RESERVOIRS WITH MARKOVIAN INFLOWS : APPLICATION OF A GENERATING FUNCTION FOR THE ASYMPTOTIC STORAGE DISTRIBUTION

By

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INTRODUCTION

A Moran reservoir of size k+m, with draft m and stationary independent inflows X_t , is a discrete-time and discrete-state stochastic system in which the object of primary interest is the distribution of the volume Z_t of contained water at time t . Here k and m are integers, X_t is a non-negative integral-valued random variable, and the range of Z_t is 0, 1, ..., k. The model operates as follows : Z_t is augmented during the time interval (t, t + 1) by the inflow X_t, and then, immediately before the epoch t + 1, m units of water are instantaneously released ; or, if the reservoir does not at the moment of release contain as many as m units, the total contents are released. (See Moran 1954 and(1959), and Gani (1957)).

In this paper we shall work in terms of a variant of this model in which the concept of an instantaneous withdrawal is not required. Instead, the supply X_t is supposed to flow in at a uniform rate during the unit time interval, and the withdrawn water is likewise supposed to br removed at a unifoım rate. We are thus concerned only with the net rate of inflow (which may be negative) It turns out (Lloyd, 1963(c)) that the defining equations differ only trivially. from those of the genuine Moran reservoir. In fact, where the Moran model requires a reseıvoir of size K+m units (to accomodate k units of water), the variant requires a size of k units. The models are otherwise formally identical. It is a slight notational advantage of the second version that one does not have to adjust the reservoir size parameter when one alters the draft m.

As a model for real reservoirs, which are multi-purpose systems operating in continuous time and subject to complicated release rules, fed by continuous, autocorrelated and seasonally-distributed inflows, the scheme that has been outlined is clearly a rather simplified first approximation. Its departures from realism derive from the mathematical difficulties that arise in attempting to construct and work with a more sophisticated model. Where one would like to derive general results on the time-dependent distribution of Z_t for a reservoir operating with continuous time, continuous quality, and seasonal (Lloyd and Odoom 1964, Lloyd 1970) autocorrelated (Lloyd 1963 b) inflows, one often has to be content with the asymptotic equilibrium distribution of Z_t in an unrealistic rservoir that is a simplified special case of the Moran model. (Lloyd 1963 a, Prabhu 1964) Favoured special cases are those having unit draft (m=1) or infinitely high walls ($k=\infty$), or both.

If one is prepared to accept the restriction to unit draft (which means that all volumes are measured as integral multiples of the actual draft) and a semi-infinite wall height (which means a reserfoir working almost always at less than maximum capacity), an explicit formula is available (Moran 1959, Prabhu 1965) for the generating function of the asymptotic distribution of reservoir levels, This will be referred to as the «generating function theorem». The structure of the equations defining this distribution is such that, by the use of a that will be referred to as the «ratio theorem», one may deduce from this generatng function the (asymptotic) distribution of levels in a *finite* reservoir.

In the Moran model, with mutually independent inflows, the errors attri butable to discreteness may be reduced by using a smaller time interval. A refinement of the time scale however makes it less plausible that serial correlation in the inflows may be neglected. It would therefore be a natural development to modify the model by attributing serial correlation to the inflows, and the natural way of doing this would seem to be in terms of a Markov Chain structure the basic theory of this has in fact been worked worked out (Lloyd 1963 (bs iand 1967), but little is known of the extent to which theorems and technique. (Lloyd and Odoom 1964 (b)) which are valid for independent inflows remain valid, or may be validated in a generalized version, for Markovian inflows) Two of the few results so far available concern the generating function theorem and the ratio theorem, both of which have been generalised in this sense. (Odoom and Lloyd 1965). Other relevant results have been obtained by Gani and Ali Khan (1968).

It is the purpose of the present paper to attempt to exploit these theorems for the investigation of the behaviour of Markovian reservoirs.

2. The variant Moran reservoir details :

With a reservoir of size k units, mutually independent inflows $\{X_t\}$, and a desired withdrawal of m units, the distributions of reservoir levels Z_t and Z_{t+1} at times t and t+1 are related as follows :

$$P(Z_{t+1} = 0) = P(0 \leqslant Z_t + X_t \leqslant m),$$

$$P(Z_{t+1} = s) = P(Z_t + X_t - m = s), \quad s = 1, 2, \ldots, k-1$$

$$P(Z_{t+1} = k) = P(Z_t + X_t - m \geqslant k).$$

or

$$\pi(0,t+1) = \sum_{u=0}^{m} \sum_{r=0}^{u} \pi(r,t)p(u-r)$$

$$\pi(s,t+1) = \sum_{r=0}^{a} \pi(r,t)p(m+s-r), \quad s=1,2,\ldots,k-1, \quad \text{where } a = \min(k,m+s)$$

$$\pi(k,t+1) = \sum_{u=0}^{\infty} \sum_{r=0}^{k} \pi(r,t)p(k+m-r+u),$$

where $\pi(s, t) = P(Z_t = s)$, $s = 0, 1, \ldots, k$, and $p(r) = P(X_t = r)$, $r = 0, 1, \ldots$. The process $\{Z_t\}$ forms a finite ergodic Markov Chain possessing a non-trivial asymptotic equilibrium distribution. This concept will occur with sufficient frequency in the sequel to justify the special notation $P_\infty(\,.\,)$ for $\lim_{t\to\infty} P(\,.)$. In the present case, writing $\pi(s) = P_\infty(Z_t = s)$, we have

$$\pi(0) = \sum_{u=0}^{m} \sum_{r=0}^{u} \pi(r)p(u-r)$$

$$\pi(s) = \sum_{r=0}^{a} \pi(r)p(m+s-r), \quad s=1,2,\ldots,k-1$$

$$\pi(k) = \sum_{u=0}^{\infty} \sum_{r=0}^{k} \pi(r)p(k+m-r+u),.$$

where p (r) denotes $P_\infty(X_t = r)$.

A reservoir which is managed in such a way that the probability of overflow may be ignored might just as well have walls of infinite height. This semi-infinite case ($k = \infty$) is worth studying since it is more tractable than the finite case and yet may serve as a first approximation to a more realistic situation.

For the semi-infinite reservoir the sequence $\{Z_t\}$ remains ergodic provided $E(X_t) < m$. When in addition $m = 1$ the defining equations simplify to

$$\pi(0,t+1) = \{\pi(0,t) + \pi(1,t)\}p(0) + \pi(0,t)p(1),$$

$$\pi(r,t+1) = \sum_{s=0}^{r+1} \pi(s,t)p(r+1-s), \qquad r=1,2,\dots$$

where $\pi(r, t) = P(Z_t = r)$ and $p(s) = P(X_t = s)$.

The generating function $g_z(\theta\,;t)$ of the probabilities $P(Z_t = r)$, $r = 0, 1, \dots$ is readily obtainable from these equations in the form

$$\theta g_z(\theta, t+1) = g_x(\theta)\, g_z(\theta, t) - (1-\theta)\, g_x(0)\, g_z(0, t)$$

where $g_x(\theta)$ is the known g.f. of the inflows. (Strictly speaking there is no need here to restrict oneself to stationary inflows, and $g_x(\theta)$ might if desired be replaced by a time-dependent $g_x(\theta, t)$). This recursion is not of much use, unfortunately, because of the presence of the term involving $g_z(0, t) = P(Z_t = 0)$. If we write $g_z(\theta) = \lim_{t \to \infty} g_z(\theta, t)$ for the g.f. of the asymptotic equilibrium distribution of levels, howevere, the equations becomes

$$g_z(\theta)\{g_x(\theta) - \theta\} = (1-\theta)\, g_x(0)\, g_z(0).$$

whence, on differentiating,

$$g_x(0)\, g_z(0) = 1 - \mu_x \tag{2.1}$$

where $\mu_x = E(X_t)$. We thus have a simple expression for the asymptotic probability of emptiness of the semi-infinite reservoir, viz. $P_\infty(Z_t = 0) = (1-\mu_x)$ $P(X = 0)$, and the equation for $g_z(\theta)$ becomes

$$g_z(\theta) = (1-u_x)(1-\theta) / \{g_x(\theta) - \theta\}. \tag{2.2}$$

This is the generating function theorem referred to in § 1.

