For almost past half century this inequality has known as the Lyapunov inequality. For more details we refer to the [20], [23]. Since we are not dealt with the continuous Lyapunov-type inequalities in this paper, so we just suggest the following motivating research papers on the continuous Lyapunov-type inequalities for interested followers and jump directly into the discrete Lyapunov-type inequalities, [14]-[17], [24], [25] and cited bibliography therein.
The first class of the discrete Lyapunov-type inequalities has presented in 1983 by Cheng, [7] where he considered the second-order difference equation

$$\Delta^2 x(k-1) + p(k)x(k) = 0, \tag{1}$$

where, $p(k)$ is a real-valued function defined on consequtive integers. Cheng, proved that if $p(k)$ be a non-negative function defined on $\mathbb{N}_a^b$, and $x(k)$ be a non-trivial solution such that $x(a-1) = 0$ and $x(b+1) = 0$, then the following sharp Lyapunov-type inequality is satisfied

$$\sum_{i=a}^{b} p(i) \geq \mu(b-a+1), \tag{2}$$

for which μ stands for a particular strictly increasing function on $\mathbb{N}$. By the use of the Lyapunov-type inequality (2), the author presented some exitence and nonexistence criteria for nontrivial solutions of the difference equation (1).
One of the most interesting branches of the discrete fractional calculus turns to the fractional ∇-difference operators and related researches. Considering the dsicrete fractional Lyapunov-type inequalities of ∇-difference type, the author in [24] studied the following two-point left and right focal fractional ∇-difference boundary value problems

$$\begin{cases} \left(\nabla_a^{\alpha} u\right)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\ \left(\nabla_a^{\alpha-1} u\right)(a+1) = 0, \ u(b) = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} \left(\nabla_a^{\alpha} u\right)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a+1) = 0, \ \left(\nabla_a^{\alpha-1} u\right)(b) = 0, \end{cases} \tag{4}$$

respectively, where, $1 < \alpha \leqslant 2$ and $h : \mathbb{N}_{a+2}^{b} \longrightarrow \mathbb{R}$. The author, making use of the Green function techique obtained corresponding Lyapunov-type inequalities for each of the ∇-difference boundary value problems 3 and 4, and in the light of these inequalities qualitative dynamics of these boundary value problems have investigated. In continuation, the same author in [25], investigated the Lyapunov-type inequality of the two-point fractional ∇-difference boundary value problem

$$\begin{cases} \left(\nabla_a^{\alpha} u\right)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a) = 0, \ \left(\nabla_a^{\beta} u\right)(b) = 0, \end{cases} \tag{5}$$

in which, $1 < \alpha \leqslant 2$, $0 \leqslant \beta \leqslant 1$ and $h : \mathbb{N}_{a+2}^{b} \longrightarrow \mathbb{R}$. In all of the fractional ∇-difference boundary value problems 3-5, $\nabla_{t_0}^{\gamma}$ denotes the fractional ∇-difference of order γ and lower terminal t_0, that will be defined exactly in the next section.
Motivated by the aforementioned works on the fractional ∇-difference boundary value problems, the multi-layer half-linear fractional ∇-difference boundary value problems of higher order have chosen to be investigated in the same way of [24] and [25], that will be introduced in the next section. Many researchers in this community are interested in the extending of the application domain related to the Lyapunov-type inequalities. Some instances can be cited here as [8], [12], [14], [15], [17]. Also, as an interesting colloection of the discrete fractional-order Lyapunov-type inequalities we suggest the reseach works [12], [24], [25] for more consultation on topic.
Disregarding the pure beauty of the analysis of dynamical systems involving the half-linear operators, these class of dynamical problems have wide potential to express some important applied phenomena such as investigation on the porous media and related topics. So, thanks to the half-linear dynamical systems we are able to study for instance the fully applicable science of the porous mediums and all of related topics. For more details we refer to [8], [9], [10], [22] and the cited bibliography therein.
At the end of this section we are going to state the organization of the rest of this investigation. Section 2, includes formulation of multilayer half-linear boundary value problems of the fractional ∇-differences. Besides, all of necessary technical background related to this investigation are given here. Next, we have Section 3 in which making use of detailed and beautiful technical analysis of the discrete fractional calculus, Lyapunov-type inequalities of the half-linear boundary value problems under investigation are obtained. Going ahead, we have Section 4 where some interesting applications of the obtained Lyapunov-type inequalities are given to demonstrate the importance of the Lyapunov-type inequalities in dynamical systems. This artice will be finalized by Section 5 as a little space that allows us to summarize findings of this investigation as well as discussion on the structure of appeared boundary conditions.

