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Research Article:

Generalized T-Norms, T-Co-Norms, and Neutrosophic Inner Product Spaces: A Comprehensive Overview

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Abstract

In this study, we delve into the fundamental concepts of generalized t-norm and t-co-norm, which play a pivotal role in the advanced mathematical framework for handling uncertainties. These definitions are meticulously detailed, serving as the cornerstone for the reconceptualization of fuzzy inner product spaces and intuitionistic fuzzy inner product spaces. By extending these concepts, we pave the way for the introduction of neutrosophic inner product spaces (NIPSs), a novel mathematical structure that accommodates the indeterminate and inconsistent information inherent in real-world problems. The newly defined NIPSs give rise to the exploration and definition of several associated properties, further enriching the theory. Additionally, the formulation and analysis of the parallelogram law within this neutrosophic context is discussed, highlighting its implications and potential applications. This paper aims to contribute to the ongoing research in fuzzy and neutrosophic systems by providing a robust mathematical foundation for future studies and practical implementations.

1. Introduction

Hilbert's space holds a significant role within functional analysis and has widespread applications in diverse fields, including representation theory concerning $\mathbb{C}^*$-algebras, non-commutative topology, and non-commutative geometry, as evidenced in (Connnes et al. 1998). Its prowess extends to mathematical physics, particularly in the realm of quantum logic, as elucidated in (Birkhoff & Neumann, 1975; Chiara et al. 2008).

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In the domain of strong quantum gravity, the measurement of space-time points assumes a nebulous character. Consequently, the determination of point locations becomes elusive, resulting in a fuzzy structure within the fabric of space-time (El Naschie,1997; El Naschie,1998). This fuzzy structure disrupts the conventional position space representation of quantum mechanics, necessitating the employment of a generalized Hilbert space defined by quasi-position eigenfunctions, a concept introduced in (Muralikrishna & Kumar, 2019). Consequently, a new array of fuzzy norms comes under consideration.

The inception of intuitionistic fuzzy normed spaces is attributed to Saadati and Park (Saadati & Park,2006), who pioneered the notation in this area. Subsequently, various authors, including (Bag & Samanta, 2003; Bag & Samanta, 2005; Bag & Samanta, 2013), explored intuitionistic fuzzy normed linear spaces. However, the realm of intuitionistic fuzzy inner product spaces remains largely uncharted, presenting a prominent challenge in intuitionistic fuzzy analysis - the identification of an intuitionistic fuzzy inner product. Such inner products find extensive applications in intuitionistic fuzzy optimization and related domains. It is noteworthy that many existing definitions of fuzzy inner products and intuitionistic fuzzy inner products rely on operations like min, $\vee, \wedge$, or max, which are not without shortcomings (El-Abyad & El-Hamouly, 1991; Goudarzi et al., 2009; Kirişci & Şimşek, 2020; Majumdar & Samanta, 2008).

Recently, the concept of the neutrosophic set (NS) has garnered attention and found numerous applications in various fields see (Elrawy, 2022; Elrawy & Abdalla, 2023; Elrawy et al., 2023 ; Nozari & Fazlpour, 2007; Şahin & Küçük, 2018; Smarandache, 2005). The neutrosophic normed space concept represents a natural extension of both fuzzy normed space and intuitionistic fuzzy normed space. Recently, in (Bera & Mahapatra, 2018), the notion of neutrosophic normed space was introduced as a broader framework encompassing fuzzy normed spaces and intuitionistic fuzzy normed spaces. In that work, they conducted an in-depth exploration of convergence and completeness properties within these spaces.

This paper aims to make two significant contributions. First, we define generalized t-norm and t-co-norm operators, which, in turn, allow us to redefine fuzzy inner product and intuitionistic fuzzy inner product in a more general context. All previous definitions of these concepts can be seen as special cases derived from these new definitions. Second, we introduce and explore the concept of NIPSs, a natural outcome of this investigation. Notably, this concept lays the foundation for defining the most crucial concept of neutrosophic Hilbert spaces.

Except for the introduction, this paper comprises four main sections. In Section 2, we revisit some fundamental definitions crucial for the entire discourse. Section 3 delves into the specifics of generalized t-norm and t-co-norm operators. In Section 4, we present and analyze the concept of NIPSs. Finally, in Section 5, we draw conclusions based on our findings.

2. Basic concepts

Here, we briefly revisit fundamental definitions and outcomes that are essential for the next sections. We assume that X is a linear space over any field.

Definition 2.1. (Osman,1987; Issac & Maya, 2012) A mapping $\ominus: [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous t-norm when the following hold:

(i) $\ominus$ is commutative, continuous, and associative,

(ii) $\ell \ominus 1 = \ell$.

(iii)$\ell \ominus b \leq c \ominus d, b \leq d, \ell \leq c,\ \ \ell, b, c, d \in [0,1]$.

Definition 2.2. (Kirişci & Şimşek, 2020; Majumdar & Samanta, 2011) A mapping $\odot: [0,1] \times [0,1] \longrightarrow [0,1]$ is say a continuous t-co-norm when the next are held:

(i) $\odot$ is commutative, associative, and continuous.

(ii) $\ell \odot 0 = \ell$.
(iii) $\ell \odot b \leq c \odot d, b \leq d, \ell \leq c, \ \ell, b, c, d \in [0,1]$.