The ratio theorem referred to earlier applies to this equilibrium distribution. For a finite reservoir of size k, with m = 1, the limiting versions of the first **k** defining equations as $t + \rightarrow \infty$ are

$$\pi(1)p(0) = \pi(0)\ \{1-p(0)-p(1)\}$$

$$\pi(r+1)p(0) = \sum_{s=0}^{r-1} \pi(s)p(r+1-s) - \pi(r)\ \{1-p(r)\},\ r=1,2,\ldots,k-1.$$

There is a (k+1)—th equation, but it is linearly dependent on its prede. cessors and so may be discarded. The equations quoted coincide exactly with the first k defining equations of a reservoir of size k' (> k) having the same inflows and draft ; and with those of the corresponding semi-infinite reservoir, provided $\mu_x < m$ (=1 in this case).

It follows that the ratios π (r) / π (0), r=1, 2, ..., k are the same for the k-valued reservoir as for the reservoir of size k' > k, including the semi-infinite case k' = . ∞ (See Prabhu, 1965).

Because it can be used in conjunction with the generating function theorem (which requires that the value of m be 1) this version of the theorem is the one of greatest «practical» importance ; but it is clear form the preceding argument that the ratio theorem holds for a general value of m, and not merely for m=1.

As a simple illustration of the combined application of the two theorems we may consider a semi-infinite reservoir subject to unit draft (m=1) and geometric inflows for which $p(r) \equiv P(X_t = r) = q\,p^r$, r = 0, 1, ..., $0 < p < 1$, q = 1 — p. The inflow generating function is $g_x(\theta) = q / (1 - p\theta)$, and $E(X) \equiv \mu_x = p / q$, whence

$$g_z(\theta) = (1 - p / q)(1 - \theta) / (q - p\theta).$$

The asymptotic distribution of levels in this semi-infinite reservoir is thus given by

$$\pi(0) = 1 - (p / q)^2$$

$$\pi(r) = (1 - p/q)(p/q)^{r+1},\quad r = 1, 2, \ldots .$$

a geometric distribution with a modified initial value. According to the ratio. theorem, the corresponding asymptotic probabilities $\pi^{(k)}(r)$ in a finite reservoir of size k are given by

$$\pi^{(k)}(r) = aq\,(p/q)^{r+1}, \quad r = 1, 2, \ldots, k$$

wheie $a = \pi^{(k)}(0)$ is a constant to be determined by the noımalisation condition

3. Markovian inflows :

We now extend the (modified Moran) model discussed in the previous section to allow the inflow distribution sequence $\{X_t\}$ to have a Markov Chain structure. That is, for a reservoir of size k, with draft m, we still assume that a quantity X_t of water enters during the interval $(t, t+1)$, both inflow and withdrawal being uniformly spread over the interval, so that the possible values of Z_t are still 0, 1, ..., k ; but we now assume in addition that $\{X_t\}$ is an ergodic Markov Chain, with $P(X_{t+1} = r \mid X_t = s) = l_{rs}$, r, s = 0, 1, ..., n. We shall denote the matrix (l_{rs}) by L, sometimes partitioned into columns as $(l_0\ l_1 \ldots l_n)$. This carries the corollary that the possible values of X_t are restricted to the set 0, 1, ..., n, where n may of course be arbitrarily large. The attribution of this particular type of serial dependence to the X_t is made in order to endow the inflows with an auto-correlation structure that is tractable in terms of the kind of model discussed earlier. The geophysical process which generates $\{X_t\}$ may plausibly be assumed to have been going on for some considerable time before the initiation of the reservoir, and we may assume that the $\{X_t\}$ chain has reached its asymptotic equilibrium phase. Thus, using (as before) p (r) to denote

$P_\infty(X_t=r)$, we have $p(r) = \sum_{s=0}^{n} \ell_{rs} p(s)$, or

$$p = Lp \qquad (3.1)$$

where $L = (l_{rs})$ is the inflow transition matrix and p the vector p (r), r=0, 1, ...,n.

It has been shown (Lloyd 1963 *b*) that the behaviour of reservoir levels Z_t may now be investigated in terms of the joint distribution of the pair (Z_t, X_t), which again forms a Markov Chain.

For a transition $Z_t \rightarrow Z_{t+1}$ in which both levels concerned are different from zero and from k, so that boundaries are not involved, the difference equation for the bivariate sequence is

$$P(Z_{t+1}=r, X_{t+1}=s) = \sum_{j=0}^{n} P(Z_t = r-j+m, X_t = j)\, P(X_{t+1}=s \mid X_t = j).$$

With obvious modifications for dryness or overflow, similar equations hold at the boundaries. It is convenient to describe the system in terms of a vector $\pi(t)$ with elements $\pi(r, s\,;t) = P(Z_t = r, X_t = s)$, $r = 0, 1, \ldots, k$, $s = 0, 1, \ldots, n$- arranged in the order

$(r, s) = (0, 0), (0, 1), \ldots, (0, n)\,;(1, 0), (1, 1)\,(1, 1), \ldots, (1, n), \ldots\,;(k, 0), \ldots, (k, n).$

This lends itself to partitioning in the form

$$\pi(t)' = (\pi_o(t)', \pi_1(t)', \ldots, \pi_k(t)')$$

where the column vector $\pi_r(t) = \{\pi(r, 0; t), \pi(r, 1; t), \ldots, \pi(r, n; t)\}'$, primes denoting matrix transposition. The Markov transition matrix R govering tran sitions $\pi(t) \rightarrow \pi(t+1)$ may then be written in the following partitioned form, where L_s is derived from L by retaining the column l_s and replacing all other columns by zeros, so that $L = \Sigma L_s$, and where $L_{ij} = L_i + L^i{}_{+1} + \ldots + L_j$:

$\pi_r(t+1)$ \ $\pi_r(t)$	r=0	1	...	m-1	m	m+1	...	k-1	k
r=0	$L_{0,m}$	$L_{0,m-1}$	...	$L_{0,1}$	L_0	0	...	0	0
1	L_{m+1}	L_m	...	L_2	L_1	L_0	...	0	0
2	L_{m+2}	L_{m+1}	...	L_3	L_2	L_1	...	0	0
3	L_{m+3}	L_{m+2}	...	L_4	L_3	L_2	...	0	0
⋮	⋮	⋮							
k-m	L_k	L_{k-1}	...				...	L_1	L_0
⋮	⋮	⋮							
k-1	L_{m+k-1}	L_{m+k-2}	...	L_k	L_{k-1}	L_{k-2}	...	L_m	L_{m-1}
k	$L_{m+k,n}$	$L_{m+k-1,n}$	...	$L_{k+1,n}$	$L_{k,n}$	$L_{k-1,n}$	...	$L_{m+1,n}$	$L_{m,n}$

In the special case of m=1 the corresponding equations become

$$\pi_0(t+1) = L_{0,1}\,\pi_0(t) + L_0\pi(t)$$

$$\pi_r(t+1) = \sum_{s=0}^{r+1} L_{r+1-s}\,\pi_s(t), \qquad r=1,2,\ldots,k-1$$

$$\pi_k(t+1) = \sum_{s=0}^{k} L_{k+1-s,n}\,\pi_s(t).$$

The asymptotic equilibrium distribution vectors $\pi_r = \lim_{t\to\infty} \pi_r(t)$, r=0, 1, ...,k therefore satisfy the relations

$$\pi_0 = L_{0,1}\pi_0 + L_0\pi_1$$

$$\pi_r = \sum_{s=0}^{r+1} L_{r+1-s}\pi_s \qquad r=0,1,\ldots,k-1$$

$$\pi_k = \sum_{s=0}^{k} L_{k+1-s,n}\pi_s.$$

Each of these k+1 matrix equations of course incorporates n scalar equations and the system bears a strong formal resemblance to the covasponding system with independent inflows. There are two significant differences however (a) the coefficient matrices L_r and L_{rs} are singular and very sparse : and (b) whereas in the case of independent inflows we may neglect as redundant (by normalisation) one of the defining equations—the last, say—for the finite reservoir of size k, the situation for autocorrelated inflows is that, of the (k+1) defining *matrix* equation for the finite reservoir we may neglect only one—the last, say—of the implied n scalar equations.

