CANTOR'S REPRESENTATION OF REAL NUMBERS AND DISTRIBUTION OF SEQUENCES

by

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1. Let (s_n) be a sequence of real numbers satisfying the conditions $0 \leqslant s_n \leqslant 1$ for all n. For any interval $0 \leqslant a \leqslant b \leqslant 1$ let $I(x)$ denote its characteristic function.

The sequence (s_n) is called uniformly distributed if

$$(1) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} I(s_k) = b - a.$$

holds for every interval $[a, b]$. It is called well distributed if

$$(2) \qquad \lim_{p \to \infty} \frac{1}{p} \sum_{k=n+1}^{n+p} I(s_k) = b - a$$

holds uniformly in n, for every interval $[a, b]$.

From this we see that all well distributed sequences are at the same time uniformly distributed.

The purpose of this work is to give an example to show that the converse is not always true. To do this we shall use Cantor's representation of real numbers given by the following theorem, [1].

Theorem 1. Let (a_i) be a given sequence of positive integers with $a_i > a_{i-1}$ for all i. Then any real number α can be represented uniquely in the form

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$$\alpha = c_0 + \sum_{i=1}^{\infty} \frac{c_i}{a_1 a_2 \dots a_i} \tag{3}$$

where the c_i are integers with $0 \leqslant c_i \leqslant a_i - 1$.

2. Let $\{x\}$ stands for $x - [x]$ where $[x]$ is the largest integer less than o equal to x. Then we have the following special case of a theorem of Weyl, [2

Theorem 2. The set of α's, $0 < \alpha \leqslant 1$, for which the sequence

$$\left(\left\{ \left(\prod_{i=1}^{k} a_i \right) \alpha \right\} \right), \quad k=1,2,\dots$$

is uniformly distributed has measure 1.

We denote the above sequence by the letter S. We shall prove the following theorem :

Theorem 3. The set of α's , $0 < \alpha \leqslant 1$, for which the sequence S is wel distributed has measure 0.

For the proof of this theorem we need the following results in set theory and measure.

Definition 1. Let μ (E) denote the measure of a set E, and let $E_1, E_2, \dots$ be measurable sets on the interval [0, 1]. The sets $E_1, E_2, \dots$ are called independent if

$$\mu\left(\bigcap_{k=1}^{n} E_k\right) = \prod_{k=1}^{n} \mu(E_k) \tag{4}$$

for every finite class of distinct sets E_k, $k = 1, 2, \dots, n$.

Definition 2. Given the arbitrary sets $E_1, E_2, \dots$, the upper limit of $(E_n)_{n=1}^{\infty}$ if given by

$$\overline{\lim_{n \to \infty}} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \tag{5}$$

The Borel-Cantelli Lemma [3]. If (E_n) is a sequence of independent sets, then

$$(6) \qquad \mu\left(\overline{\lim_{n\to\infty}}\, E_n\right) = 0$$

if and only if

$$(7) \qquad \sum_{n=1}^{\infty} \mu(E_n) < \infty$$

The law of nought or one, [4]. Suppose that E is a measurable set on [0, 1], such that a relation of the form $(x_1, x_2, \ldots) \in E$ remains true when a finite number of the x_i are replaced by others. Then the set E has either the measure 0 or the measure 1.

In particular if E is a measurable set on [0, 1] with the property that « if $t = .\, b_1\, b_2\, b_3 \ldots$ is the infinite dyadic expansion of a point $t \in E$; then the point obtained by altering a finite number of the b_i also belongs to E » , then E has either the measure 0 or measure 1.

We also need the following theorem due to Keogh, Lawton, and Petersen [5].

Theorem 4. The sequence S defined before is well distributed if, and only if, the sequence

$$\left(\frac{c_k}{a_k}\right)$$

is well distributed.

