ON THE EXTENDED CESARO MEANS

By

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INTRODUCTION

Let $\{\beta_n\}$ be a sequence of real numbers satisfying the conditions :

(1.1) $$\beta_0 = 0,\ 0 < p < \beta_{n+1} - \beta_n < q \text{ for all } n.$$

A summability matrix method $A = (a_{m,n})$ was defined by Burkill, (1), by means of the following equations :

(1.2) $$a_{m,n} = \left[\frac{\prod_{r=1}^{k} (\beta_{m+r} - \beta_n) - \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+1})}{\prod_{r=1}^{k} \beta_{m+r}}\right], \text{ for } n \leq m,$$

and

$a_{m,n} = 0$ for $n > m$.

A sequence $\{s_n\}$ is called (C, β_n, k) limitable to the limit s if

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_{m,n} s_n = s$$

Putting $\beta_n = n$ for all n in (1.2) we get the well known Cesàro means of order k.

Burkill, considered the relation between these new methods and each of the Riemann's methods (R, k, β_n) and the Reisz methods (R, β_n, k). For definition of (R, k, β_n) and (R, β_n, k) see, for instance, (2)).

From the relations, obtained by Burkill, many properties for the (R, k, β_n) and (R, β_n, k) methods, can be extended at once to the (C, β_n, k) methods.

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Here we wish to discuss some properties of the methods (C, β_n, k), of which we shall obtain proofs depending, directly, on the definition of these methods.

For reference we give the proof of regularity of these methods in the following section.

2. Regularity of the methods : From the definition of (β_n) and the matrix A, it follows that :

(2.1) $a_{m,n} \geq 0$ for all m and n.

Also we have

$$p + \beta_n < \beta_{n+1} < q + \beta_n$$

from which we see that β_n increases to ∞.

Using these facts we get :

$$(2.2) \quad \sum_{n=0}^{m} |a_{m,n}| = \sum_{n=0}^{m} a_{m,n} = 1 \text{ for all } m ,$$

and

$$(2.3) \quad |a_{m,n}| \leq 2 |1 - \beta_n / \beta_{m+1}| |1 - \beta_n / \beta_{m+2}| \ldots |1 - \beta_n / \beta_{m+k}|$$

$$\longrightarrow 0 \text{ as } m \longrightarrow \infty \text{ for every } n$$

In other words, the methods (C, β_n, k) defined by (1.2), with conditions (1.1) are regular.

3. Strong regularity of tpe methods : A sequence $\{s_n\}$ is called, by Lorentz, (3), almost convergent to s if

$$\lim_{p \to \infty} \frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p} = s$$

holds uniformly in n.

A method which sums all almost convergent sequences is called strongly regular.

The following theorem is due to Lorentz, (3) :

Theorem 3.1 In order that the regular matrix method $A = (a_{m,n})$ be strongly regular, it is necessary and sufficient that

$$(3.1) \quad \lim_{m \to \infty} \sum_{n=0}^{\infty} \left| a_{m,n} - a_{m,n+1} \right| = 0 .$$

Now we prove the following theorem :

Theorem 3.2. Let the sequence $\{\beta_n\}$ satisfy, besides the conditions (1.1), additional condition

$$\beta_{n+1} - \beta_n \geq \beta_{n+2} - \beta_{n+1} \quad \text{for all } n$$

i.e.

$$(3.2) \quad 2\beta_{n+1} \geq \beta_n + \beta_{n+2} .$$

Then, under these conditions, the method $A = (a_{m,n})$ defined by (1.2) is strongly regular.

Proof : To prove this theorem we first show that the condition (3.2) implies that $a_{m,n} \geq a_{m,n+1}$ for all $n \leq m$. We have :

$$a_{m,n} - a_{m,n+1} = \frac{1}{\prod_{r=1}^{k} \beta_{m+r}} \Big[\prod_{r=1}^{k} (\beta_{m+r} - \beta_n) + \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+2}) - 2 \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+1}) \Big]$$

For all positive integral values of k, the quantity between square brackets is positive. This can be proved by induction as follows :

For $k = 1$ we get

$$\beta_{m+1} - \beta_n + \beta_{m+1} - \beta_{n+2} - 2(\beta_{m+1} - \beta_{n+1}) = 2\beta_{n+1} - \beta_n - \beta_{n+2} > 0, \text{ using (3.2).}$$

Suppose this result is true for k. Then we have :

$$(\beta_{m+k+1} - \beta_n) \prod_{r=1}^{k} (\beta_{m+r} - \beta_n) + (\beta_{m+k+1} - \beta_{n+2}) \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+2}) - 2(\beta_{m+k+1} - \beta_{n+1}) \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+1})$$

$$= \left[(\beta_{m+k+1} - \beta_{n+1}) + (\beta_{n+1} - \beta_n)\right] \prod_{r=1}^{k} (\beta_{m+r} - \beta_n)$$

$$+ \left[(\beta_{m+k+1} - \beta_{n+1}) - (\beta_{n+2} - \beta_{n+1})\right] \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+2})$$

$$-2\,(\beta_{m+k+1} - \beta_{n+1}) \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+1})$$

$$> (\beta_{n+1} - \beta_n) \prod_{r=1}^{k} (\beta_{m+r} - \beta_n) - (\beta_{n+2} - \beta_{n+1}) \prod_{r=1}^{k} (\beta_{m+r} - \beta_{n+2}) > 0 .$$

Therefor $a_{m,n} \geqslant a_{m,n+1}$ for all $n \leqslant m$.