Definition 2.3. (Singh, 2019) An NS $\mathcal{N}$ over $\mathbb{N}$ is introduced as

$$\mathcal{N} = \{< k, \mathcal{N}_1(k), \mathcal{N}_2(k), \mathcal{N}_3(k) >: k \in \mathbb{N}\},$$

where $\mathcal{N}_1(k), \mathcal{N}_2(k), \mathcal{N}_3(k) \longrightarrow [0,1]$.

Definition 2.4. (Majumdar & Samanta, 2011) The NS $\mathcal{N}$ on a linear space U over $\mathbb{R}$ is called a neutrosophic norm $(U, \mathbb{R}, \ominus, \odot)$ when the next axioms hold:

1) $0 \leq \mathcal{N}_1(\ell, \tau), \mathcal{N}_2(\ell, \tau), \mathcal{N}_3(\ell, \tau) \leq 1$.
2) $0 \leq \mathcal{N}_1(\ell, \tau) + \mathcal{N}_2(\ell, \tau) + \mathcal{N}_3(\ell, \tau) \leq 3$.
3) $\mathcal{N}_1(\ell, \tau) = 0,\ \mathcal{N}_2(\ell, \tau) = 1, and\ \mathcal{N}_3(\ell, \tau) = 1$ with $\tau \leq 0$.
4) $\mathcal{N}_1(\ell, \tau) = 1, \mathcal{N}_2(\ell, \tau) = 0$ and $\mathcal{N}_3(\ell, \tau) = 0$ with $\tau > 0$ iff $\ell = 0$.
5) $\mathcal{N}_1(c\ell, \tau) = \mathcal{N}_1\left(\ell, \frac{\tau}{|c|}\right),\ \mathcal{N}_2(c\ell, \tau) = \mathcal{N}_2\left(\ell, \frac{\tau}{|c|}\right)$ and $\mathcal{N}_3(c\ell, \tau) = \mathcal{N}_3\left(\ell, \frac{\tau}{|c|}\right) \forall c \neq 0$, $\tau > 0$.
6) $\mathcal{N}_1(\ell, s) \ominus \mathcal{N}_1(b, \tau) \leq \mathcal{N}_1(\ell + b, s + \tau)\ \ \forall s, \tau \in \mathbb{R}$.
7) $\mathcal{N}_1(\ell, .)$ is continuous non-decreasing function for $\tau > 0$, $\lim_{\tau\to\infty} \mathcal{N}_1(\ell, \tau) = 1$.
8) $\mathcal{N}_2(\ell, s) \odot \mathcal{N}_2(b, \tau) \geq \mathcal{N}_2(\ell + b, s + \tau)$.
9) $\mathcal{N}_2(\ell, .)$ is continuous non-increasing function for $\tau > 0$, $\lim_{\tau\to\infty} \mathcal{N}_2(\ell, \tau) = 0$.
10) $\mathcal{N}_3(\ell, s) \odot \mathcal{N}_3(b, \tau) \geq \mathcal{N}_3(\ell + b, s + \tau)$.
11) $\mathcal{N}_3(\ell, .)$ is continuous non-increasing function for $\tau > 0$, $\lim_{\tau\to\infty} \mathcal{N}_3(\ell, \tau) = 0$.

3. Generalized t-norm and t-co-norm

In this section, we define a generalized t-norm and generalize t-conorm which generalize (Majumdar & Samanta, 2011; Issac & Maya, 2012; Kirişci & Şimşek, 2020). Also, we use this concept to redefine fuzzy inner product and intuitionistic fuzzy inner product space.

Definition 3.1. A binary $\star: [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous generalized t-norm when the following satisfied:
(i) $\star$ is commutative, associative, and continuous.
(ii) $a \star x = \begin{cases} a & \text{if } x = 1 \\ x, & \text{otherwise} \end{cases}, \forall a, x \in [0,1]$.
(iii) $a \star b \leq c \star d, for\ b \leq d, a \leq c,\ a, b, c, d \in [0,1]$.

Definition 3.2. A binary $\circledast: [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous generalized co-t-norm when the following satisfied:
(i) $\circledast$ is commutative, associative, and continuous.
(ii) $a \circledast x = \begin{cases} a & \text{if } x = 0 \\ x, & \text{otherwise} \end{cases}, \forall a, x \in [0,1]$.
(iii) $a \circledast b \leq c \circledast d,\ for\ b \leq d, a \leq c,\ a, b, c, d \in [0,1]$.

According to **Definition 3.1.** and **Definition 3.2.**, we can redefine the fuzzy inner product and the intuitionistic fuzzy inner product as follows:

Definition 3.3. The fuzzy subset (Y, μ) is said to be a fuzzy real inner product space, if $\forall a, b, c \in Y$ and $\tau, s, k \in R$, with $\star$ and satisfy the following:

FI1) $0 \leq \mu(a, b, \tau) \leq 1 \quad \forall \tau < 0$,

FI2) $\mu(a + b, c, \tau + s) \geq \mu(a, c, \tau) \star \mu(b, c, s)$,

FI3) $\mu(a, b, s\ \tau) \geq \mu(a, a, s^2) \star \mu(b, b, \tau^2)$,

FI4) $\mu(a, b, \tau) = \mu(b, a, \tau)$,

FI5) $\mu(ka, b, \tau) = \mu(a, b, \frac{\tau}{|k|})$,

FI6) $\lim_{\tau\to+\infty} \mu(a, b, \tau) = 1$,

FI7) $\mu(a, a, \tau) = 1\ (\tau > 0)$ iff $a = 0$.
Where $\mu: Y \times Y \times \mathbb{R} \to [0,1]$.