The manner in which the equations determine the π_{rs} will be made clear by inspecting a few of the equations in scalar form. Here π_{rs} is the s—th element of the vector π_r, and denotes P_∞ (Z_t = r, X_t = s). For the case k = 4, for example, the first two blocks of equations (starting at the lower boundary) are as follows :

π_{00}	π_{01}	π_{02}	π_{03}	π_{10}	π_{11}	π_{12}	π_{13}	π_{20}	π_{21}	π_{22}	π_{23}
$t_{00}-1$	t_{01}	0	0	t_{00}	0	0	0	0	0	0	0
t_{10}	$t_{11}-1$	0	0	t_{10}	0	0	0	0	0	0	0
t_{20}	t_{21}	-1	0	t_{20}	0	0	0	0	0	0	0
t_{30}	t_{31}	0	-1	t_{30}	0	0	0	0	0	0	0
0	0	t_{02}	0	-1	t_{01}	0	0	t_{00}	0	0	0
0	0	t_{12}	0	0	$t_{11}-1$	0	0	t_{10}	0	0	0
0	0	t_{22}	0	0	t_{21}	-1	0	t_{20}	0	0	0
0	0	t_{32}	0	0	t_{31}	0	-1	t_{30}	0	0	0

The last two blocks of equations (at the upper boundary) are as follows :

π_{rs}																
r	k-3				k-2				k-1				k			
s	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
	0	0	0	t_{03}	0	t_{02}	0	0	-1	t_{01}	0	0	t_{00}	0	0	0
	0	0	0	t_{13}	0	t_{12}	0	0	0	$t_{11}-1$	0	0	t_{10}	0	0	0
	0	0	0	t_{23}	0	t_{22}	0	0	0	t_{12}	-1	0	t_{20}	0	0	0
	0	0	0	t_{33}	0	t_{32}	0	0	0	t_{13}	0	-1	t_{30}	0	0	0
	0	0	0	0	0	0	0	t_{03}	0	0	t_{02}	t_{03}	-1	t_{01}	t_{02}	t_{03}
	0	0	0	0	0	0	0	t_{13}	0	0	t_{12}	t_{13}	0	$t_{11}-1$	t_{12}	t_{13}
	0	0	0	0	0	0	0	t_{23}	0	0	t_{22}	t_{23}	0	t_{21}	$t_{22}-1$	t_{23}
	0	0	0	0	0	0	0	t_{33}	0	0	t_{32}	t_{33}	0	t_{31}	t_{32}	$t_{33}-1$

The first four (in general, n) equations define the ratios π_{01} , $\pi_{02}, \ldots, \pi_{on}, \pi_{10}$ in terms of π_{00}; the second set of n equations similarly define $\pi_{11}, \ldots , \pi_{1n} , \pi_{20}$, in terms of their predecessors and hence in terms of π_{00} and so on ; and the k—th set will be seen to define $\pi_{k-1,1}$, $\pi_{k-1,2}$, ... $\pi_{k-1,n}$, $\pi_{k,0}$. The equations involved up to this point coincide exactly with the equations that apply to a reservoir of size k' $>$ k having the same inflow and draft : In particular they coincide with the equations for the corresponding semi-infinite reservoir ($k=\infty$). When however it comes to the determination of π_{k1} ,π_{k2} ,π_{kn}in terms of π_{oo} $\rightarrow$ for the reservoir of size k, the equations no longer coincide with those applying to a larger reservoir.

Thus the ratio theorem for reservoirs with independent inflows (Odoom and Lloyd 1965) has the following analogue for Markovian inflows :

Let $\pi^{(k)}(r, s)$ denote the joint probability, under asymptotic equilibrium conditions, that $Z_t = r$ and $X_t = s$ in a reservoir of size k ; then, for any $k' > k$,

$$\frac{\pi^{(k)}(r,s)}{\pi^{(k)}(0,0)} = \frac{\pi^{(k')}(r,s)}{\pi^{(k')}(0,0)} \; ; \quad \begin{array}{l} r=0,1,\ldots,k-1 \\ s=0,1,\ldots,n, \end{array} \qquad (3.2)$$

and this holds also when $k' = \infty$ provided that the infinite reservoir is ergodic, that is provided $\mu_x < m$.

We are likely to be interested in the distribution of Z_t rather than the joint distribution of Z_t and X_t. The ratio theorem (3.2) implies that $\pi^{(k)}(r, s) = a_{rs}\pi^{(k)}(0.0)$ where the constant of proportionally a_{rs} depends on r and s but not on k. It follows that for the asymptotic marginal distribution of Z_t, in a reservoir of size k, denoting $P_\infty(Z_r = r)$ by $\pi^{(k)}(r, .)$, we have

$$\pi^{(k)}(r,.) = \sum_s \pi^{(k)}(r,s) = \pi^{(k)}(0,0) \sum_s a_{rs}$$
$$= a_r \pi^{(k)}(0,0) \text{ say,}$$

$$\text{whence } \frac{\pi^{(k)}(r,.)}{\pi^{(k)}(0,0)} = \frac{\pi^{(k')}(r,1)}{\pi^{(k')}(0,.)} \, , \quad k' > k \qquad (3.3)$$
$$= \frac{\pi^{(\infty)}(r,.)}{\pi^{(\infty)}(0,.)} \text{ provided } \mu_x < m, \text{ for } r=0,1,\ldots,k-1.$$

The generalisation of the generating function theorem proceeds as follows :

We are concerned with the equilibrium distribution of levels in a semi-infinite reservoir, with unit draft, and with Markovian inflows for which the transition matrix is $L = \sum_{0}^{n} L_r$. The distribution vector $p = \{p(r)\}$ satisfies $Lp = p$, and the mean inflow rate is $\mu < 1$. Thus joint distribution vectors $\pi(0), \pi(1), \ldots,$ where $\pi(r) = \{\pi(r, 0), \pi(r, 1), \ldots, \pi(r, n)\}$,[1] and where $\pi(r, s) = p_\infty(Z_t = r, X_t = s)$, satisfy the equations.

$$\pi(0) = (L_0+L_1)\pi(0)+L_0\pi(1)$$

$$\pi(r) = \sum_{s=0}^{r+1} L_{r+1-s}\pi(s), \qquad r=1,2,\dots,k-1$$

$$\pi(k+r) = \sum_{s=0}^{k} L_{k-s}\pi(r+s), \qquad r=1,2,\dots .$$

Introducing the vector generating functions

$$H(\theta) = \Sigma L_r \theta^r \tag{3.4}$$

and

$$g_{z,x}(\theta) = \Sigma \pi(r) \theta^r \tag{3.5}$$

we find from these equations that

$$\theta g_{z,x}^{'}(\theta) = (\theta-1) L_o \pi_o + H(\theta) g_{z,x}(\theta)$$

whence $\{H(\theta) - I\theta\} g_{zx}(\theta) = (1-\theta) L_o \pi_o$

$$= (1-\theta) l_o \pi_{oo}.$$

Using the facts that $\sum_r l_{ro} = 1$, and that $g_{z,x}(1) = p$, we find on premultiplying this equation by $1 = (1, \dots, 1)$ that

$$\ell_{00} = 1-\mu,$$

so that $$\{H(\theta) - I\theta\}g_{Z,X}(\theta) = (1-\mu)(1-\theta)\ell_0.$$

The generating function $g_z(\theta)$ for the distribution of Z may be obtained as $g_z(\theta) = 1^1 g_{z,x}(\theta)$. Thus

$$g_Z(\theta) = (1-\mu)(1-\theta)1'\{H(\theta) - I\theta\}^{-1}\ell_0 \tag{3.6}$$

$$= -(1-\mu)(1-\theta) \frac{\begin{vmatrix} 0 & I' \\ \ell_0 & H(\theta)-I\theta \end{vmatrix}}{|H(\theta)-I\theta|}, \tag{3.7}$$

the latter version being a consequence of the standard expression for a bilinear form :

$$x'A^{-1}y = - \frac{\begin{vmatrix} O & x' \\ y & A \end{vmatrix}}{|A|} .$$

The extent to which these theorems may be applied to the evaluation of probabilities in particular cases will depend on whether an explicit inver.e can be found for the matrix H (θ)—Iθ, or, equivalently, whether suitable expansions can be obtained for the determinants