2. Formulation of the Main Problems and Related Background

This section essentially is a bi-devision environment. The first part is devoted to introduce the main problems to be investigated and second one contains all of technical requirements that will enable us to reach the claimed results. So, we begin this section with formulation of the following half-linear ∇-difference boundary value problems:

$$\text{BVP. 1}\quad \begin{cases} \nabla^{\alpha}_{a^+}\left(\Theta_{\beta_2}\left\{\nabla^{\alpha}_{b_-}\left(\Theta_{\beta_1}\left\{\nabla^{\alpha}_{a^+}u\right\}\right)\right\}\right)(t)=q(t)(\Theta_{\beta_1\beta_2}u)(t),\\ n-1<\alpha\leq n,\ n\in\mathbb{N},\\ \\ \nabla^{\alpha+k-n}_{a^+}\left(\Theta_{\beta_2}\left\{\nabla^{\alpha}_{b_-}\left(\Theta_{\beta_1}\left\{\nabla^{\alpha}_{a^+}u\right\}\right)\right\}\right)(a)=0,\ \nabla^{\alpha+k-n}_{a^+}u(a)=0,\\ k=0,1,2,...,n-1, \end{cases} \tag{6}$$

and

$$\text{BVP. 2}\quad \begin{cases} \nabla^{\alpha}_{a^+}\left(p_2\Theta_{\beta_2}\left\{\nabla^{\alpha}_{b_-}\left(p_1\Theta_{\beta_1}\left\{\nabla^{\alpha}_{a^+}u\right\}\right)\right\}\right)(t)=q(t)(\Theta_{\beta_1\beta_2}u)(t),\\ n-1<\alpha\leq n,\ n\in\mathbb{N},\\ \\ \nabla^{\alpha+k-n}_{a^+}\left(p_2\Theta_{\beta_2}\left\{\nabla^{\alpha}_{b_-}\left(p_1\Theta_{\beta_1}\left\{\nabla^{\alpha}_{a^+}u\right\}\right)\right\}\right)(a)=0,\ \nabla^{\alpha+k-n}_{a^+}u(a)=0,\\ k=0,1,2,...,n-1. \end{cases} \tag{7}$$

The elements of the fractional difference boundary value problems (6) and (7) are described as follows:

- (i) The notation $\nabla^{\alpha}_{a^+}$ and $\nabla^{\alpha}_{b_-}$ stand for the left and right sided fractional ∇-differences of order α, respectively.
- (ii) $\Theta_{\mu}u$ denotes the half-linear signed-power operator
$$\Theta_{\mu}u=|u|^{\mu-1}u,\quad \mu\in(0,+\infty).$$
- (iii) $p_1(t),\ p_2(t)$ are positive real valued functions with $p_2(t)$ increasing.
- (iv) $q(t)$ is an appropriate real valued function.
- (v) The lower and upper terminals a and b, obey the following property:
$$a<b,\quad a\geq 1,\quad b\geq 3,\quad b-a\geq 2,\quad a,b\in\mathbb{Z}.$$

Here we are at the beginning of the second devision of this section, in which all of technical equipments needed in the next two sections are provided.

Definition 2.1. *[*[18]*, Chap. 3] Assume $m\in\mathbb{Z}^+$. Then, the rising function of t is given by*

$$t^{\overline{m}}=\prod_{i=0}^{m-1}(t+i),\quad t^{\overline{0}}=1.$$

In the case that the positive integer m is generalized by the real number α, such generalization is called as the fractional rising function defined by

$$t^{\overline{\alpha}}=\frac{\Gamma(t+\alpha)}{\Gamma(t)},\quad t\in\mathbb{R}-\mathbb{Z}_{\leq 0}. \tag{8}$$

As special cases of the fractional rising functions one may mention the followings:

$$0^{\overline{\alpha}} = 0, \qquad \nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}. \tag{9}$$

In the following Figure 1, one can observe spacial cases of the fractional riding functions.

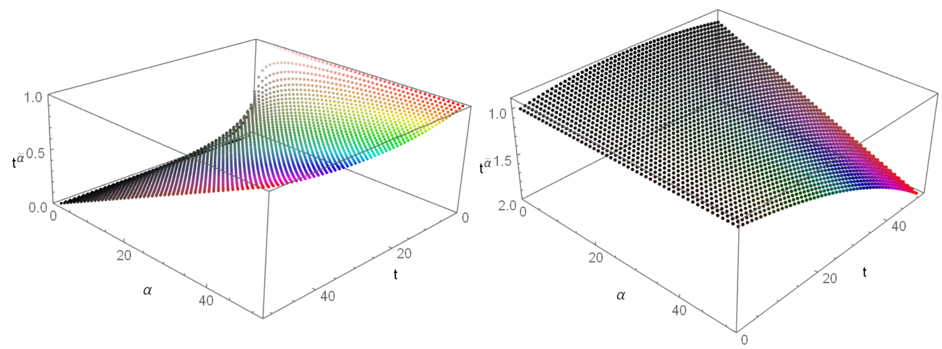

FIGURE 1. Illustration of fractional rising function $t^{\overline{\alpha}}$ for $t, \alpha \in (0,1)$ (left) and $t \in (1,2)$, $\alpha \in (0,1)$ (right)

In order to clarify the meaning of some notation that will be used widely in the future, for each $a, b \in \mathbb{R}$, one has

$$\mathbb{N}_a = \{a, a+1, a+2, ...\},\ {}_b\mathbb{N} = \{b, b-1, b-2, ...\},\ \mathbb{N}_a^b = \{a, a+1, ..., b-1, b\}.$$

Thanks to the aforementioned information, we now are ready to define the fractional ∇-sum operators in what follows.