Example 3.1. Consider $(Y, <..,..>)$ is inner product space and define a fuzzy subset $\mu: Y \times Y \times \mathbb{R} \to [0,1]$ by $\mu(a, b, \tau) = \begin{cases} 0 & \text{if } \tau \leq |< a, b >| \\ 1 & \text{if } \tau > |< a, b >| \end{cases}$ with $\mu(a) \star \mu(b) = \mu(a) \wedge \mu(b)$. Then (Y, μ) is an FIPS.

Definition 3.4. Let $\mu,\ \mu^{\star}: Y \times Y \times \mathbb{C} \to [0,1]$ be two mappings and $\forall a,b,c \in Y$ and $\tau, s \in \mathbb{C}$, with $\star,\ \circledast$. Then (Y,μ,μ^*) is said to be an intuitionistic fuzzy inner product space if the following are satisfied:

IFI1) $\mu(a,b,\tau)+\mu^*(a,b,\tau)\le 1$,

IFI2) $\mu(a+b,c,|\tau|+|s|)\ge \mu(a,c,|\tau|)\star\mu(b,c,|s|)$,

IFI3) $\mu(a,b,|s\tau|)\ge \mu(a,a,|s|^2)\star\mu(b,b,|\tau|^2)$,

IFI4) $\mu(a,b,\tau)=\mu(b,a,\bar{\tau})$,

IFI5) $\mu(\alpha a,b,\tau)=\mu\left(a,b,\frac{\tau}{|\alpha|}\right), \alpha(\neq 0)\in\mathbb{C}$,

IFI6) $\mu(a,a,\tau)=1 \forall\ \tau>0$ iff $a=\underline{0}$,

IFI7) $\mu(a,a,\bullet): R\to I=[0,1]$ is a monotonic non-decreasing function and $\lim_{\tau\to\infty}\mu(a,a,\tau)=1$,

IFI8) $\mu^*(a+b,c,|\tau|+|s|)\le \mu^*(a,c,|\tau|)\circledast \mu^*(b,c,|s|)$,

IFI9) $\mu^*(a,b,|s\tau|)\le \mu^*(a,a,|s|^2)\circledast\mu^*(b,b,|\tau|^2)$,

IFI10) $\mu^*(a,b,\tau)=\mu^*(b,a,\bar{\tau})$,

IFI11) $\mu^*(\alpha a,b,\tau)=\mu^*\left(a,b,\frac{\tau}{|\alpha|}\right), \alpha(\neq 0)\in\mathbb{C}$

IFI12) $\forall\tau\in\mathbb{C}\backslash R^+, \mu^*(a,a,\tau)=1$,

IFI13) $\mu^*(a,a,\bullet): R\to I=[0,1]$ is a monotonic non-increasing function and $\lim_{\tau\to\infty}\mu^*(a,a,\tau)=0$.

4. Neutrosophic inner product space

In this section, we establish the notion of neutrosophic inner product space over a real linear space. Also, we shall present some properties and results.

Definition 4.1. Presume $\mathcal{H}_{\mathcal{N}}=(\mathcal{H}_{\mathcal{N}_1},\mathcal{H}_{\mathcal{N}_2},\mathcal{H}_{\mathcal{N}_3})$, where.

$\mathcal{H}_{\mathcal{N}_1}: =< \mathrm{x},\mathrm{y},\tau >_{\mathcal{N}_1}: Y\times Y\times\mathbb{R}^+\to[0,1]$,
$\mathcal{H}_{\mathcal{N}_2}: =< \mathrm{x},\mathrm{y},\tau >_{\mathcal{N}_2}: Y\times Y\times\mathbb{R}^+\to[0,1]$,
$\mathcal{H}_{\mathcal{N}_3}: =< \mathrm{x},\mathrm{y},\tau >_{\mathcal{N}_3}: Y\times Y\times\mathbb{R}^+\to[0,1]$,

with $\star$ and $\circledast$ continuous generalize t-co-norm. Then $(Y,\mathcal{H}_{\mathcal{N}})$ is called an NIPSs if for all $x,y,z\in Y, \tau,s\in\mathbb{R}^+$ the next axiom is held:

1. $0\le \mathcal{H}_{\mathcal{N}_1},\mathcal{H}_{\mathcal{N}_2},\mathcal{H}_{\mathcal{N}_3}\le 1$.
2. $0\le \mathcal{H}_{\mathcal{N}_1}+\mathcal{H}_{\mathcal{N}_2}+\mathcal{H}_{\mathcal{N}_3}\le 3$.
3. $< x,x,\tau >_{\mathcal{N}_1}=0, < x,x,\tau >_{\mathcal{N}_2}=1$ and $< x,x,\tau >_{\mathcal{N}_3}=1$ with $\tau\le 0$.
4. $< x,x,\tau >_{\mathcal{N}_1}=1, < x,x,\tau >_{\mathcal{N}_2}=0$ and $< x,x,\tau >_{\mathcal{N}_3}=0$ with $\tau>0$ iff $x=0$.
5. $< \lambda x,y,\tau >_{\mathcal{N}_1}=< x,y,\frac{\tau}{|\lambda|} >_{\mathcal{N}_1}, < \lambda x,y,\tau >_{\mathcal{N}_2}=< x,y,\frac{\tau}{|\lambda|} >_{\mathcal{N}_2}$, *and* $< \lambda x,y,\tau >_{\mathcal{N}_3}=< x,y,\frac{\tau}{|\lambda|} >_{\mathcal{N}_3}\ \forall\lambda\neq 0$, $\tau>0$.
6. $< x+y,z,\tau+s >_{\mathcal{N}_1}\ge < x,z,\tau >_{\mathcal{N}_1}\star < y,z,s >_{\mathcal{N}_1}$ $\forall s,\tau\in\mathbb{R}$.
7. $\overline{< x,y,\tau >_{\mathcal{N}_1}}=< y,x,\tau >_{\mathcal{N}_1}$, $\overline{< x,y,\tau >_{\mathcal{N}_2}}=< y,x,\tau >_{\mathcal{N}_2}$, and $\overline{< x,y,\tau >_{\mathcal{N}_3}}=< y,x,\tau >_{\mathcal{N}_3}$.
8. $< x,x,. >_{\mathcal{N}_1}:\mathbb{R}\to[0,1]$ is a monotonic non-decreasing function of $\mathbb{R}$ and $\lim_{\tau\to\infty} < x,x,\tau >_{\mathcal{N}_1}=1$.
9. $< x+y,z,\tau+s >_{\mathcal{N}_2}\le < x,z,\tau >_{\mathcal{N}_2}\circledast < y,z,s >_{\mathcal{N}_2}\ \ \forall s,\tau\in\mathbb{R}$.
10. $< x,x,. >_{\mathcal{N}_2}:\mathbb{R}\to[0,1]$ is a monotonic non-increasing function of $\mathbb{R}$ and $\lim_{\tau\to\infty} < x,x,\tau >_{\mathcal{N}_2}=0$.
11. $< x+y,z,\tau+s >_{\mathcal{N}_3}\le < x,z,\tau >_{\mathcal{N}_3}\circledast < y,z,s >_{\mathcal{N}_3}\ \forall s,\tau\in\mathbb{R}$.
12. $< x,x,. >_{\mathcal{N}_3}:\mathbb{R}\to[0,1]$ is a monotonic non-increasing function of $\mathbb{R}$ and $\lim_{\tau\to\infty} < x,x,\tau >_{\mathcal{N}_3}=0$.
13. $< x,y,\tau s >_{\mathcal{N}_1}\ge < x,x,\tau^2 >_{\mathcal{N}_1}\star < y,y,s^2 >_{\mathcal{N}_1}$.
14. $< x,y,\tau s >_{\mathcal{N}_2}\le < x,x,\tau^2 >_{\mathcal{N}_2}\circledast < y,y,s^2 >_{\mathcal{N}_2}$.

15. $< x,y,\tau s >_{\mathcal{N}_3} \leq < x,x,\tau^2 >_{\mathcal{N}_3} \circledast < y,y,s^2 >_{\mathcal{N}_3}$.

Example 4.2. Let $\mathcal{H}_{\mathcal{N}_1}, \mathcal{H}_{\mathcal{N}_2}, \mathcal{H}_{\mathcal{N}_3}: Y \times Y \times \mathbb{R} \to [0,1]$ be defined as follows:

$$\mathcal{H}_{\mathcal{N}_1} = \begin{cases} 1 & \text{if } \tau \geq < x,y > \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{H}_{\mathcal{N}_2} = \begin{cases} 1 & \text{if } \tau \leq < x,y > \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{H}_{\mathcal{N}_3} = \begin{cases} 1 & \text{if } \tau \leq < x,y > \\ 0 & \text{otherwise} \end{cases}$$

with $x \star y = x \wedge y$, $x \circledast y = x \vee y$ where $x, y \in [0,1]$. Then $\mathcal{H}_{\mathcal{N}}$ is a NIPS.

Theorem 4.3. Let $\mathcal{N}$ and $(Y, \mathcal{H}_{\mathcal{N}})$ be a neutrosophic norm and a NIPS, respectively. Then

$$\mathcal{N}_1(x,\tau) := \begin{cases} < x,x,\ \tau^2 >_{\mathcal{N}_1} & \text{if } \tau > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{N}_2(x,\tau) := \begin{cases} < x,x,\ \tau^2 >_{\mathcal{N}_2} & \text{if } \tau > 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{N}_3(x,\tau) := \begin{cases} < x,x,\ \tau^2 >_{\mathcal{N}_3} & \text{if } \tau > 0 \\ 1 & \text{otherwise} \end{cases}$$

Proof. In fact, all the conditions of **Definition 2.4** are obvious, but we explain some of these conditions:

$$\mathcal{N}_1(cx,\tau) = < cx, cx,\ \tau^2 >_{\mathcal{N}_1}$$

$$= < cx, x, \frac{\tau^2}{|c|} >_{\mathcal{N}_1}$$

$$= < x, x, \frac{\tau^2}{|c^2|} >_{\mathcal{N}_1}$$

$$= < cx, x, (\frac{\tau^2}{|c|^2}) >_{\mathcal{N}_1}$$

$$= \mathcal{N}_1(x, \frac{\tau}{|c|}).$$

$$\mathcal{N}_1(x+y,\tau+s) = < x+y, x+y, (\tau+s)^2 >_{\mathcal{N}_1}$$

$$= < x+y, x+y, \tau^2 + \tau s + s\tau + s^2 >_{\mathcal{N}_1}$$

$$\geq < x,x,\tau^2 >_{\mathcal{N}_1} \star < x,y,\tau s >_{\mathcal{N}_1} \star < y,x,s\tau >_{\mathcal{N}_1} \star < y,y,s^2 >_{\mathcal{N}_1}$$

