THE EGYPTIAN STATISTICAL JOURNAL
ISSR, CAIRO UNIV., VOL.(35), No.2, 1991

IMPROVING AN APPROXIMATION OF THE DIGAMMA FUNCTION WITH APPLICATIONS IN STATISTICS

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The DiGamma function-known also as the Psi function- is defined by

$$\psi(x) = d\{\ln \Gamma(x)\}/dx = \Gamma'(x)/\Gamma(x)$$

for a real positive x, where $\Gamma(x)$ is the gamma function. Workers in the field of mathematical statistics often encounter the $\psi(.)$ function, particularly when gamma or beta densities are involved. For values of x not too small, the approximation

$$\psi(x) \approx \ln(x - \tfrac{1}{2}) \qquad (1)$$

appears in the literature. For example Johnson and Kotz [2], suggested using this approximation to obtain approximate values of the ML estimates of the two shape parameters of the beta distribution. In this reference, approximations of the $\psi^{n}(.)$ function by the corresponding derivatives of the function $\ln(x-\frac{1}{2})$, are suggested and used for values of x not too small.

The purpose if this note is to introduce an improvement of this approximation and use the new approximation in two statistical applications.

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The improved approximation

From the relation

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x + \tfrac{1}{2})$$

we obtains that

$$\psi(2x) = \ln 2 + \tfrac{1}{2}\psi(x) + \tfrac{1}{2}\psi(x + \tfrac{1}{2}).$$

Equivalently, we have

$$\psi(x) = 2\psi(2x) - \psi(x + \tfrac{1}{2}) - 2\ln 2 \qquad (2)$$

Using the approximation given by (1) for the two $\psi(.)$ functions on the right hand side of (2), we obtain, after simplification.

$$\psi(x) \approx \ln(x - \tfrac{1}{2} + \frac{1}{16x}) \qquad (3)$$

which offers a better approximation than (1). To give an idea about the performance of (3) relative to that of (1) we calculated both approximations and compared them with exact values of the $\psi(.)$ function obtained from tables given in [1].

The results appear in table la for values of x equal 1,1.5, 2,2.5,...,4

For small values of x we can use the recurrence formula

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

with the approximation in (3) to approximate $\psi(x+1)$, and obtain for $x < 1$,

Table 1a

Exact and Aproximate Values of the Psi Function

($x \geq 1$)

x	$\psi(x)$ Exact	$\ln(x-\frac{1}{2})$	$\ln(x-\frac{1}{2}+\frac{1}{16x})$
1	-.577216	-0.6931	-.5754
1.5	.036490	0	.0408
2	.422784	.4055	.4261
2.5	.703157	.6931	.7056
3	.922784	.9163	.9246
3.5	1.103157	1.0986	1.1045
4	1.256118	1.2528	1.2572

$$\psi(x) \approx \ln(x + \tfrac{1}{2} + \frac{1}{16(x+1)}) - \frac{1}{x} \qquad (3')$$

To illustrate the performance of this approximation for small x, we compare values calculated from (3') with exact values from the tables and produce table 1b

Table 1b

Exact and Approximate Values of the Psi Function

($x \leqq 1$)

x	ψ(x) Exact	ψ(x) Approx.
.1	-10.42375	-10.42035
.2	- 5.28904	- 5.28491
.3	- 3.50252	- 3.49812
.4	- 2.56138	- 2.55695
.5	- 1.96351	- 1.95918
.6	- 1.54062	- 1.53646
.7	- 1.22002	- 1.21607
.8	- .96501	- .96128
.9	- .75493	- .75141
1	- .57722	- .57391

The approximation in (3) can be used to obtain approximations of the derivatives of the $\psi(.)$ function. Hence we have

$$\psi'(x) \approx \frac{2}{x-1/4} - \frac{1}{x} \qquad (4)$$

and generally for n = 1,2,3,.....

