ON THE THEORY OF SEMI STABLE DISTRIBUTION LAWS

By

HUSSEIN FAHIM

Faculty of Economics and Political Science, Cairo University

INTRODUCTION

The central problem of probability theory is the problem of finding the most general conditions for the convergence of laws of sequences of sums of random variables. Throughout the paper $X_1, X_2, \ldots, X_n, \ldots$ denete mutually independent random varaibles, defined on the probability space $\{\Omega, F, P\}$.

Let $S_n = \sum_{k=1}^{n} X_k$ be consecutive sums. Thus the problem beco-

search for conditions under which the law of convergence hold for the

formed sums $\frac{S_n}{B_n} - A_n$ where A_n, $B_n > 0$ are such normed

constants.

The classical limit problem deals with random variables which have finite first moments and, in the normal convergence case, with finite second moments as well.

Due to the powerful tools of characteristic functions and the efforts of Kolmogorov A., Levy F., Feller W., Bawly G., Khinchine A., Gnedenko E., Doblin W., (1931 - 1938), probability theory can give the answer of the following ques- : what laws, in addition to the normal law, may be limit laws for sums of independent random varaibles.

The solution of the problem is due to the introduction, by de Finetti of the dinfinietly divisible family of laws, and to the discovery of their explicit representa- by Kolmogorov A., in the case of finite second moments, and by Levy P., in the general case.

Recieved October 8, 1971.

The real liberation which gave birth to the central limit problem with a new approach due to Levy P. He stated and solved the following problem find the family of all possible limit laws of normed sums of independent and identically distributed random variables.

We are concerned now with some definitions which we are needed throughout the paper.

Definition of infinitely divisible laws τ :

A distribution function F (x) is called infinitely divisible, if for any positive integer n, its characteristic function is the n *th* power of some other characteristic function. It is already proved that [1] the class of limit laws for sums of independent random variables coincides with the class of infinitely divisible laws.

Definition of quasi stable laws τ :

Due to Levy P. (see [2] page «2.8») a distribution function F (x) is called quasi stable if for any constants $b_1 > o$, $b2 > o$ C1 and C2 there exists the constants $b > o$ and C succh that

$$F\left(\frac{x-c_1}{b_1}\right) * F\left(\frac{x-c_2}{b_2}\right) = F\left(\frac{x-c}{b}\right)$$

or expressed in terms of characteristic functions

$$f(b_1 t)\ f(b_2 t) = f(bt)\ e^{i\gamma t}$$

where b and γ are the constants which we can find. It is proved [2] that quasi characteristic functions have the form

$$\ln f(t) = \begin{cases} iat - i\beta c t \frac{2}{\pi} \ln|t| - c|t| \\ iat - c|t|^{\alpha} \left\{ 1 + i\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right) \right\}, \alpha \neq 1 \end{cases}$$

where $c \geq o$, $|\beta| \leqslant$, $o < \alpha \leqslant 2$ and a are real numbers. This type of characteristic functions are called by Feller W. ([3] page 166) stable in the broad sense, and is called by Loeve M.

([4] page 326), Gnedenko B., Kolmogorov A. ([1])

stablecharacteristic functions.

Definition of stable laws τ_2 :

Due to Levy P. ([2] page 94 , paragraph 30 and page 193-202) a distribu-function F (x) is called stable if for any constants $b_1 > o$, $b_2 > o$, there stable a costant b such that

$$F\left(\frac{x}{b_1}\right) \star F\left(\frac{x}{b_2}\right) = F\left(\frac{x}{b}\right)$$

expressed in terms of charateristic functions

$$f(b_1 t)\ f(b_2 t) = f(b t)$$

there are a difference between the two classes τ_1 and τ_2 , when $\alpha = 1$ cauchy type. Stable characteristic functions have the form,

$$\ln f(t) = \begin{cases} iat - c|t| \\ -c|t|^{\alpha}\left\{1 + i\beta \frac{t}{|t|} \tan\left(\frac{\pi\alpha}{2}\right)\right\}, \alpha \neq 1 \end{cases}$$

where $C \geq o$, $|\beta| \leq 1$, $o < \alpha \leq 2$ and a are real numbers. This tyoe of characteristic functions are called stable in the strict senes by Feller W. ([3] page 166), and are called strongly stable by Gnedenko B., Kolmogorov A. ([1]). We can easily prove that the class τ_2 is containde in the class τ_1