Definition 2.2. [2], *[*[18]*, Chap. 3] The left and right sided ∇-sums of fractional order $\alpha > 0$ are defined as*

$$\nabla_{a^+}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t - \delta(s))^{\overline{\alpha-1}} f(s), \tag{10}$$

$$\nabla_{b_-}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b} (s - \delta(t)^{\overline{\alpha-1}} f(s), \tag{11}$$

where $\delta(s) = s - 1$.

Next, we have some basic properties of fractional ∇-sum operators as below.

Remark 1. [2],*[*[18]*, Chap. 3] Fractional left and right sided ∇-sums of order $\alpha > 0$, given by (10) and (11) satisfy the following properties:*

(*i*) $\nabla_{a^+}^{-\alpha}$ *maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.*
(*ii*) $\nabla_{b_-}^{-\alpha}$ *maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.*

Now, in the light of fractional ∇-sum operators, we are ready to define the left and right sided fractional ∇-difference operators as follows.

Definition 2.3. [2],[[18], *Chap. 3] Fractional left and right sided* ∇*-differences of order* $0 \leq n-1 < \alpha \leq n$ *for* $n \in \mathbb{N}$ *are defined by*

$$\nabla^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \nabla^n_t \left(\sum_{s=a}^{t} (t-\delta(s))^{\overline{n-\alpha-1}} f(s) \right), \tag{12}$$

$$\nabla^{\alpha}_{b_-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \Delta^n_t \left(\sum_{s=t}^{b} (s-\delta(t)^{\overline{n-\alpha-1}} f(s) \right), \tag{13}$$

for which $\alpha > 0,\ n = [\alpha] + 1$ *and* Δ_t *stands for the forward difference operator on the variable* t.

Similar to the fractional ∇-sum operators, here we are going to present the following basic properties for the fractional ∇-difference operators.

Remark 2. [2],[[18], *Chap. 3] Fractional left and right sided* ∇*-difference operators of order* $\alpha > 0$, *given by* (12) *and* (13) *admit the following properties:*

(*i*) $\nabla^{\alpha}_{a^+}$ *maps functions defined on* $\mathbb{N}_a$ *to functions defined on* $\mathbb{N}_{a+n}$,
(*ii*) $\nabla^{\alpha}_{b_-}$ *maps functions defined on* ${}_b\mathbb{N}$ *to functions defined on* ${}_{b-n}\mathbb{N}$,

in which $n = [\alpha] + 1$.

Technical requirements is continued by the following composition and power rules for the fractional ∇-discrete operators.

Lemma 2.1. [2],[[18], *Chap. 3] Suppose* f *denotes a real valued function and* $\alpha > 0,\ 0 \leq n-1 < \beta \leq n$. *In this case, the following rules are satisfied:*

(N_1) $\nabla^{-\alpha}_{a^+} \nabla^{-\beta}_{a^+} f(t) = \nabla^{-(\alpha+\beta)}_{a^+} f(t) = \nabla^{-\beta}_{a^+} \nabla^{-\alpha}_{a^+} f(t)$,
(N_2) $\nabla^{-\beta}_{a^+}\ \nabla^{\beta}_{a^+} f(t) = f(t) + c_1 (t-a)^{\overline{\beta-1}} + c_2 (t-a)^{\overline{\beta-2}} + ... + c_n (t-a)^{\overline{\beta-n}}$, $c_i \in \mathbb{R},\ i = 1, 2, ..., n$.
(N_3) $\nabla^{\beta}_{a^+}\ \nabla^{-\beta}_{a^+} f(t) = f(t)$.
(N_4) $\nabla^{\beta}_{a^+} (t-a)^{\overline{\alpha}} = \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} (t-a)^{\overline{\alpha-\beta}},\ \alpha - \beta + 1 \notin \mathbb{Z}_{\leq 0}$.

This section is finalized by presenting discrete versions of the integration by parts rule and the Holder inequality.

Lemma 2.2. *Suppose* $\alpha \in \mathbb{R}^+$ *and* $a, b \in \mathbb{R}$ *with* $a < b,\ b \equiv a$ (*mod 1*). *In this case,*

$$\sum_{a}^{b} f(t) \nabla^{\alpha}_{a^+} g(t) = \sum_{a}^{b} g(t) \nabla^{\alpha}_{b_-} f(t), \tag{14}$$

where, $f(t)$ *is defined on* $\mathbb{N}_a$ *and* $g(t)$ *is defined on* ${}_b\mathbb{N}$.