4. Application of the generating function theorcm for Markovian infiows.

4. 1 *The covariance of levels with inflows* :

When the inflows are independent, the generating function theorem enables one to evaluate the mean level $E_\infty (Z_t) \equiv \lim_{t \to \infty} E(Z_t) = \mu_z$, say, as $g_z(1)$. It is in fact necessary to differentiate $g_z(\theta)$ twice to obtain this, and one then easily obtains the result that

$$\tfrac{1}{2} E(X_t (X_t - 1)) = (1 - \mu_x) \mu_z$$

or, in an obvious notation,

$$\mu_z = \tfrac{1}{2} (\sigma_x^2/(1 - \mu_x) - \mu_x). \tag{4.1}$$

It is obviously of interest to apply the same technique to the generating function when the inflows are Markovian. For this the convenient form of the theorem appears to be :

$$\{H(\theta) - 1\theta\} g_{z,x}(\theta) = (1 - \mu_z)(1 - \theta) 1_o$$

If we differentiate twice with respect to θ we find

$$\ddot{H}(\theta) g_{z,x}(\theta) + 2 \{\dot{H}(\theta) - I\} \dot{g}_{z,x}(\theta) + \{H(\theta) - I\theta\} \ddot{g}_{z,x}(\theta) = O,$$

where $\ddot{H}(\theta) = \Sigma\, r\, L_r \theta^{r-1}$ etc.

Premultiply by $1'$ and set $\theta = 1$, noting that $1'L_r = e_r' = (0, 0, \ldots, 0, 1, 0, 0)$ the unit element being in the $(r + 1)$—th position, to obtain

$$1'\ddot{H}(1)g_{Z,X}(1) = \sum_r r(r-1)e_r' \sum_s \pi_s = \sum_r \sum_s r(r-1)\pi_{sr}$$

$$= E_\infty\{X_t \; X_{t-1}\},$$

$$1'\ddot{H}(1)\dot{g}_{Z,X}(1) = \sum_r re_r' \sum_s s\pi_s = \sum_r \sum_s rs\pi_{sr}$$

$$= E_\infty\{Z_t X_t\},$$

$$1'\dot{g}_{Z,X}(1) = \dot{g}_Z(1) = \mu_Z,$$

and

$$1' (H(1)—I) = 0,'$$

whence we obtain

$$\tfrac{1}{2} E_\infty \; (X_t \; (X_t - 1)) = \mu_z - E_\infty \; (Z_t \; X_t) \qquad (4.2)$$

as the generalisation of (4. 1).

The technique does not, therefore, provide a value for μ_z, but if μ_z is known we may compute cov (Z_t, X_t).

4.2 *Computation of distribution of levels in a finite reservoir : a simple illustrative example :*

Suppose the inflow transition matrix to a semi-infinite reservoir with unit draft to be

X_{t+1} \ X_t	0	1	2
0	$1-\alpha$	0	β
1	0	1	0
2	α	0	$1-\beta$,

$0 < \alpha < \beta < 1.$

Then the asymptotic distribution of X_t is determined by the conditions

$$(1-\alpha) \, p(0) + \beta \, p(2) = p(0)$$

$$p(1) = 1 - p(0) - p(2)$$

where $p(r) = P_\infty(X_t = r)$, $r = 0, 1, 2$. We then have

$$P_o = \lambda\beta, \quad p_1 = 1 - \lambda(\alpha + \beta), \quad p_2 = \lambda\alpha$$

for some suitable parameter λ. If we choose $\lambda = 1/(\alpha + \beta)$ we obtain the simple two-valued inflow distribution :

$$P_\infty(X_t = 0) = \beta/(\alpha + \beta), \; P_\infty(X_t = 2) = \alpha/(\alpha + \beta),$$

the mean inflow being $\mu_x = 2\alpha/(\alpha + \beta) < 1$. It is convenient in this simple case to work with a modified generating function vector

$$h(\theta) = \Sigma_o^\infty u_r \theta^r$$

where $u_r = (\pi_{ro}, \pi_{r2})'$, We find from the generating function theorem

$$h(\theta) = -\frac{(1-\theta)(\beta-\alpha)}{\beta+\alpha} \begin{pmatrix} 1-\alpha-\theta & \beta\theta^2 \\ \alpha & (1-\beta)\theta^2-\theta \end{pmatrix}^{-1} \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix}$$

$$= a(1+b\theta+b^2\theta^2+\ldots) \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} - \begin{pmatrix} 1-\alpha-\beta \\ 0 \end{pmatrix} \theta$$

where $a = (\beta - \alpha)/(\beta + \alpha)(1 - \alpha)$, $b = (1-\beta)/(1-\alpha)$.

Then u_r, the coefficient of θ^r in the expansion of $h(\theta)$, is given by

$$u_0 = a \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix}$$

$$u_r = cb^{r-1} \begin{pmatrix} 1-\alpha \\ 1-\beta \end{pmatrix}, \quad r=1,2,\ldots$$

where $c = a\alpha/(1-\alpha)$.

Thus for the semi-infinite reservoir we have

$$\pi_{oo} = a(1-\alpha), \; \pi_{02} = a\alpha,$$

and, for r = 1, 2, ...,

$$\pi_{r0} = P_\infty \ (Z_t = r, X_t = 0) = cb^{r-1} \ (1-\alpha)$$

and

$$\pi_{r2} = P_\infty \ (Z_t = r, X_t = 2) = cb^{r-1} \ (1-\beta\ ).$$

Then $P_\infty \ (Z_t = 0) = \pi_{0.} = a,$

and

$$P_\infty \ (Z_t = r) = \pi_{r.} = 2cb^{r-1} \ (1-\frac{\alpha + \beta}{2}\ ). \quad r = 1, 2, \ldots$$

We now invoke the ratio theorem to obtain the corresponding probabilities for a reservoir of size k subject to the same inflows and draft. We have

$$\pi_{rs}^{(k)} = A \ \pi_{rs}, \qquad r = 0, 1, \ldots, k-1$$

and

$$\pi_{r.}^{(k)} = A \ \pi_{r}$$

for a suitable constant A. To determine $\pi_{k.}^{(k)}$ we must examine the final row-block of the (partitioned) transition matrix for the complete vetor $\pi^{(k)}$ of the finite reservoir. (A typical element of this vector is $\pi_{rs}^{(k)} = P_\infty \ (Z_t = r, X_t = s)$.) This produces the equations

$\pi_{rs}^{(k)}$

r	k-1			k				
s	0	1	2	0	1	2		
	0	0	β	−1	0	β		0
	0	0	0	0	0	0	=	0
	0	0	1−β	0	0	1−β		0

Thus $\begin{pmatrix} 0 & \beta \\ 0 & 1-\beta \end{pmatrix} u_{k-1}^{(k)} - \begin{pmatrix} 1 & -\beta \\ 0 & \beta \end{pmatrix} u_{k}^{(k)} = 0$

where $u_r^{(k} = (\pi_{r0}^{(k)}, \pi_{r2}^{(k)})$. Using the known result that

$$u_{k-1}^{(k)} = Acb^{k-2} \begin{pmatrix} 1-\alpha \\ 1-\beta \end{pmatrix} .$$

where A is a constant to be determined, we thus have

$$u_{k}^{(k)} = Acb^{k-2} \begin{pmatrix} 1 & -\beta \\ 0 & \beta \end{pmatrix}^{-1} \begin{pmatrix} 0 & \beta \\ 0 & 1-\beta \end{pmatrix} \begin{pmatrix} 1-\alpha \\ 1-\beta \end{pmatrix}$$

$$= Acb^{k-2} \frac{(1-\beta)}{\beta} \begin{pmatrix} \beta \\ 1-\beta \end{pmatrix}$$

and thus

$$P_{\infty} (Z_t = k) = u_k^{(k)} = Acb^{k-2} (1-\beta) / \beta .$$

Combining this with the earlier results

$$P_{\infty} = (Z_t = 0) = Aa$$

$$P_{\infty} (Z_t = r) = Acb^{r-1} (2-\alpha-\beta), r = 1, 2, ..., k-1$$

we finally have for the finite reservoir, of size k,

$$P_{\infty} (Z_t = 0) = Aa$$

$$P_{\infty} (Z_t = r) = Ac (2-\alpha-\beta) b^{r-1} , \quad r = 1, 2, ..., k-1$$

$$P_{\infty} (Z_t = k) = Ac (1-\beta) b^{k-2} / \beta$$

where A is the normalisation constant, and a,b, and c are as defined above.