$$\geq < x,x,\tau^2 >_{\mathcal{N}_1} \star < x,x,\tau^2 >_{\mathcal{N}_1} \star < x,x,\tau^2 >_{\mathcal{N}_1} \star < y,y,s^2 >_{\mathcal{N}_1} < y,y,s^2 >_{\mathcal{N}_1} \star < y,y,s^2 >_{\mathcal{N}_1}$$

$$\geq < x,x,\tau^2 >_{\mathcal{N}_1} \star < y,y,s^2 >_{\mathcal{N}_1}$$

$$\geq \mathcal{N}_1(x,\tau) \star \mathcal{N}_1(y,s).$$

$$\mathcal{N}_2(x+y,\tau+s) = < x+y, x+y, (\tau+s)^2 >_{\mathcal{N}_2}$$

$$= < x+y, x+y, \tau^2 + \tau s + s\tau + s^2 >_{\mathcal{N}_2}$$

$$\leq < x,x,\tau^2 >_{\mathcal{N}_2} \circledast < x,y,\tau s >_{\mathcal{N}_2} \circledast < y,x,s\tau >_{\mathcal{N}_2} \circledast < y,y,s^2 >_{\mathcal{N}_2}$$

$$\leq < x,x,\tau^2 >_{\mathcal{N}_2} \circledast < x,x,\tau^2 >_{\mathcal{N}_2} \circledast < x,x,\tau^2 >_{\mathcal{N}_2} \circledast$$

$$< y,y,s^2 >_{\mathcal{N}_2} \circledast < y,y,s^2 >_{\mathcal{N}_2} \circledast < y,y,s^2 >_{\mathcal{N}_2}$$

$$\leq < x,x,\tau^2 >_{\mathcal{N}_2} \circledast < y,y,s^2 >_{\mathcal{N}_2}$$

$$\leq \mathcal{N}_2(x,\tau) \circledast \mathcal{N}_2(y,s).$$

Finally,

$$\mathcal{N}_3(x+y,\tau+s) = < x+y, x+y, (\tau+s)^2 >_{\mathcal{N}_3}$$

$$= < x+y, x+y, \tau^2 + \tau s + s\tau + s^2 >_{\mathcal{N}_3}$$

$$\leq < x,x,\tau^2 >_{\mathcal{N}_3} \circledast < x,y,\tau s >_{\mathcal{N}_3} \circledast < y,x,s\tau >_{\mathcal{N}_3} \circledast < y,y,s^2 >_{\mathcal{N}_3}$$

$$\leq < x,x,\tau^2 >_{\mathcal{N}_3} \circledast < x,x,\tau^2 >_{\mathcal{N}_3} \circledast < x,x,\tau^2 >_{\mathcal{N}_3} \circledast < y,y,s^2 >_{\mathcal{N}_3} \circledast < y,y,s^2 >_{\mathcal{N}_3} \circledast < y,y,s^2 >_{\mathcal{N}_3}$$

$$\leq \mathcal{N}_3(x,\tau) \circledast \mathcal{N}_3(y,s).$$

The following proposition illustrates the composition of a family of NIP is NIP.

Proposition 4.4. If $\mathcal{H}^i_{\mathcal{N}}$, $i = 1,2,\ldots,n$ is a family of NIPs on X, then.

$$\mathcal{H}_{\mathcal{N}} := (\mathcal{H}^1_{\mathcal{N}_1} \star \ldots \star \mathcal{H}^n_{\mathcal{N}_1}, \mathcal{H}^1_{\mathcal{N}_2} \circledast \ldots \circledast \mathcal{H}^n_{\mathcal{N}_2}, \mathcal{H}^1_{\mathcal{N}_3} \circledast \ldots \circledast \mathcal{H}^n_{\mathcal{N}_3}),$$

is a NIP on X.

Proof. Let $\mathcal{H}_{\mathcal{N}} = (\mathcal{H}_{\mathcal{N}_1}, \mathcal{H}_{\mathcal{N}_2}, \mathcal{H}_{\mathcal{N}_3})$. First by using **Definition 3.1** and **3.2**, we find.

$$\mathcal{H}_{\mathcal{N}_1} := \mathcal{H}^1_{\mathcal{N}_1} \star \ldots \star \mathcal{H}^n_{\mathcal{N}_1} = \mathcal{H}^i_{\mathcal{N}_1},$$

$$\mathcal{H}_{\mathcal{N}_2} := \mathcal{H}^1_{\mathcal{N}_2} \circledast \ldots \circledast \mathcal{H}^n_{\mathcal{N}_2} = \mathcal{H}^i_{\mathcal{N}_2},$$

$$\mathcal{H}_{\mathcal{N}_3} := \mathcal{H}^1_{\mathcal{N}_3} \circledast \ldots \circledast \mathcal{H}^n_{\mathcal{N}_3} = \mathcal{H}^i_{\mathcal{N}_3},$$

where $i \in \{1,2,\ldots,n\}$. So, we show some (1), (2), (6), (9) and (15) conditions. For $x, y, z \in X$ and $t, s \in \mathbb{R}$. Now,

(1) By **Definition 4.1.** and for some i, we say $0 \leq \mathcal{H}^i_{\mathcal{N}_1}, \mathcal{H}^i_{\mathcal{N}_2}, \mathcal{H}^i_{\mathcal{N}_3} \leq 1$.