Lemma 2.3. *Assume* $a, b \in \mathbb{R}$ *with* $a < b,\ b \equiv a$ (*mod 1*). *For any real valued functions* $f(t)$ *and* $g(t)$, *the discrete Hoder inequality*

$$\sum_{a}^{b} |f(t)g(t)| \leq \left(\sum_{a}^{b} |f(t)|^p \right)^{\frac{1}{p}} \left(\sum_{a}^{b} |g(t)|^q \right)^{\frac{1}{q}}, \tag{15}$$

holds, in which, $p, q \in \mathbb{R},\ p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main Results

As described above, in this section we make use of the discrete fractional calculus techniques to chracterise the Lyapunov-type inequalities of both of the half-linear fractional ∇-differece boundary value problems (6) and (7). To this aim, we first consider the 3-layer nested ∇-difference boundary value problem (6) to be investigated.

Theorem 3.1. *Suppose that $u(t)$ is a nontrivial solution for the fractional half-linear ∇-difference boundary value problem* (6)*. Then, the following Lyapunov-type inequality is satisfied:*

$$\sum_{a}^{b}|q(t)| > \frac{1}{\left\{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} .(b-a+1)^{\beta_2(\beta_1+1)}}. \tag{16}$$

Proof. The first attempt in the use of assumptions, turns to the boundary conditions

$$\nabla_{a^+}^{\alpha+k-n}u(a) = 0, \quad k = 0, 1, 2, ..., n-1. \tag{17}$$

To this aim, without loss of generality and just for the convenience we suppose that $u(t)$ ba a positive solution of fractional ∇-difference boundary value problem (6), and is defined on $\mathbb{N}_a^b$. In this case, clearly there is a $t_0 \in \mathbb{N}_a^b$ such that

$$m = u(t_0) = \max\left\{u(t)|\ t \in \mathbb{N}_a^b\right\}.$$

In this case, the boundary conditions (17) yields the following

$$m = \nabla_{a^+}^{-\alpha}\nabla_{a^+}^{\alpha}u(t_0) = \frac{1}{\Gamma(\alpha)}\sum_{a}^{t_0}(t_0-s+1)^{\overline{\alpha-1}}\nabla_{a^+}^{\alpha}u(s). \tag{18}$$

Next, we need to consider the following monotonicity results.

$$\Delta_t(t-s+1)^{\overline{\alpha-1}} < 0, \quad 0<\alpha<1, \quad s \le t, \tag{19}$$

$$\Delta_s(t-s+1)^{\overline{\alpha-1}} > 0, \quad 0<\alpha<1, \quad s \le t, \tag{20}$$

$$\Delta_t(t-s+1)^{\overline{\alpha-1}} \ge 0, \quad \alpha>1, \quad a \le s \le t \le b, \tag{21}$$

$$\Delta_s(t-s+1)^{\overline{\alpha-1}} \le 0, \quad \alpha>1, \quad a \le s \le t \le b. \tag{22}$$

Relying on the recent monotonicity results, one can derive

$$\max_{t,s\in\mathbb{N}_a^b}(t-s+1)^{\overline{\alpha-1}} = \begin{cases} \Gamma(\alpha); & 0<\alpha<1,\ s\le t, \\ (b-a+1)^{\overline{\alpha-1}}; & \alpha \ge 1,\ s \le t. \end{cases} \tag{23}$$

Now, relying on (18) and (23), we come to the conclusion that

$$m < \max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\sum_{a}^{b}|\nabla_{a^+}^{\alpha}u(t)| \tag{24}$$

It is time to use the discrete Holder inequality (15). To this aim, we need the following setting:

$$f(t) = \nabla_{a^+}^{\alpha}u(t), \quad g(t) = 1, \quad p = \beta_1+1, \quad q = 1+\frac{1}{\beta_1}. \tag{25}$$

Having the seeting (25) in hand and thanks to the inequality (24), the Hoder inequality (15) helps us to arrive at the following inequality

$$\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}} < (b-a+1)^{\dfrac{\beta_1}{\beta_1+1}} \left(\sum_a^b \left|\nabla_{a^+}^{\alpha} u(t)\right|^{\beta_1+1}\right)^{\dfrac{1}{\beta_1+1}}. \quad (26)$$

Some direct manipulation on the inequality (26), gives us

$$\frac{1}{(b-a+1)^{\beta_1}} \left(\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1} < \sum_a^b \left|\nabla_{a^+}^{\alpha} u(t)\right|^{\beta_1+1}. \quad (27)$$

In continuation we are preparing ourselves to use the fractional ∇-difference summation by parts formula (14). To this aim, we need to use the signed-power operators $\Theta_\nu(z) = |z|^{\nu-1}z,\ \nu \in (0, +\infty)$ as follows:

$$\left|\nabla_{a^+}^{\alpha} u(t)\right|^{\beta_1+1} = \nabla_{a^+}^{\alpha} u(t) \Theta_{\beta_1} \left(\nabla_{a^+}^{\alpha} u(t)\right)(t). \quad (28)$$