4.3 *Probability of emptiness in a semi-infinite reservoir with general inflow transition matrix*

In this section we shall restrict discussion to semi-infinite reservoirs with unit draft.

In the simple illustrative example considered in the previous section the inversion of the matrix H (θ) — Iθ was a trivial task. For more general inflow patterns however this is not so, and it seems more hopeful to work with the deter-minantal version of the formula.

Consider a general inflow X_t with possible values 0,1, ..., n, and expectation μ_x ($<$ 1), generated by the trasition matrix L of order n + 1 :

$$L = (\ell_{rs}) \qquad r,s = 0,1,\ldots,n$$

with

$$\sum_{s=0}^{n} \ell_{rs} = 1, \qquad r=0,1,\ldots,n.$$

The generating function g_z (θ) of the asymptotic equilibrium distribution of levels is

$$g_z(\theta) = -(1-\mu_x)(1-\theta)\,|A(\theta)|\,/\,|B(\theta)|$$

where $|B(\theta)|$ is a determinant of order not exceeding n + 1, with

$$B(\theta) = H(\theta) - I\theta = \Sigma_o^n L_r \theta^r - I\theta,$$

and $|A(\theta)|$ is the determinat of order n + 2 obtained from this by bordering thus:

$$|A(\theta)| = \begin{vmatrix} 0 & 1 \\ 2_0 & B(\theta) \end{vmatrix} .$$

The first point to be noted is that g_z (θ) is a rational function of θ . We may write B (θ) as b_{rs} (θ), r's = 1, 2, ..., n, where

$$b_{rs}(\theta) = l_{rs}\,\theta^s,\ r + s$$

and

$$b_{rr}(\theta) = l_{rr}\,\theta^r - \theta\ .$$

Thus $|B(\theta)|$ is a polynomial in *a*, of degree ½ n (n + 1) + 1. We note that this polynomial has (1-θ) as a factor, since $|B(1)| =$ L-I is zero, each column sum being zero ; it also has as a factor. Thus $|B(\theta)|\,/\,(1-\theta)$ is a polynomial of degree ½ (n—1). We write

$$|B(\theta)| = (1-\theta)\theta^n \sum_0^{\frac{1}{2}n(n-1)} b_r \theta^r .$$

The bordered determinant $|A(\theta)|$ is clearly a polynomial of degree ½ n (n + 1) in θ, and it will be seen later that it has a factor θ^n. Let

$$|A(\theta)| = \theta^n \sum_0^{\frac{1}{2}n(n-1)} a_r \theta^r .$$

Thus
$$\begin{aligned} g_Z(\theta) &= -(1-\mu_X)\{\sum_0^{\frac{1}{2}n(n-1)} a_r\theta^r\}/\{\sum_0^{\frac{1}{2}n(n-1)} b_r\theta^r\} \\ &= -(1-\mu_X)\{c_0 + R(\theta)\} \end{aligned}$$

where c_0 is an appropriate constant (or polynomial in cases where $|B(\theta)|$ has less than maximal degree) and $R(\theta)$ is a proper rational fraction in θ.

In principle, the resolution of $R(\theta)$ into its partial fractions, which may then be expanded by the binomial theorem as power series in θ furnishes a method by which the coefficient of θ^r may be determined. From the expansion it will be seen that $P_\infty(Z_t = r)$, given by the coefficient of θ^r, is made up of a sum of purely geometric terms $\Sigma_i a_i \lambda_i^{-r}$ corresponding to the real roots λ_i of the denominator of $R(\theta)$, and a sum of trigonometric terms $\Sigma_j b_j \rho_j^{-r} \cos(r\,\phi_j + e_j)$ corresponding to complex roots $\lambda_j = \rho_j\, e_x{}^{i\phi_j}$. In practice this will be reaso nably tractable only in special cases, but it will always be the case that

$$P_\infty(Z_t = 0) = g_z(0) = -(1-\mu_x)\,a_0 / b_0 .$$

and
$$P_\infty(Z_t = 1) = g_z'(0) = -(1-\mu_x)\,(a_1 b_0 - a_0 b_1)/b_0^2, \quad (4.3.1)$$

with analogous but increasingly unwieldy expressions for $P_\infty(Z_t = r)$,

$r = 2, 3, \ldots, k$.

Now the ratio $|A(\theta)| / |B(\theta)| = \begin{vmatrix} 0 & 1' \\ l_0 & B(\theta) \end{vmatrix} / |B(\theta)|$ has a structure to which an expansion theorem of Schweins is applicable. The Schweinsian expansion is exemplified by :

$$|a_1\, b_2\, c_3\, d_4| \,/\, |b_2\, c_3\, d_4| = a_1 - a_2\, b_1 / b_2 - |a_2\, b_3| \; |b_1\, c_2| \; | \\ b_2\, |b_2\, c_3| - |a_2\, b_3\, c_4| \; |b_1\, c_2\, d_3| \,/\, |b_2\, c_3| \; |b_2\, c_3\, d_4| \qquad (4.3.2)$$

where, for example, $|b_2\, c_3\, d_4|$ denotes the third order determinant

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}$$

In our case the determinant $|A(\theta)|$ of order $n+2$, is

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ \ell_{00} & \ell_{00}-\ell & \theta\ell_{01} & \theta^2\ell_{02} & \cdots & \theta^n\ell_{0n} \\ \ell_{10} & \ell_{10} & \theta\ell_{11}-\theta & \theta^2\ell_{12} & \cdots & \theta^n\ell_{1n} \\ \ell_{20} & \ell_{20} & \theta\ell_{21} & \theta^2\ell_{22}-\theta & \cdots & \theta^n\ell_{2n} \\ & & & & & \\ \ell_{n0} & \ell_{n0} & \theta\ell_{n1} & \theta^2\ell_{n2} & \cdots & \theta^n\ell_{nn}-\theta \end{vmatrix} \qquad (4.3.3)$$

For our purpose it is convenient to write this in an alternative form obtained by multiplying the first row by θ and then adding all the other rows to it. This alters the first row to

$$\theta^{-1} \quad (1 \quad 1 \quad \theta \quad \theta^2 \quad \ldots \quad \theta^n).$$

leaving the other rows unaffected. We may now remove a factor θ from each column other than the first two, whence

$$|A(\theta)| = \theta^{n-1} \quad |C(\theta)|$$

where $|C(\theta)|$ differs from (4.3.3) in that its first row is $(1, 1, 1, \theta, \theta^2, \ldots, \theta^{n-1})$ and in that its columns, from the first on, excluding their first entries, have each had a factor θ removed. Likewise $|B(\theta)|$, the minor of the leading term in (4.3.3), may be written as

$$|B(\theta)| = \theta^n \quad |D(\theta)|,$$

where | D (θ) | is the minor of the leading terms in | C (θ) | . We then have

$$g_Z(\theta) = -(1-\mu_X)\,\frac{(1-\theta)}{\theta}\,\frac{|C(\theta)|}{|D(\theta)|},$$

and a Schweinsian expansion applied to the determinantial ratio produces

$$\frac{|C(\theta)|}{|D(\theta)|} = 1 - \ell_{00}/(\ell_{00}-\theta) - \begin{vmatrix}\ell_{00} & \ell_{00}-\theta\\ \ell_{10} & \ell_{10}\end{vmatrix}\begin{vmatrix}1 & 1\\ \ell_{00}-\theta & \ell_{01}\end{vmatrix}\Big/(\ell_{00}-\theta)\begin{vmatrix}\ell_{00}-\theta & \ell_{01}\\ \ell_{10} & \ell_{11}-1\end{vmatrix} - \ldots$$