Definition of semi stable laws τ 3 :

stableDue to Levy P. ([2] page 204) a distfibution function F (x) is called semi , if its characteristic function satisfies the following difference equation :

$$\sigma\ (dt) = d^{\alpha}\ (t)$$

or any real $q \neq o$, $d \neq 1$ where

$$\sigma\ (t) = 4\ f\ (t)$$

It is known ([2]) that σ (t) have the following type :

$$\sigma(t) = |t|^{\alpha} \left[- P_0 (\ln |t|) + \frac{it}{|t|} P_1 (\ln |t|) \right]$$

where P_0 and P_1 are periodic functions with period ln q. This type of characteristic functions are used by Zinger A. ([5]) to obtain the class of distributions, when among the distributions of the sum of random variables Xi there are only r different ones.

Definition of quasi - semi stable laws τ_4:

Due to Levy p. ([2] page 208) a sditribution function F (X) is called quausi-semi stable if its characteristic function satisfy the following difference equation,

$$\sigma (dt) = d^{\alpha} \sigma (t) + i \gamma t$$

for any real q, where σ (t) = In f (t) and α , γ

are constants. It is proved that ([2])

$$f(t) = \left[f\left(\frac{t}{n^{1/\alpha}} \right) \exp \left\{ \frac{i\gamma t}{n^{1+1/\alpha}} \right\} \right]^n$$

It is easily to prove the following [6] page 116 - 118 7

Lemma : Let for any real $X > o$ and $y > o$, we have,

1.k (X) is measurable function, k (x) $\neq$ o.

2.k (Xy) = k (X) k (y)

then the only function which satify these conditions is the function

$$R (x) = X^{\alpha}$$

for any $\infty < \alpha < \infty$

In this paper we study the summation of a random number of independent stochastic quantities, and the investigation of limiting distributions for such sums. Consider a s quence of integer random variables ν_1, ν_2, ..., ν_n, ... defined on the same probability space of events $\{ \Omega, F, P, \}$ for any n , with positive integer values and tending to $+\infty$ for $n \to \infty$ all investigations it was sumed that ν_n is for any n indepent of the random variables X_k. Let

$$P_{nk} = P(\nu = k), \sum_{k=1}^{\infty} P_{nk} = 1$$

The *main result* :

The aime of this paper is to obtin the semi stable laws as limiting distribu-ion for random sums. Let

$$\frac{1}{B_n} \sum_{k=1}^{\nu_n} X_k$$

be a sequence of normed sums, aving a non-degenerate limit distribution for appropriatly selected constants $B_n > o$. Assume tha the random variables in the sum have a distribution functions $F_1^{(n)}(x)$,

$F_2^{(n)}(x)$, with characteristic functuons , $f_1^{(n)}(t)$, $f_2^{(n)}(t)$,

where,

$$f_1^{(n)}(t) = \int_{-\infty}^{\infty} e^{itx} dF_1^{(n)}(x), \quad f_2^{(n)}(t) = \int_{-\infty}^{\infty} e^{itx} dF_2^{(n)}(x)$$

we introduce the notations r_{1n} , r_{2n} for the number of random variables in $S[k_n]$, having a distribution function $F_1^{(n)}(x)$ and $F_2^{(n)}(x)$. We have

$$[k_n] = r_{1n} + r_{2n}$$

It is proved in [7], [8] the following theorem :