This strategy leads us to reach the following inequality

$$\frac{1}{(b-a+1)^{\beta_1}} \left(\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1} < \sum_a^b \left\{\nabla_{a^+}^{\alpha} u(t) \Theta_{\beta_1} \left(\nabla_{a^+}^{\alpha}(t)\right)(t)\right\}. \quad (29)$$

Here we apply the summation by parts rule on the right hand side of the inequality (29). In this case, one has

$$\begin{aligned} &\frac{1}{(b-a+1)^{\beta_1}} \left(\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1} \\ &\qquad < \sum_a^b \left\{u(t) \nabla_{b_-}^{\alpha} \Big(\Theta_{\beta_1}\left(\nabla_{a^+}^{\alpha} u\right)\Big)(t)\right\}. \end{aligned} \quad (30)$$

In this position we turn to the fractional nested ∇-difference equation

$$\nabla_{a^+}^{\alpha} \left(\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha} u\right\}\right)\right\}\right)(t) = q(t)(\Theta_{\beta_1\beta_2} u)(t). \quad (31)$$

Appying fractional ∇-summation $\nabla_{a^+}^{-\alpha}$ on both sides of (31), and consequently making use of the boundary conditions

$$\nabla_{a^+}^{\alpha+k-n} \left(\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha} u\right\}\right)\right\}\right)(a) = 0, \quad k = 0, 1, 2, ..., n-1, \quad (32)$$

we arrive at the following half-linear fractional ∇-difference equations

$$\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}(t)=\frac{1}{\Gamma(\alpha)}\sum_{a}^{t}(t-s+1)^{\overline{\alpha-1}}\nabla_{a^+}^{\alpha}q(s)(\Theta_{\beta_1\beta_2}u)(s).$$

To preceed one more step, first of all we consider the following inversion of the signed-power operators

$$\Theta_{\nu}^{-1}=\Theta_{\nu^{-1}},\quad \nu\in(0,+\infty). \tag{33}$$

Now, combination of (18), (24) and (33) yields the following inequality

$$\begin{aligned}
&\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)(t)\\
&\quad<\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\frac{1}{\beta_2}}\Theta_{\beta_2^{-1}}\left(\sum_{a}^{b}q(s)\left(\Theta_{\beta_1\beta_2}u\right)(s)\right)\\
&\quad\le m^{\beta_1}\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\frac{1}{\beta_2}}\Theta_{\beta_2^{-1}}\left(\sum_{a}^{b}|q(s)|\right).
\end{aligned}\tag{34}$$

We are approaching the desired conclusion. Indeed, substituting (34) into (30) we get the following

$$\begin{aligned}
&\frac{1}{(b-a+1)^{\beta_1}}\left(\frac{m}{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1}\\
&\quad<m^{\beta_1}\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\frac{1}{\beta_2}}\sum_{a}^{b}u(t)\Theta_{\beta_2^{-1}}\left(\sum_{a}^{b}|q(s)|\right)\\
&\quad\le m^{\beta_1+1}\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\frac{1}{\beta_2}}(b-a+1)\Theta_{\beta_2^{-1}}\left(\sum_{a}^{b}|q(s)|\right)\\
&\quad\le m^{\beta_1+1}\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\frac{1}{\beta_2}}(b-a+1)\left(\sum_{a}^{b}|q(s)|\right)^{\frac{1}{\beta_2}}.
\end{aligned}\tag{35}$$

Some direct manipulation on the recent inequality (35), gives us

$$\sum_{a}^{b}|q(t)|>\frac{1}{\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1}.(b-a+1)^{\beta_2(\beta_1+1)}}.$$

So, the proof is completed. □

Having the Lyapunov-type inequality (16) in hand, it is time to find Lyapunov-type inequality of the more generalized fractional half-linear ∇-difference boundary value problem (7). To this aim we have the following theorem.

Theorem 3.2. *Suppose that $u(t)$ is a nontrivial solution for the fractional half-linear ∇-difference boundary value problem (7). Then, the following Lyapunov-type inequality is satisfied:*

$$\sum_a^b |q(t)| > \frac{1}{\left\{\max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} \left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1\beta_2} \left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right)^{\beta_2}}. \tag{36}$$

Proof. Prior to begin the proof procedure, we mention this point that since some parts of the proof are same as presented in the proof of Theroem 3.1. So, these parts are essentially sketched to avoid unnecessary repeat. Let us begin the proof. Assume $u(t)$ is a positive solution of the half-linear fractional ∇-difference boundary value problem (7). Hence, there is a $t_0 \in \mathbb{N}_a^b$ such that

$$m = u(t_0) = \max\left\{u(t) \middle| \, t \in \mathbb{N}_a^b\right\}.$$

Thanks to the composition rule (N_2) in Lemma 2.1 in combination with the boundary conditions

$$\nabla_{a^+}^{\alpha+k-n} u(a) = 0, \quad k = 0, 1, 2, ..., n-1,$$

we get that

$$m < \max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\} \sum_a^b p_1(t)^{-\frac{1}{\beta_1+1}} \left(p_1(t)^{\frac{1}{\beta_1+1}} \left|\nabla_{a^+}^{\alpha} u(t)\right|\right). \tag{37}$$