(2) By **Definition 4.1.** and for some i, we say $0 \leq \mathcal{H}^i_{\mathcal{N}_1} + \mathcal{H}^i_{\mathcal{N}_2} + \mathcal{H}^i_{\mathcal{N}_3} \leq 3$.

(6) $< x+y, z, \tau+s >_{\mathcal{N}_1} = < x+y, z, \tau+s >_{\mathcal{N}_1} \star < x+y, z, \tau+s >_{\mathcal{N}_1} \star \ldots \star < x+y, z, \tau+s >^n_{\mathcal{N}_1}$

$$= \left[< x,z,\tau >^1_{\mathcal{N}_1} \star \ldots \star < x,z,\tau >^n_{\mathcal{N}_1}\right] \star \ldots \star \left[< y,z,s >^1_{\mathcal{N}_1} \star \ldots \star < y,z,s >^n_{\mathcal{N}_1}\right]$$

$$\geq [< x,z,t >^1_{\mathcal{N}_1} \star < y,z,s >^1_{\mathcal{N}_1}] \star \ldots \star [< x,z,t >^n_{\mathcal{N}_1} \star < y,z,st >^n_{\mathcal{N}_1}]$$

$$= < x,z,\tau >^n_{\mathcal{N}_1} \star < y,z,s >^n_{\mathcal{N}_1}.$$

(9) $< x+y, z, \tau+s >_{\mathcal{N}_2} = < x+y, z, \tau+s >^1_{\mathcal{N}_2} \circledast \ldots \circledast < x+y, z, \tau+s >^n_{\mathcal{N}_2}$

$$\leq [< x,z,\tau >^1_{\mathcal{N}_2} \circledast < y,z,s >^1_{\mathcal{N}_2}] \circledast \ldots \circledast [< x,z,\tau >^n_{\mathcal{N}_2} \circledast < y,z,s\tau >^n_{\mathcal{N}_2}]$$

$$= [< x,z,\tau >^1_{\mathcal{N}_2} \circledast \ldots \circledast < x,z,\tau >^n_{\mathcal{N}_2}] \circledast \ldots \circledast [< y,z,s >^1_{\mathcal{N}_2} \circledast \ldots \circledast < y,z,s >^n_{\mathcal{N}_2}]$$

$$= < x,z,t >^n_{\mathcal{N}_2} < y,z,s >^n_{\mathcal{N}_2}.$$

Finally, we show.

(15) $< x,y,\tau s >_{\mathcal{N}_3} = < x,y,\tau s >^1_{\mathcal{N}_3} \circledast \ldots \circledast < x,y,\tau s >^n_{\mathcal{N}_3}$

$$\leq \left[< x,x,\tau^2 >^1_{\mathcal{N}_3} \circledast < y,y,s^2 >^1_{\mathcal{N}_3}\right] \circledast \ldots \circledast \left[< x,x,\tau^2 >^n_{\mathcal{N}_3} \circledast < y,y,s^2 >^n_{\mathcal{N}_3}\right]$$

$$= \left[< x,x,\tau^2 >^1_{\mathcal{N}_3} \circledast \ldots \circledast < x,x,\tau^2 >^n_{\mathcal{N}_3}\right] \circledast \ldots \circledast \left[< y,y,s^2 >^1_{\mathcal{N}_3} \circledast \ldots \circledast < y,y,s^2 >^n_{\mathcal{N}_3}\right]$$

$$= < x,x,\tau^2 >^n_{\mathcal{N}_3} \star < y,y,s^2 >^n_{\mathcal{N}_3}.$$

5. let $(Y, \mathcal{H}_{\mathcal{N}})$ be NIPS. Then the following are valid

(i) For all $x, b, z \in Y$ and $s, \tau \in \mathbb{R}$

$$< x, b+z, \tau+s >_{\mathcal{N}_1} \geq < x,b,\tau >_{\mathcal{N}_1} \star < x,z,s >_{\mathcal{N}_1},$$

$$< x, b+z, \tau+s >_{\mathcal{N}_2} \leq < x,b,\tau >_{\mathcal{N}_2} \circledast < x,z,s >_{\mathcal{N}_2},$$

$$< x, b+z, \tau+s >_{\mathcal{N}_3} \leq < x,b,\tau >_{\mathcal{N}_3} \circledast < x,z,s >_{\mathcal{N}_3}.$$

(ii) For $c \neq 0$,

$$< cx, b, \tau >_{\mathcal{N}_1} = < x, cb, \tau >_{\mathcal{N}_1},$$

$$< cx, b, \tau >_{\mathcal{N}_2} = < x, cb, \tau >_{\mathcal{N}_2},$$

$$< cx, b, \tau >_{\mathcal{N}_3} = < x, cb, \tau >_{\mathcal{N}_3}.$$

Proof. The proof is as a previous approach.

We will invest **Theorem 4.3** and **Proposition 4.4** to define neutrosophic α-norms generated from NIPS.