It follows that.

$$\sum_{n=0}^{m} \left| a_{m,n} - a_{m,n+1} \right| = a_{m,0} - a_{m,n+1}$$
$$= a_{m,0} - 0$$
$$\longrightarrow 0 \text{ as } \quad n \longrightarrow \infty$$

So, according to theorem 3.1, the method is strongly regular.

4, *The Borel property of the methods.* A summability method is said, by Hill, to possess the Borel property if it sums almost all sequences of 0's and 1,s to the value ½. The following theorem, by Hill, gives sufficient condtitons for a method to have this property :

Theorem 4.1. A summability method $A = (a_{m,n})$ has the Borel property if it satisfies the following conditions :

$$(4.1) \quad \sum_{n=0}^{\infty} a_{m,n} \longrightarrow 1$$

$$(4.2) \quad \sum_{n=0}^{m} a_{m,n}^2 = O\left(\frac{1}{\log m}\right).$$

Now we prove the following theorem :

Theorem 4.2. The summability method (C,β_n,k), defined by (1.2) has the Borel property.

Preof : First, condition (4.1) is satisfied from (2.2).

For condition (4.2), we have :

$$|a_{m,n}| = \left| \prod_{r=1}^{k} (1 - B_n / B_{m+r}) - \prod_{r=1}^{k} (1 - B_{n+1} / B_{m+r}) \right|$$

$$= \Big| (B_{m+1} - B^n) \sum_{r=1}^{k} (1/B_{m+r}) - (B^2_{n+1} - B^2_n) \sum_{\mu \neq \nu} (1/B_{m+\mu} B_{m+\nu})$$

$$+ \dots + (-1)^k (B^k_{n+1} - B^k_n) / B_{m+1} B_{m+2} \cdots B_{m+k} \Big|$$

$$< qk/B_{m+1} + q(k-1)/B_{m+1} + \dots + q/B_{m+1}$$

$$= qk/B_{m+1} = O(1/m+1).$$

It follows that

$$a^2_{m,n} = |a_{m,n}| \, O(1/m+1)$$

Therefore :

$$\sum_{n} a^2_{m,n} = O(1/m+1) \sum |a_{m,n}|$$

$$= O(1/m+1)$$

$$= O(1/\log m).$$

So, according to theorem 4.1, we have the required result.

5. *Summability of* $\{s_n \alpha_n (y)\}$: If (s_n) is a given sequence, a biunique mapping of its infinite subsequences (s_{n_i}) onto the interval $Y = (0 < y \leqslant 1)$ may be obtained by defining $y = o, \alpha_1 \alpha_2 \alpha_3 \dots$ (radix 2) by means of the equations:

$$\alpha_n = 1 \ (n = n_i) \text{ and } \alpha_n = 0 \ (n \neq n_i).$$

The inverse correspondence is evident if we agree to use only the infinite representation of y. The phrase» almost all subsequences of (s_n) » will then mean that the corresponding subset of Y has measure one.

When $\{s_{n_i}\}$ is replaced by $\{s_n \alpha_n (y)\}$ where $\alpha_n (y)$ is the n *th* digit in the infinite dyadic expansion of y we have the following theorem of Hill :

Theorem 5.1. Let A be a summability method satisfying the two conditions of Theorem 4.1. Then a bounded sequence $\{s_n\}$ is summable A if and only if almost all of the sequences $\{s_n \alpha_n (y)\}$ are summable A.

Since we have proved that the method (C, β_n, k) satisfies the two conditions mentioned in theorem 5.1, (Theorem 4.2), we can state the following theorem:

Theorem 5.2. A bounded sequence $\{s_n\}$ is (C, β_n, k) summable if and only if the sequences $\{s_n \alpha_n(y)\}$ are summable (C, β_n, k).

REFERENCES

1. H. BURKILL, On Riesz and Riemann summability, Proc. Cambridge Philos. Soc. 57 (1961), 55...60.
2. G.H. HARDY, Divergent series, Oxford (1949).
3. G.G. LORENTZ, A contribution to the theory of divergent series, Acta. Math. 80 (1948) 167...190.
4. J.D. HILL, Remarks on the Borel property, Pacific J. Math. 4 (1954). 227...242.