The proof procedure goes ahead with the use of the discrete Hoder inequality (15) and choosing the setting

$$p = 1 + \frac{1}{\beta_1+1}, \quad q = \beta_1 + 1.$$

In this case, keeping in mind the definition of sigend-power operators we arrive at the following inequality

$$\left(\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1} < \left(\sum_a^b p_1(t)^{-\dfrac{1}{\beta_1}}\right)^{\beta_1} \left(\sum_a^b \nabla_{a^+}^{\alpha} u(t)\left\{p_1(t)\Theta_{\beta_1}\left(\nabla_{a^+}^{\alpha} u\right)(t)\right\}\right). \tag{38}$$

Here is the place that we have to use the fractional ∇-summation by parts rule on the right hand side of the inequality (38). In this case, we conclude the following inequality

$$\left(\frac{m}{\max\left\{1, \dfrac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}}\right)^{\beta_1+1} \frac{1}{\left(\sum_a^b p_1(t)^{-\dfrac{1}{\beta_1}}\right)^{\beta_1}} < \sum_a^b u(t)\nabla_{b_-}^{\alpha}\left\{p_1(t)\Theta_{\beta_1}\left(\nabla_{a^+}^{\alpha} u\right)(t)\right\}. \tag{39}$$

In continuation we need to step back into the governing half-linear fractional ∇-difference boundary value problem (7). Applying $\nabla_{a^+}^{-\alpha}$ on both sides of (7) and then imposing the boundary conditions

$$\nabla_{a^+}^{\alpha+k-n}\left(p_2\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(p_1\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}\right)(a) = 0, \quad k = 0, 1, 2, ..., n-1,$$

thanks to the composition rule (N_2) in Lemma 2.1, we get that

$$\nabla_{b_-}^{\alpha}\left\{p_1(t)\Theta_{\beta_1}\left(\nabla_{a^+}^{\alpha} u\right)(t)\right\} < m^{\beta_1}\left(\max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right)^{\dfrac{1}{\beta_2}} \times \times p_2(t)^{-\dfrac{1}{\beta_2}} .\Theta_{\beta_2^{-1}}\left(\sum_a^b |q(t)|\right). \tag{40}$$

Substituting the recent inequality (40) into the right hand side of (39), gives us

$$\frac{\left(\max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right)^{-(\beta_1+1)}}{\left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1}} < \left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right) \Theta_{\beta_2^{-1}} \left(\sum_a^b |q(t)|\right). \quad (41)$$

In order to complete the proof we need to use the following identity. Suppose f and g both are real valued functions on $\mathbb{N}_a^b$. In this case,

$$\frac{1}{2}\sum_{s=a}^{b}\sum_{t=a}^{b}(f(t)-f(s))(g(t)-g(s)) = (b-a+1)\sum_a^b f(t)g(t) - \sum_a^b f(t)\sum_a^b g(t).$$

Now, let f and g be decreasing and increasing functions on $\mathbb{N}_a^b$, respectively. Then, one has

$$(b-a+1)\sum_a^b f(t)g(t) \leq \sum_a^b f(t)\sum_a^b g(t). \quad (42)$$

Since p_2 and $\Theta_\nu,\ \nu \in (0,+\infty)$ both are increasing functions, so, $p_2^{-\frac{1}{\beta_2}}$ and $\Theta_\nu,\ \nu \in (0,+\infty)$ are decreasing and increasing functions, respectively. In this case, applying the inequality (42) on (41), we arrive at the following inequality

$$\frac{\left(\max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right)^{-(\beta_1+1)}}{\left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1}} < \left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right) \left(\sum_a^b |q(t)|\right)^{\frac{1}{\beta_2}}. \quad (43)$$

Taking the power of β_2 on both sides of (43) and some manipulation, gives us the conclusion

$$\sum_a^b |q(t)| > \frac{1}{\left\{\max\left\{1, \frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} \left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1\beta_2} \left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right)^{\beta_2}}.$$

So, the proof is completed. □

Remark 3. *Let us consider the generalized half-linear fractional ∇-difference boundary value problem* (7) *and its Lyapunov-type inequality, namely,* (36)*. We note that if we choose the setting $p_1(t) = p_2(t) = 1$, in this case, the half-linear fractional ∇-difference boundary value problem* (7) *coincides into the fractional ∇-difference boundary value problem* (6)*, and consequently, the Lyapunov-type inequality* (36) *coincides into the Lyapunov-type inequality* (16)*.*