Definition 4.6. Let $(Y, \mathcal{H}_{\mathcal{N}})$ be a NIPS, we define α-norms generated from NIPS on X as follows:

$$\| y \|^{\mathcal{N}_1}_{\alpha} = \text{Inf}\{\tau > 0 : < y,y,\tau^2 >_{\mathcal{N}_1} \geq 1-\alpha\},$$
$$\| y \|^{\mathcal{N}_2}_{\alpha} = \text{Sup}\{\tau > 0 : < y,y,\tau^2 >_{\mathcal{N}_2} \leq \alpha\},$$
$$\| y \|^{\mathcal{N}_3}_{\alpha} = \text{Sup}\{\tau > 0 : < y,y,\tau^2 >_{\mathcal{N}_3} \leq \alpha\},$$

with $\alpha \in (0,1)$, $y \in Y$ and $\tau \in \mathbb{R}$.

In the follow-up, the following axioms must be observed.

I. $< x+y, x+y, 2\tau^2 >_{\mathcal{N}_1} \star < x-y, x-y, 2s^2 >_{\mathcal{N}_1} \geq < x, x, \tau^2 >_{\mathcal{N}_1} \star < y, y, s^2 >_{\mathcal{N}_1}$.

II. $< x+y, x+y, 2\tau^2 >_{\mathcal{N}_2} \circledast < x-y, x-y, 2s^2 >_{\mathcal{N}_2} \leq < x, x, \tau^2 >_{\mathcal{N}_2} \circledast < y, y, s^2 >_{\mathcal{N}_2}$.

III. $< x+y, x+y, 2\tau^2 >_{\mathcal{N}_3} \circledast < x-y, x-y, 2s^2 >_{\mathcal{N}_3} \leq < x, x, \tau^2 >_{\mathcal{N}_3} \circledast < y, y, s^2 >_{\mathcal{N}_3}$.

Next proposition obvious the Parallelogram law in NIPS.

Proposition 4.7. Let $\mathcal{H}_{\mathcal{N}}$ be a NIPS on Y with $\alpha \in (0,1)$, and $(\| y \|_\alpha^{\mathcal{N}_1}, \| y \|_\alpha^{\mathcal{N}_2}, \| y \|_\alpha^{\mathcal{N}_3})$ be the neutrosophic α-norms on Y, then

$$\left(\| a-b \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| a+b \|_\alpha^{\mathcal{N}_1}\right)^2 = 2[\left(\| a \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| b \|_\alpha^{\mathcal{N}_1}\right)^2],$$
$$\left(\| a-b \|_\alpha^{\mathcal{N}_2}\right)^2 + \left(\| a+b \|_\alpha^{\mathcal{N}_2}\right)^2 = 2[\left(\| a \|_\alpha^{\mathcal{N}_2}\right)^2 + \left(\| b \|_\alpha^{\mathcal{N}_2}\right)^2],$$
$$\left(\| a-b \|_\alpha^{\mathcal{N}_3}\right)^2 + \left(\| a+b \|_\alpha^{\mathcal{N}_3}\right)^2 = 2[\left(\| a \|_\alpha^{\mathcal{N}_3}\right)^2 + \left(\| b \|_\alpha^{\mathcal{N}_3}\right)^2].$$

Proof. Firstly, for $\tau, s \in \mathbb{R}^+$

$$\left(\| a-b \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| a+b \|_\alpha^{\mathcal{N}_1}\right)^2 = [\star\{\tau^2 : \mathcal{N}_1(a-b,\tau) \geq 1-\alpha\}] + [\star\{s^2 : \mathcal{N}_1(a+b,s) \geq 1-\alpha\}]$$

$$= \star\{\tau^2 + s^2 : \mathcal{N}_1(a-b,\tau) \geq 1-\alpha,\ \mathcal{N}_1(a+b,s) \geq 1-\alpha\} \qquad (1)$$

Also, the other side for $p, q \in \mathbb{R}^+$

$$2\left[\left(\| a \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| b \|_\alpha^{\mathcal{N}_1}\right)^2\right] = 2([\star\{p^2 : \mathcal{N}_1(a,p) \geq 1-\alpha\}] + [\star\{q^2 : \mathcal{N}_1(b,q) \geq 1-\alpha\}])$$

$$= 2(\star\{p^2 + q^2 : \mathcal{N}_1(x,p) \geq 1-\alpha,\ \mathcal{N}_1(y,q) \geq 1-\alpha\}). \qquad (2)$$

Therefore

$$\left(\| a-b \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| a+b \|_\alpha^{\mathcal{N}_1}\right)^2 \leq 2[\left(\| a \|_\alpha^{\mathcal{N}_1}\right)^2 + \left(\| b \|_\alpha^{\mathcal{N}_1}\right)^2] \qquad (3)$$

Once more

$$2[(\| a \|_\alpha^{\mathcal{N}_1})^2 + (\| b \|_\alpha^{\mathcal{N}_1})^2] = 2[(\| \frac{a+b}{2} + \frac{a-b}{2} \|_\alpha^{\mathcal{N}_1})^2 + (\| \frac{a+b}{2} - \frac{a-b}{2} \|_\alpha^{\mathcal{N}_1})^2]$$

$$= \frac{1}{2}[(\| (a+b)+(a-b) \|_\alpha^{\mathcal{N}_1})^2 + (\| (a+b)-(a-b) \|_\alpha^{\mathcal{N}_1})^2]$$

$$\leq (\| (a+b) \|_\alpha^{\mathcal{N}_1})^2 + (\| (a-b) \|_\alpha^{\mathcal{N}_1})^2 \qquad (4)$$

Then we obtain the first result.