whence, after a little straightforward manipulation,

$$g_Z(\theta) = (1-\mu_X)(1-\theta)\left[\frac{q_0}{Q_0(\theta)} + \theta\,\frac{q_1}{Q_1(\theta)} + \theta^2\,\frac{q_2}{Q_2(\theta)} + \ldots + \theta^{n-1}\,\frac{q_{n-1}}{Q_{n-1}(\theta)}\right]$$

where $q_0 = \begin{vmatrix}1 & 1\\ \ell_{10} & \ell_{11}-1\end{vmatrix}$, $Q_0(\theta) = \begin{vmatrix}\ell_{00}-\theta & \ell_{01}\\ \ell_{10} & \ell_{11}-1\end{vmatrix}$,

$$q_1 = \begin{vmatrix}\ell_{10} & \ell_{11}-1\\ \ell_{20} & \ell_{21}\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\ \ell_{00}-\theta & \ell_{01} & \ell_{02}\\ \ell_{10} & \ell_{11}-1 & \ell_{12}\end{vmatrix}$$

$$Q_1(\theta) = \begin{vmatrix}\ell_{00}-\theta & \ell_{01}\\ \ell_{10} & \ell_{11}-1\end{vmatrix}\begin{vmatrix}\ell_{00}-\theta & \ell_{01} & \theta\ell_{02}\\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12}\\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1\end{vmatrix}.$$

$$q_2 = \begin{vmatrix}\ell_{10} & \ell_{11}-1 & \theta\ell_{12}\\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1\\ \ell_{30} & \ell_{31} & \theta\ell_{32}\end{vmatrix}\begin{vmatrix}1 & 1 & 0 & 1\\ \ell_{00}-\theta & \ell_{01} & \theta\ell_{02} & \ell_{03}\\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12} & \ell_{13}\\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1 & \ell_{23}\end{vmatrix}.$$

$$Q_2(\theta) = \begin{vmatrix}\ell_{00}-\theta & \ell_{01} & \theta\ell_{02}\\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12}\\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1\end{vmatrix}\begin{vmatrix}\ell_{00}-\theta & \ell_{01} & \theta\ell_{02} & \theta^2\ell_{03}\\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12} & \theta^2\ell_{12}\\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1 & \theta^2\ell_{23}\\ \ell_{30} & \ell_{31} & \theta\ell_{32} & \theta^2\ell_{33}-1\end{vmatrix}$$

and so on, with finally,

$$q_{n-1} = \begin{vmatrix} \ell_{10} & \ell_{11}-1 & \cdots & \theta^{n-2}\ell_{1,n-1} \\ \ell_{20} & \ell_{21} & \cdots & \theta^{n-2}\ell_{2,n-1} \\ \vdots & \vdots & & \vdots \\ \ell_{n-1,0} & \ell_{n-1,1} & \cdots & \theta^{n-2}\ell_{n-1,n-1}-1 \\ \ell_{n0} & \ell_{n1} & \cdots & \theta^{n-2}\ell_{n,n-1} \end{vmatrix} \begin{vmatrix} 1 & 1 & \theta & 1 & \cdots & 1 \\ \ell_{00}-\theta & \ell_{01} & \theta\ell_{02} & \ell_{03} & & \ell_{0n} \\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12} & \ell_{13} & & \ell_{1n} \\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1 & \ell_{23} & & \ell_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \\ \ell_{n-1,0} & \ell_{n-1,1} & \theta\ell_{n-1,2} & \ell_{n-1,3} & \cdots & \ell_{n-1,n} \end{vmatrix} .$$

$$Q_{n-1}(\theta) = \begin{vmatrix} \ell_{00}-\theta & \ell_{01} & \theta\ell_{02} & \cdots & \theta^{n-2}\ell_{0,n-1} \\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12} & \cdots & \theta^{n-2}\ell_{1,n-1} \\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1 & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ \ell_{n-1,0} & \ell_{n-1,1} & \theta\ell_{n-1,2} & \cdots & \theta^{n-2}\ell_{n-1,n-1}-1 \end{vmatrix} \begin{vmatrix} \ell_{00}-\theta & \ell_{01} & \theta\ell_{02} & \cdots & \theta^{n-1}\ell_{0n} \\ \ell_{10} & \ell_{11}-1 & \theta\ell_{12} & & \theta^{n-1}\ell_{1n} \\ \ell_{20} & \ell_{21} & \theta\ell_{22}-1 & & \\ \vdots & \vdots & \vdots & & \vdots \\ \ell_{n0} & \ell_{n1} & \theta\ell_{n2} & \cdots & \theta^{n-1}\ell_{nn}-1 \end{vmatrix} .$$

It is true that the expansion of $g_z(\theta)$ given above in is not a power series in θ : it is not a formula from which one may readily obtain the distribution of Z_t , but it is in a form which makes it easily possible to obtain $g_z(0)$, the probability of emptiness, thus :

$$P_\infty(Z_t=0) = (1-\mu_X)q_0/Q_0(0)$$

$$= (1-\mu_X)\begin{vmatrix} 1 & 1 \\ \ell_{10} & \ell_{11}-1 \end{vmatrix} \begin{vmatrix} \ell_{00} & \ell_{01} \\ \ell_{10} & \ell_{11}-1 \end{vmatrix} .$$

It will be noted that this probability depends on the inflow distribution through its mean value and the leading 2 x 2 submatrix of L only.

We may similarly evaluate.

$$P_\infty(Z_t=1) = g_Z'(0) =$$

$$= -(1-\mu_X)\left\{\begin{vmatrix} 1 & 1 \\ \ell_{10} & \ell_{11}-1 \end{vmatrix} \begin{vmatrix} \ell_{00}-1 & \ell_{01} \\ \ell_{10} & \ell_{11}-1 \end{vmatrix} + \begin{vmatrix} \ell_{10} & \ell_{11}-1 \\ \ell_{20} & \ell_{21} \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ \ell_{00} & \ell_{01} & \ell_{02} \\ \ell_{10} & \ell_{11}-1 & \ell_{12} \end{vmatrix}\right\} \begin{vmatrix} \ell_{00} & \ell_{01} \\ \ell_{10} & \ell_{11}-1 \end{vmatrix}^{-2}$$

A similar formula is available for P_∞ (Z=2), but the expressions thereafter become rather cumbersome.

4.4 Mean level, for a highly specialised inflow transition matrix :

In the previous section, dealing with a general inflow transition matrix, it was possible to obtain little more than the probability of emptiness. Here, by way of contrast, we examine the drastically simplified (indeed quasi-deterministic) transition matrix of order n + 1.

$$L = \begin{pmatrix} p & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ q & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$0 < p < 1$, $q = 1-p$, for which the equilibrium inflow distribution is

$$P(X_t = 0) = a,\ P(X_t = r) = qa,\ r = 1, 2, \dots, n.$$

where

$$a = 1 / (1 + nq),$$

which is a uniform distribution with a modified first term. The mean inflow is

$$\mu_x = \tfrac{1}{2} n\,(n+1)\, q \,/\, (1+nq).$$

For an ergodic semi-infinite reservoir we must have $0 < \mu_x < 1$, whence $q < 2 / n (n - 1)$.