We finalize this section with following two interesting corollaries. Let us consider the case when the lower terminal a^+ is replaced with the upper terminal b_- in the half-linear fractional ∇-difference boundary value problems (6) and (7), and vice versa. In this case, relying on the monotonicity properties

$$\Delta_t (t-s+1)^{\overline{\alpha-1}} > 0, \quad 0<\alpha<1, \quad s \le t, \tag{44}$$

$$\Delta_s (t-s+1)^{\overline{\alpha-1}} < 0, \quad 0<\alpha<1, \quad s \le t, \tag{45}$$

$$\Delta_t (t-s+1)^{\overline{\alpha-1}} \le 0, \quad \alpha>1, \quad a \le s \le t \le b, \tag{46}$$

$$\Delta_s (t-s+1)^{\overline{\alpha-1}} \ge 0, \quad \alpha>1, \quad a \le s \le t \le b, \tag{47}$$

it can be proved that both of the Lyapunov-type inequalities (16) and (36) are also valid for the new terminal setting, respectively. If we choose the special case $0<\alpha\le 1$, then, the Lyapunov-type inequalities (16) and (36) are reduced to the following Lyapunov-type inequalities:

$$\sum_a^b |q(t)| > \left(\frac{1}{b-a+1}\right)^{\beta_2(\beta_1+1)}, \tag{48}$$

$$\sum_a^b |q(t)| > \left(\frac{1}{\left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1}\left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right)}\right)^{\beta_2}, \tag{49}$$

respectively. Let us consider the case when $0<\alpha\le 1$, and $\beta_k=1,\ k=1,2$. In this case, Corollary 3 gives us the following simpler Lyapunov-type inequalities:

$$\sum_a^b |q(t)| > \left(\frac{1}{b-a+1}\right)^2, \tag{50}$$

$$\sum_a^b |q(t)| > \frac{1}{\sum_a^b p_1(t)\sum_a^b p_2(t)}, \tag{51}$$

We have to notice this fact that in the case $\beta_k=0,\ k=1,2$, both of the boundary value problems (6) and (7) lose their half-linearity and therefore we are just dealt with two classes of the 3-layer *nabla*-difference boundary value problems of fractional-order.

4. Applications

In this section the applicability of the Lyapunov-type inequalities (16) and (36) will be examined. In this way, two applications for these Lyapunov-type inequalities is presented. The first application turns to the key role of Lyapunov-type inequalities to identify the eigenvalue region of the corresponding eigenvalue problems.

Consider the half-linear fractional ∇-difference eigenvalue problem

$$\begin{cases} \nabla_{a^+}^{\alpha}\left(\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}\right)(t)-\lambda(\Theta_{\beta_1\beta_2}u)(t)=0, \\ n-1<\alpha\le n,\ n\in\mathbb{N}, \\ \\ \nabla_{a^+}^{\alpha+k-n}\left(\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}\right)(a)=0, \quad \nabla_{a^+}^{\alpha+k-n}u(a)=0, \\ k=0,1,2,...,n-1. \end{cases} \tag{52}$$

Making use of the Lyapunov-type inequality (16), the eigenvalues of this fractional eigenvalue problem are distributed within the infinite interval $\mathbb{R}-[-\Lambda,\Lambda]$, where

$$\Lambda=\frac{1}{\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1}.(b-a+1)^{\beta_2(\beta_1+1)}}. \tag{53}$$

Similarly, the eigenvalue region of the half-linear fractional ∇-difference eigenvalue problem

$$\begin{cases} \nabla_{a^+}^{\alpha}\left(p_2\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(p_1\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}\right)(t)-\lambda(\Theta_{\beta_1\beta_2}u)(t)=0, \\ n-1<\alpha\le n,\ n\in\mathbb{N}, \\ \\ \nabla_{a^+}^{\alpha+k-n}\left(p_2\Theta_{\beta_2}\left\{\nabla_{b_-}^{\alpha}\left(p_1\Theta_{\beta_1}\left\{\nabla_{a^+}^{\alpha}u\right\}\right)\right\}\right)(a)=0, \quad \nabla_{a^+}^{\alpha+k-n}u(a)=0, \\ k=0,1,2,...,n-1, \end{cases} \tag{54}$$

is identified as $\mathbb{R}-[-\Xi,\Xi]$, where

$$\Xi=\frac{1}{\left\{\max\left\{1,\frac{(b-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1}\left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1\beta_2}\left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right)^{\beta_2}}. \tag{55}$$

Let us consider the special case $0<\alpha\le 1$. In this case, according to the Corollary 3, we have the following eigenvalue regions for the fractional eigenvalue problems (52) and (54), respectively.

$$\mathbb{R}-[-\Lambda,\Lambda], \quad \Lambda=\left(\frac{1}{b-a+1}\right)^{\beta_2(\beta_1+1)}, \tag{56}$$

$$\mathbb{R}-[-\Xi,\Xi], \quad \Xi=\left(\frac{1}{\left(\sum_a^b p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1}\left(\sum_a^b p_2(t)^{-\frac{1}{\beta_2}}\right)}\right)^{\beta_2}. \tag{57}$$

Choosing the numerical setting $a = 1,\ b = 3,\ \beta_1 = \beta_2 = \beta \in (0,1)$ for (56) and $\beta_1, \beta_2 \in (0,1),\ p_1(t) = \sin(t),\ p_2(t) = \exp(t)$ for (57), the corresponding eigenvalue regions are illustrated as Figure 2 below.