Secondly, for $\tau, s \in \mathbb{R}^+$

$$(\| a-b \|_\alpha^{\mathcal{N}_2})^2 + (\| a+b \|_\alpha^{\mathcal{N}_2})^2 = [\circledast\{\tau^2 : \mathcal{N}_2(a-b,t) \leq \alpha\}] + [\circledast\{s^2 : \mathcal{N}_2(a+b,s) \leq \alpha\}]$$

$$= \circledast\{\tau^2 + s^2 : \mathcal{N}_2(a-b,\tau) \leq \alpha,\ \mathcal{N}_2(a+b,s) \leq \alpha\} \qquad (5)$$

Also, the other side for $p, q \in \mathbb{R}^+$

$$2[(\| x \|_\alpha^{\mathcal{N}_2})^2 + (\| y \|_\alpha^{\mathcal{N}_2})^2] = 2([\circledast\{p^2 : \mathcal{N}_2(x,p) \leq \alpha\}] + [\circledast\{q^2 : \mathcal{N}_1(y,q) \leq \alpha\}])$$
$$= 2(\circledast\{p^2 + q^2 : \mathcal{N}_2(x,p) \leq \alpha,\quad \mathcal{N}_2(y,q) \leq \alpha\}), \qquad (6)$$

Now, we obtain from (5) and (6)

$$(\| x-y \|_\alpha^{\mathcal{N}_2})^2 + (\| x+y \|_\alpha^{\mathcal{N}_2})^2 \leq 2[(\| x \|_\alpha^{\mathcal{N}_2})^2 + (\| y \|_\alpha^{\mathcal{N}_2})^2] \qquad (7)$$

Once more

$$2[(\| x \|_\alpha^{\mathcal{N}_2})^2 + (\| y \|_\alpha^{\mathcal{N}_2})^2]$$
$$= 2[(\| \frac{x+y}{2} + \frac{x-y}{2} \|_\alpha^{\mathcal{N}_2})^2 + (\| \frac{x+y}{2} - \frac{x-y}{2} \|_\alpha^{\mathcal{N}_2})^2]$$
$$= \frac{1}{2}[(\| (x+y)+(x-y) \|_\alpha^{\mathcal{N}_2})^2 + (\| (x+y)-(x-y) \|_\alpha^{\mathcal{N}_2 2})^2]$$
$$\leq (\| (x+y) \|_\alpha^{\mathcal{N}_2})^2 + (\| (x-y) \|_\alpha^{\mathcal{N}_2})^2, \qquad (8)$$

Then from (6) and (8) we obtain the second result.

Finally, for $t, s \in \mathbb{R}^+$

$$(\| x - y \|_{\alpha}^{\mathcal{N}_3})^2 + (\| x + y \|_{\alpha}^{\mathcal{N}_3})^2$$

$$= [\circledast\{t^2 : \mathcal{N}_3(x - y, t) \leq \alpha\}] + [\circledast\{s^2 : \mathcal{N}_3(x + y, s) \leq \alpha\}]$$
$$= \circledast\{t^2 + s^2 : \mathcal{N}_3(x - y, t) \leq \alpha,\ \mathcal{N}_3(x + y, s) \leq \alpha\}, \quad (9)$$

Also, the other side for $p, q \in \mathbb{R}^+$

$$2[(\| x \|_{\alpha}^{\mathcal{N}_3})^2 + (\| y \|_{\alpha}^{\mathcal{N}_3})^2]$$

$$= 2([\circledast\{p^2 : \mathcal{N}_3(x, p) \leq \alpha\}] + [\circledast\{q^2 : \mathcal{N}_3(y, q) \leq \alpha\}])$$
$$= 2(\circledast\{p^2 + q^2 : \mathcal{N}_3(x, p) \leq \alpha, \quad \mathcal{N}_3(y, q) \leq \alpha\}), (10)$$

Next, we obtain from (9) and (10)

$$(\| x - y \|_{\alpha}^{\mathcal{N}_3})^2 + (\| x + y \|_{\alpha}^{\mathcal{N}_3})^2 \leq 2[(\| x \|_{\alpha}^{\mathcal{N}_3})^2 + (\| y \|_{\alpha}^{\mathcal{N}_3})^2] \quad (11)$$

Once more

$$2[(\| x \|_{\alpha}^{\mathcal{N}_3})^2 + (\| y \|_{\alpha}^{\mathcal{N}_3})^2]$$

$$= 2[(\| \tfrac{x+y}{2} + \tfrac{x-y}{2} \|_{\alpha}^{\mathcal{N}_3})^2 + (\| \tfrac{x+y}{2} - \tfrac{x-y}{2} \|_{\alpha}^{\mathcal{N}_3})^2]$$
$$= \tfrac{1}{2}[(\| (x+y) + (x-y) \|_{\alpha}^{\mathcal{N}_3})^2 + (\| (x+y) - (x-y) \|_{\alpha}^{\mathcal{N}_3})^2]$$
$$\leq (\| (x+y) \|_{\alpha}^{\mathcal{N}_3})^2 + (\| (x-y) \|_{\alpha}^{\mathcal{N}_3})^2, \quad (12)$$

Then from (11) and (12) we obtain the result.

4. Conclusions

This paper focuses on extending the concepts of t-norm and t-co-norm. It introduces generalized definitions for fuzzy inner products and intuitionistic inner products. Additionally, the study explores the framework of neutrosophic inner product spaces. Future work will aim to demonstrate this application in detail and observe the application of neutrosophic inner products in decision scenarios, especially when uncertainty, vagueness and incomplete data play a significant role. We will also investigate the refined neutrosophic sets t-norm and t-conorm.

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