The generating function theorem requires the evaluation of $| H(\theta) - I\,\theta |$, of order n+1, which in this case is

$$\theta^n \begin{vmatrix} p-\theta & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & \theta & 0 & & 0 & 0 \\ 0 & 0 & -1 & \theta^2 & & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & -1 & \theta^{n-1} \\ q & 0 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}$$

$$= (-1)^n \theta^n \,(p-\theta + q\theta^{\frac{1}{2}n(n-1)}).$$

The bordered determinant $\begin{vmatrix} 0 & 1' \\ 1_0 & H(\theta)-I\theta \end{vmatrix}$ is not quite so transparent, but by row and column manipulations it may be shown to be equal to

$$- \mid H(\theta) - I\theta \mid - (-1)^n \theta^n (\theta + qh_n)$$

where h_n is the determinant of order n having 1's in the first row, θ^r (r = 0, 1, ..., n—1) in the r—th diagonal position, —1's in the sub-diagonal , and zero everywhere else ; it will be seen that

$$h_n = \theta^{n-1} h_{n-1} + 1$$

whence

$$h_n = 1 + \sum_{r=1}^{n-1} \theta^{\frac{1}{2}(n-r)(n+r+1)} .$$

We finally obtain for the bordered determinant the value

$$(-1)^{n+1} \theta^n (1 + q \sum_{r=1}^{n-1} \theta^{\frac{1}{2}r(2n-r-1)}),$$

whence

$$g_z(\theta) = (1-\mu_x)(1-\theta)(1+q\sum_{r=1}^{n} \theta^{\frac{1}{2}r(2n-r-1)}/(p-\theta + q\theta^{\frac{1}{2}n(n-1)}).$$

Using the quoted value of μ_x we note that $g_z(1) = 1$ as is required and then obtain for this reservoir (a) the probability of emptiness

$$P_\infty (Z_t = 0) = g_z (0) = (1-\mu_x) / p$$

$$= \{1-½n (n-1) q\} / p (1+nq)$$

and (b) the average level $E_\infty (Z_t) = g'_z (1) =$

$$(n-1) n (n+1) q \{3n+2-n(n+2)q\} / 12 (1+nq) (2 - n \{n-1) q\}.$$

The variance may be obtained from the formula

$$a(n) g''_2(1) = (n+1)n(n-1) (n-2)q(4(n+2) (5n^3+1) + n(5n^3-14n^2-29n$$

$$-42)q + n^2 (n-1) (n^2+ n^2 + 8) q^2).$$

where $\quad a (n) = 120 (1 + nq) \{2 - n (n-1) q \}^2$

This is a case in which, knowing μ_z, we may invoke (4.2) to detrmine $E_\infty (Z_t X_t)$, hence the covariance of Z_t and X_t. The foregoing formulae determine σ^2_z, so that if required the correlation coefficient between Z_t and X_t may be obtained.

It was remarked in § 4.3 that for the reservoirs under consideration the storage distribution has a rational generating function which in principle may be resolved into its partial fractions and hence expanded in powers of θ. In the present example the generating function is of a rather simple structure and is to some extent amenable to this treatment, or rather to the following approximation to it.

Let R (θ) be a rational function of the form

$$R (\theta) = T (\theta) / V (\theta) = Q (\theta) + U (\theta) / V (\theta)$$

where the degree of T is at least equal to that of V, the polynomial Q (θ) is the quotient of T by V, and U (θ) is the remainder obtained on dividing T by V, so that U (θ) / V (θ) is a «proper» rational fraction, that is, the degree of the numerator is less than that of the denominator.

Now consider the zeros of V (θ), and let θ_0 be the one with smallest absolute value, assumed to be a single root, and positive. It is well-known (Feller, 1959) that U (θ)/V (θ) is well approximated by the single partial fraction $\lambda_0/ (\theta - \theta_0)$, all other zeros being neglected. Here $\lambda_0 = V (\theta_0) / V' (\theta_0)$.

By definition, $T(\theta) \equiv Q(\theta)V(\theta) + U(\theta)$ so that $U(\theta_0) = T(\theta_0)$ (since $V(\theta_0) = 0$). Thus

$$\lambda_0 = T(\theta_0) / V'(\theta_0)$$

and our approximation becomes

$$R(\theta) = T(\theta)/V(\theta) \simeq Q(\theta) + \lambda_0 \sum_{r=0}^{\infty} \theta^r/\theta_0^{r+1} .$$

In our case we note that the denominator of $R(\theta)$ is

$$p - \theta + q\theta^k = (1-\theta)\{1 - q(1 + \theta + \theta^2 + \dots + \theta^{k-i})\}$$

where $k = \frac{1}{2}n(n-1)$. On removing the common factor $(1-\theta)$ between numerator and denominaior we have

$$\frac{g_Z(\theta)}{1-\mu} = R(\theta) \text{ say}$$
$$= \frac{T(\theta)}{V(\theta)}$$

where $T(\theta) = 1 + q\sum_{r=1}^{n} \theta^{\frac{1}{2}r(2n-r-1)} = 1+\dots+q\theta^k$

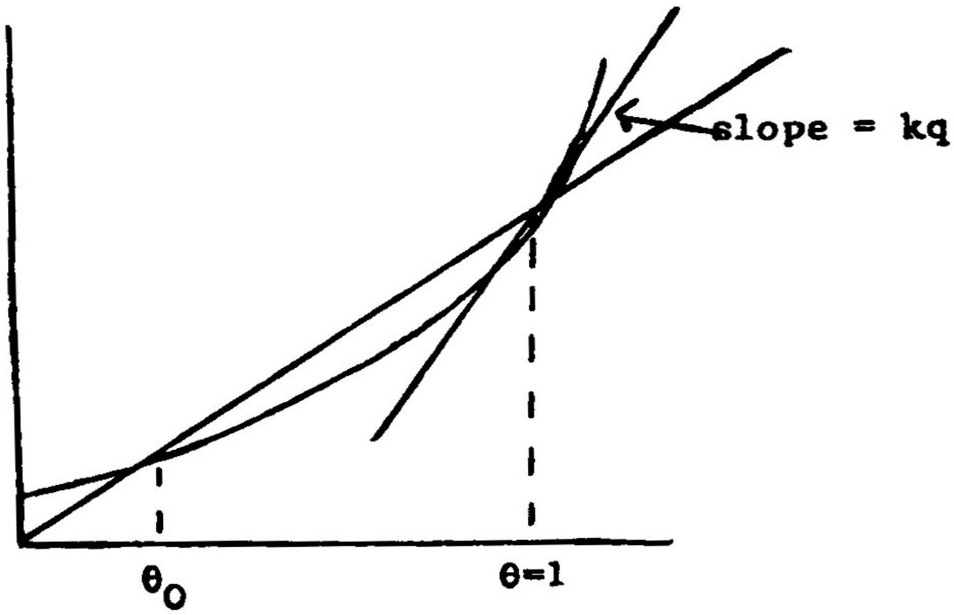
and $V(\theta) = 1 - q\sum_{r=0}^{k-1}\theta^r = 1 - \dots - q\theta^{k-1}$.

Thus, $Q(\theta) = \theta + c$, for some suitable constant c.

The foregoing theory requires that $V(\theta)$ should have a unique positive root θ_0 of minimal absolute value, and our $V(\theta)$ does indeed have such a root. For consider the polynomial $(1-\theta)V(\theta) = p - \theta + q\theta^k$. Its zeros satisfy the equation

$$p + q\theta^k = \theta.$$

This certainly has a root $\theta = 1$. Since $p + q\,\theta^k$ is convex it will in general have a second positive root θ_0, with $0 < \theta_0 < 1$ if $k\,q > 1$ and $\theta_0 > 1$ if $kq < 1$ (and $\theta_0 = 1$ if $kq = 1$).



The root θ_0 is of course a root of $V(\theta) = 1 - q(1 + \theta + \ldots + \theta^k)$ and it is in fact the unique positive root of minimal absolute value, since

$$
\begin{aligned}
|V(\theta)| &\leq q(1 + |\theta| + \ldots + |\theta|^k) \\
&< q(1 + \theta_0 + \ldots + \theta_0^k) \qquad \text{if } |\theta| < \theta_0 \\
&= 1
\end{aligned}
$$

so that no number θ for which $|\theta| < \theta_0$ can be a zero of $V(\theta)$.

As regards the value of θ_0, if p is small a first approximation will be 0, a second one, using the iteration $\theta(j+1) = f(\theta^{(j)})$, will be p, and a third $p+qp^k$. This will be adequate for most purposes. (But will fail when $p > 1 - 1/k$, when $\theta_0 \geqslant 1$).