$$\psi^{(n+1)}(x) = \frac{d^n}{dx^n} \left\{ \frac{2}{x-1/4} - \frac{1}{x} \right\}$$

$$= \frac{2\,(-1)^n\, n!}{(x-1/4)^{n+1}} - \frac{(-1)^n\, n!}{x^{n+1}}$$

Applications

1. For the two parameter Gamma distribution with density given by

$$f(x) = [\Gamma(\alpha)\beta^{\alpha}]^{-1}\, x^{\alpha-1}\, e^{-x/\beta} \qquad x > 0$$

$$\alpha > 0, \beta > 0$$

the MLE of both α and β are obtained by solving the two equations

$$\ln G = \psi(\alpha) + \ln\beta$$

$$\bar{X} = \alpha\,\beta$$

where $\bar{X}$ and G denote the sample arithmetic and geometric means respectively.

The estimate for α is obtained by solving iteratively the equation

$$\ln(\bar{X}/G) = \ln\alpha - \psi(\alpha)$$

If α is not too small, an approximate value of the estimate of α can be obtained by replacing $\psi(\alpha)$ by a suitable approximation. Using the approximation given by (1) we have after simplification.

$$\hat{\alpha}_1 \approx \frac{H}{2(H-1)}$$

where $H = \bar{X}/G$.

On the other hand, if the improved approximation given by (3) is used, we obtain

$$\hat{\alpha}_2 = \frac{\sqrt{H}}{4(\sqrt{H}-1)}$$

To judge the performance of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ we prepare table 2 which gives the exact MLE $\hat{\alpha}$ and both $\hat{\alpha}_1$ and $\hat{\alpha}_2$ for various values of H

2- In a method of estimating the two shape parameters of the Beta distribution, introduced by Selim [3] and improved by Selim and Saad [4], $\psi'^{-1}(.)$ (the inverse of the $\psi'(.)$ function) was needed to calculate the estimates. If few estimates are to be evaluated, Tables of the $\psi'(.)$ function may be used. If a large number of such estimates are needed, such as the case in a Monte Carlo study of the properties of those estimators, it would be practical to use a reliable approximation of the $\psi'^{-1}(.)$ function.

Table (2)

A Comparison Between Two Approximations of ML Estimate of the Parameter α

H	α	$\hat{\alpha}_1$	$\hat{\alpha}_2$
1.7811	1	1.1401	.9972
1.4463	1.5	1.6204	1.4840
1.3104	2	2.1108	1.9774
1.2375	2.5	2.6048	2.4732
1.1922	3	3.1015	2.9710
1.1614	3.5	3.5979	3.4682
1.1390	4	4.0971	3.9681
1.1088	5	5.0956	4.9674
1.0659	8	8.0873	7.9602

For not too small-values, we utilize the approximation of the $\psi'(.)$ function, given in (4), to derive an approximation of its inverse as follows:

Let

$$y = \psi'(x)$$

then (4) can be written as

$$y = \frac{2}{x-1/4} - \frac{1}{x}$$

Simplifying we obtain

$$x^2y + x(1+y/4) - 1/4 = 0,$$

which is a quadratic equation in x. The approximation is given by the real root of this equation. That is

$$\psi_*'^{-1}(x) = 1/2y + \frac{1}{8} + 1/2y\ [y^2/16 + 3y/2 + 1]^{\frac{1}{2}} \qquad (5)$$

To get an idea about the accuracy of this approximation, we present table 3 which gives for some values of x the exact values of $\psi'^{-1}(x)$ obtained from tables of the $\psi'(.)$ function and the corresponding approximation calculated from 5.

Table (3)

Exact and Approximate Values of

the $\psi'^{-1}(.)$ Function

x	$\psi'^{-1}(x)$ Exact	$\psi_*'^{-1}(x)$ Approx.
.2213	5	4.995
.2487	4.5	4.495
.2838	4	3.994
.3304	3.5	3.493
.3949	3	2.994
.6449	2	1.995
1.6449	1	1.009
2.5420	.75	.771

In conclusion, we find these approximations useful for practical purposes.

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