3. *Proof of theorem* 3. Let (ν_k) be a sequence of natural numbers satisfying the condition

$$\nu_{k+1} \geqslant \nu_k + [\log_2 k] \qquad k = 1, 2, \ldots$$

If the sequence S is well distributed then we cannot have, for instance

$$(8) \qquad \frac{e_n}{a_n} \leqslant \frac{1}{2} \text{ for } \nu_k \leqslant n \leqslant \nu_k + [\log_2 k]$$

for infinitely many k. For if (8) were satisfied and I (x) is the characteristic function of the interval $[0, \frac{1}{2}]$, we have :

$$(9)\qquad \frac{1}{[\log_2 k]} \sum_{n=\nu_k+1}^{\nu_k+[\log_2 k]} I\left(\frac{c_n}{a_n}\right) = 1$$

for infinitely many k and the sequence $\left(\frac{c_n}{a_n}\right)$ is not well distributed. According to theorem 4, this means that the sequence S is not well distributed.

For a fixed integer n, it is clear that the probability for an α for which $\frac{c_n}{a_n} \leqslant \frac{1}{2}$ is equal to $\frac{1}{2}$. At the same time, this probability is equal to the measure of the set of α's for which $\frac{c_n}{a_n} \leqslant \frac{1}{2}$. It follows that the probability for an α for which

$$(10)\qquad \frac{c_n}{a_n} \leqslant \frac{1}{2}, \quad \nu_{k+1} \leqslant n \leqslant \nu_{k+1} + [\log_2(k+1)]$$

is equal to $(1/2)^{[\log_2(k+1)]}$

This is equal to the measure of the set of α's for which (10) holds. Denoting this set by E_k we have :

$$(11)\qquad \begin{aligned} \mu(E_k) &= (1/2)^{[\log_2(k+1)]} \\ &\geqslant (1/2)^{\log_2(k+1)} \\ &= \frac{1}{k+1}, \quad \text{for all } k = 1,2,3,\ldots \end{aligned}$$

Now let

$$\begin{aligned} I_m &= \bigcap_{k=1}^{m} E_k \\ &= \left\{\alpha \,\middle|\, \frac{c_n}{a_n} \leqslant 1/2,\ \nu_{k+1} \leqslant n \leqslant \nu_{k+1} + [\log_2(k+1)],\ k=1,2,\ldots,m\right\} \end{aligned}$$

According to the choice of the sequence (ν_k) we have :

$$\begin{aligned} \mu(I_m) &= (1/2)^{[\log_2 2] + [\log_2 3] + \cdots + [\log_2(m+1)]} \\ &= \prod_{k=1}^{m} \mu(E_k) . \end{aligned}$$

So that, by definition 1, the sets (E_k) are independent.

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Also we have by (11)

$$\sum \mu(E_k) \geqslant \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$$

So, by the Boral-Cantelli lemma, we have

$$\mu\left(\overline{\lim_{k\to\infty}}\, E_k\right) > 0$$

That is the set of α for which (10) is satisfied for infinitely many k has positive measure.

However, we have seen that if (10) is satisfied for infinitely many k, the sequence S is not well distributed. If we denote by T the set of α's for which the sequence S is not well distributed, it follows that

$$\overline{\lim_{k\to\infty}}\, E_k \subset T$$

But we have proved that

$$\mu\left(\overline{\lim_{k\to\infty}}\, E_k\right) > 0$$

So T has also a positive measure.

Now we prove that T satisfies the hypotheses of the 0 or 1 law. If the dyadic expansion of a point $t \in T$ is changed in a finite number of places, this means the addition (or subtraction) of a rational to the point t. But all rationals when expanded as in (3) have only finitely many terms. That is $c_n \neq 0$ for only finitely many n. Hence when the dyadic expansion of $t \in T$ is altered in finitely many places, the expansion (3) is altrred in finitely many places. But if t_1 is obtained from t by altering finitely many terms in (3) $t_1 \in T$ as the sequence of fractions $\left(\frac{c_n}{a_n}\right)$ will be un-altered [save for a finite set of terms. Hence, according to the particular case of the law of 0 or 1, the set T satisfies the hypotheses of this law and consequently has either the measure 0 or measure 1. But we have proved that it has a positive measure. So this measure must be 1.

This completes the proof of theorem 3.

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