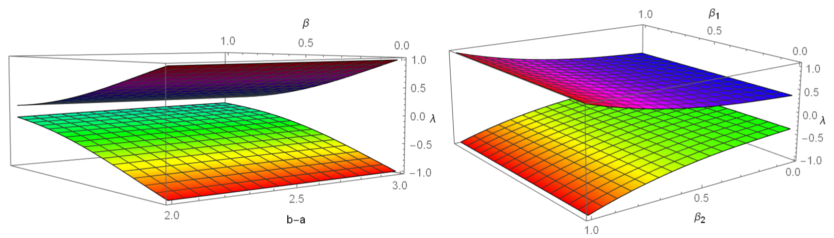

FIGURE 2. Left: eigenvalue distribuation of the eigenvalue problem (52), Right: eigenvalue distribuation of the eigenvalue problem (54). The spaces restricted between surfaces do not include any eigenvalue

As second application of the Lyapunov-type inequalities (16) and (36), one may derive nonexistence criteria for the half-linear fractional ∇-difference boundary value problems (6) and (7). In this way, we have the following applications. Suppse $q(t)$ is a real valued function on $\mathbb{N}_{t_1}^{t_2}$, for which

$$t_1, t_2 \in \mathbb{R}, \quad t_2 \equiv t_1 (\text{mod } 1), \quad t_1 \geq 1,\ t_2 \geq 3, \quad t_2 - t_1 \geq 2.$$

Furthermore, assume tha the following finite sum inequality is satisfied:

$$\sum_{t_1}^{t_2} |q(t)| \leq \frac{1}{\left\{\max\left\{1, \dfrac{(t_2 - t_1 + 1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} . (t_2 - t_1 + 1)^{\beta_2(\beta_1+1)}}. \quad (58)$$

In this case, the half-linear fractional ∇-difference boundary value problem (6) has no nontrivial solution on $\mathbb{N}_{t_1}^{t_2}$.

Proof. The proof is presented based on a contradiction. So, suppose on the contrary that the half-linear fractional ∇-difference boundary value problem (6) has at least one nontrivial solution on $\mathbb{N}_{t_1}^{t_2}$. In this case, according to the above assumptions and thanks to Theorem 3.1, we have the following Lyapunov-type inequality:

$$\sum_{t_1}^{t_2} |q(t)| > \frac{1}{\left\{\max\left\{1, \dfrac{(t_2 - t_1 + 1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} . (t_2 - t_1 + 1)^{\beta_2(\beta_1+1)}}. \quad (59)$$

Since the recent Lyapunov-type inequality contradicts the assumed inequality (58), it has proven that the contradictory assumption of existence of at least one nontrivial solution for fractional boundary value problem (6) is invalied and consequently, under the aforementioned assumption, there is no nontrivial solution for the fractional boundary value problem (6) on $\mathbb{N}_{t_1}^{t_2}$. □

If we consider the generalized half-linear fractional ∇-difference boundary value problem (7) instead of (6), in the similar way one may derive the following nonexistence criterion.
Suppse $q(t)$ is a real valued function on $\mathbb{N}_{t_1}^{t_2}$, for which

$$t_1, t_2 \in \mathbb{R}, \quad t_2 \equiv t_1 (\text{mod } 1), \quad t_1 \geq 1,\ t_2 \geq 3, \quad t_2 - t_1 \geq 2.$$

Furthermore, assume tha the following finite sum inequality is satisfied:

$$\sum_{t_1}^{t_2} |q(t)| \leq \frac{1}{\left\{\max\left\{1, \frac{(t_2-t_1+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right\}\right\}^{\beta_2(\beta_1+1)+1} \left(\sum_{t_1}^{t_2} p_1(t)^{-\frac{1}{\beta_1}}\right)^{\beta_1\beta_2} \left(\sum_{t_1}^{t_2} p_2(t)^{-\frac{1}{\beta_2}}\right)^{\beta_2}}. \tag{60}$$

In this case, the half-linear fractional ∇-difference boundary value problem (7) has no nontrivial solution on $\mathbb{N}_{t_1}^{t_2}$.

5. Concluding Remarks

Here we are in such a position to summarize the outline of this article. In this investigation, two classes of half-linear 3-layer fractional ∇-difference equations with fully left sided boundary conditions have chosen to be investigated. Thanks to the discrete Holder inequality and fractional ∇-difference summation by parts, Lyapunov-type inequalities of these boundary value problems have successfully been extracted. The obtained Lyapunov-type inequalities enabled us to study some qualitative behaviour of the under consideration half-linear problems. The first step turns to the distribution of the eigenvalues for the corresponding half-linear fractional ∇-difference eigenvalue problems, and second step is to present nonexistence results for the nontrivial solutions of the half-linear boundary value problems under investigation.

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