We thus have

$$\frac{g_Z(\theta)}{1-\mu} \simeq \theta + c + \frac{\lambda_0}{\theta-\theta_0}$$

$$= \theta + c - \lambda_0 \sum_{r=0}^{\infty} \theta^r/\theta_0^{r+1}$$

where $\theta_0 \simeq p + qp^k$

and

$$\lambda_0 = - \frac{1 + q \sum_{r=1}^{n} \theta_0^{\frac{1}{2}r(2n-r-1)}}{q \sum_{r=0}^{k-2} r\theta_0^{r-1}} ,$$

so that, to our degree of approximation,

$$P_\infty (Z = r) = (1 - \mu)\, \lambda_0 \,/\, \theta_0^{\,r+1}.$$

4.5.—A more general inflow transition matrix

Consider the (n x n) inflow transition matrix

$$L = \begin{pmatrix} 1-f_0 & 1-f_1 & 1-f_2 & \cdots & 1-f_{n-1} & 1 \\ f_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_2 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & f_{n-1} & 0 \end{pmatrix}$$

This will generate an inflow distribution $P_\infty (X_t = r) = P_r$, r=0, 1, ..., n such that

$$p_r = f_{r-1}\, p_{r-1}, \quad r = 1, 2, \ldots, n,$$

whence

$$p_r = f_0\, f_1 \ldots f_{r-1}\, p_0, \quad r = 1, 2, \ldots, n.$$

where $$1/p_0 = 1 + \sum_{r=0}^{n-1} f_0 f_1 \ldots f_r.$$

This, although far from possessing any generality, is a useful matrix to work with, partly because of its adaptability, and partly because under suitable condition it allows us to remove the restriction of the inflows to a finite set of values.

The mean inflow μ_x must satisfy $\mu_x < 1$, where

$$\mu_X = \sum_0^n r p_r = p_0 \sum_0^n r\, f_0 f_1 \ldots f_{r-1}$$

whence, we must have

$$\sum_1^{n-1} r\, f_0 f_1 \ldots f_r < 1$$

for ergodicity. If we allow $n \to \infty$ we must then ensure that the f_r are such that

$$\sum_1^{\infty} r f_0 f_1 \ldots f_r < 1.$$

For the finite inflow case we obtain the following results for the determinantal evaluations :

$$| H(\theta) - I\theta | = (-1)^n \theta^n Q(\theta).$$

where

$$Q(\theta) = -\theta + (1-f_0) + \sum_{r=1}^{n-1} (1-f_r) f_0 f_1 \dots f_{r-1} \theta^{\frac{1}{2}r(r-1)} + f_0 f_1 \dots f_{n-1} \theta^{\frac{1}{2}n(n-1)},$$

and

$$\begin{vmatrix} 0 & 1' \\ f_0 & H - I\theta \end{vmatrix} = (-1)^{n+1} \theta^n P(\theta)$$

where

$$P(\theta) = 1 + \sum_{r=1}^{n} f_0 f_1 \dots f_{r-1} \theta^{\frac{1}{2}r(r-1)},$$

so that the generating function for the distribution of reservoir levels is

$$g_z(\theta) = (1-\mu_x)(1-\theta) P(\theta) / Q(\theta).$$

4.4.1 : Geometric inflows :

We now consider the special case of this model when

$f_0 = f_1 = \dots = f_n = \alpha$. We then have for the inflows.

$$P_\infty (X_t = r) = \alpha^r p_0, \quad r = 0, 1, \dots, n,$$

a finite geometric distribution, which becomes a standard geometric distribution when $n \to \infty$. For this case

$$p_r = \beta \alpha^r, \quad r = 0, 1, \dots, , \beta = 1 - \alpha$$

ane

$$\mu_x = \alpha / \beta .$$

whence, for ergodicity, $\alpha \leqslant \frac{1}{2}$.

For such a geometric inflow distribution we find that

$$g_Z(\theta) = (\beta-\alpha)(1-\theta)J(\theta)/\beta\{\beta J(\theta)-\theta\}$$

where

$$J(\theta) = \sum_{r=0}^{\infty} \alpha^r \theta^{\frac{1}{2}r(r-1)}.$$

From this formula we readily obtain the mean level :

$$E_\infty (Z_t) = g_z' (1) = \alpha^2 (1+ \alpha) / \beta^2 (\beta - \alpha).$$

(It is interesting to compare the corresponding result in the absence of autocorrelation in the inflow distribution : It is $\alpha^2 (1-\alpha) / \beta^2 (\beta-\alpha)$. With somewhat greater effort we may similarly obtain the variance of Z_t

$$\text{var}_\infty (Z_t) = \alpha^2 (1+7\alpha) / \beta^4 + 2 \alpha^3 (15 \alpha+\beta) / \beta^4 (\beta - \alpha) + 3\alpha^5 (1+\alpha) / \beta^4 (\beta - \alpha)^2 .$$

To obtain the distribution of Z_t we need to expand $g_z(\theta)$ as a power seiies in θ.

We have

$$g_Z(\theta)\{qJ(\theta) - \theta\} = (1-\mu_X)(1-\theta)J(\theta).$$

Suppose

$$g_Z(\theta) = \sum_0^\infty c_r\theta^r.$$

Differentiating repeatedly and setting $\theta = 0$, we obtain the following ecurrence ɹelationship for the c_r :

$$\beta J(0)c_n + \{\beta J^{(1)}(0) - 1\}c_{n-1} + \beta J^{(2)}(0)c_{n-2}\, 2/2\,!$$
$$+ \beta J^{(3)}(0)c_{n-3}/3! + \ldots + \beta J^{(n-1)}(0)c_1/(n-1)!$$
$$= -(1-\mu_X)J^{(n-1)}(0)/(n-1)!\,, \quad n=1,2,\ldots$$

where $J(0) = 1+\alpha,\ J^{(1)}(0) = \alpha^2,\ J^{(2)}(0) = 0,$
$J^{(3)}(0) = 3!\alpha^3,\ J^{(4)}(0) = 0,\ J^{(5)}(0) = 0,\ J^{(6)}(0) = 6!\alpha^4,$

and, in general,

$$H^{(\frac{1}{2}r(r+1))}(0) = (\tfrac{1}{2}r(r+1))!\ \alpha^r,\ r = 1, 2, \ldots$$
$$H^{(r)}(0) = 0 \quad \text{otherwise.}$$

We find

$$c_0 = (\beta-\alpha)/\beta^2, \quad c_1 = (\beta-\alpha)\alpha^2/\beta^3(1+\alpha),$$

$$c_2 = (\beta-\alpha)\alpha^3/\beta^4(1+\alpha)^2,$$

$$c_3 = (\beta-\alpha)\alpha^3(1-\beta\alpha^2)/\beta^5(1+\alpha)^3,$$

$$c_4 = (\beta-\alpha)\alpha^4(1-\beta\alpha^4)/\beta^6(1+\alpha)^4,\ldots \quad .$$

4.5.2 : Poisson inflows

If in the adaptable matrix of (4.5) we set

$f_0 = \alpha$, $f_1 = \alpha/2$, $f_3 = \alpha/3$, ..., $f_{n-1} = \alpha/n$ we obtain for the equilibrium inflow distribution a truncated Poisson distribution. On allowing $n \rightarrow \infty$ we obtain the Poisson distribution with parameter α. ($\alpha \leqslant 1$).

In this case the generating function for the levels may be expressed in the form

$$g_z(\theta) = (1-\alpha)(1-\theta)\, k'(\theta,\alpha) / \{k'(\theta,\alpha) - K(\theta,\alpha) - \theta\},$$

where

$$K(\theta,\alpha) = \sum_{r=0}^{\infty} \alpha^{r+1}\theta^{\frac{1}{2}r(r-1)}/(r+1)!,$$

and $k' = dK/d\alpha$.

After suitable manipulations we find for the probability of emptiness

$$g_z^1(0) = (1-\alpha^2)/(1-\tfrac{1}{2}\alpha^2)$$

and mean storage

$$g_z^{\cdot}(1) = \alpha^2/2(1-\alpha).$$

(It may be noted that in the corresponding reservoir with independent inflows the probability of emptiness is $(1-\alpha)\,e^{\alpha}$, and the mean storage $\alpha^2/2(1-\alpha)$.)

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SUMMARY

After drawing attention to the contrast between the assumptions made in simple resevoir theory and the requirements of a realistic model the paper concentrates on a compromise model involving (as a simplification) semi-infinite capacity and (as a step towards realism) Markovian inflows, and considers for this model a number of applications of the Odoom-Lloyd matrix generating function for the asymptotic storage distribution.

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