Theorem : If for $n \to \infty$

$$P\left(\frac{S_{\nu_n}}{B_n} < x\right) \longrightarrow \Phi(x)$$

where $\varphi(X)$ is non-degenerate limit distribution, and if there exists a sequence of positive numbers k_n, such that $k_n \to \infty$, and

$$\left| \frac{\nu_n}{k_n} - 1 \right| \xrightarrow{P} 0$$

Then for every sequence ($B_n \to \infty$) and for every n,

$$Q\left(\frac{t}{B_n}\right) = \exp\left\{\sigma_n\left(\frac{t}{B_n}\right)\right\} + o(1)$$

in e5ery finite interval t, where

$$\sigma_n(t) = \sum_{j=1}^{2} \sigma_{jn}(t)$$

and

$$\sigma_{jn}(t) = r_{jn}\left[e^{f_j^{(n)}(t)-1} - 1\right]$$

Theorem: Let for any sequence $B_n \to \infty$, there exists a sequence $K_n \to \infty$ such that

$$\lim_{n\to\infty} \sigma_n\left(\frac{t}{B_n}\right) = \lim_{n\to\infty} k_n \left\{f\left(\frac{t}{B_n}\right) - 1\right\} = \sigma(t) \neq 0$$

Then, there exists α such that for any a we have

$$\sigma(a\,t) = a^{\alpha}(t)$$

Proof : We have

$$\sigma(t') = \lim_{n\to\infty} k_n \left\{f\left(\frac{t'}{B_n}\right) - 1\right\}$$

$$= \lim_{n\to\infty} \frac{f_1\left(\frac{t'}{B_n}\right) - 1}{\delta\left(\frac{1}{B_n}\right)} + i\,\frac{f_2\left(\frac{t'}{B_n}\right)}{\delta\left(\frac{1}{B_n}\right)}$$

where f_1 is the real part of the characteristic function f. and f_2 is the imaginary part, and $\frac{1}{k_n} = \delta\left(\frac{1}{\beta_n}\right)$ is continuous function,

$$\sigma(t') = \lim_{n\to\infty} \frac{f_1\left(\frac{t'}{tB_n}\cdot t\right)-1}{\delta\left(\frac{t'}{t}\cdot\frac{1}{B_n}\right)} \quad \frac{\delta\left(\frac{t'}{t}\cdot\frac{1}{B_n}\right)}{\delta\left(\frac{1}{B_n}\right)}$$

$$+ i \lim_{n\to\infty} \frac{f_2\left(\frac{t'}{tB_n}\cdot t\right)}{\delta\left(\frac{t'}{t\,B_n}\right)} \quad \frac{\delta\left(\frac{t'}{t}\cdot\frac{1}{B_n}\right)}{\delta\left(\frac{1}{B_n}\right)}$$

$$= \sigma(t) \lim_{n\to\infty} \frac{\delta\left(\frac{t'}{t}\cdot\frac{1}{B_n}\right)}{\delta\left(\frac{1}{B_n}\right)}$$

or

$$\sigma(t') = \sigma(t)\, k\left(\frac{t'}{t}\right)$$

and put $t' = at$, then

$$\sigma(a\,t) = \sigma(t)\,k(a)$$

instead of t put b^t , we have

$$\sigma(a\,b\,t) = \sigma(bt)\,k(a) = \sigma(t)\,k(b)\,k(a)$$

$$= \sigma(t)\,k(a\,b)$$

$$k(a\,b) = k(a)\,k(b)$$

in the lemma, we have

$$k(a) = a^{\alpha}$$

where α is a constant, and than we have,

$$\sigma(a\,t) = a^{\alpha}\,\sigma(t)$$

Remark 1 : It is easy to prove that, if we have the relation

$$\sigma(a\,t) = a^{\alpha} \quad (t)\ ,\ a \neq 0,\ a \neq 1,\ a > 0$$

then f (t) is infinitely divisible characteristic function. Realy put in the above Relation $a^{1/\alpha}$ instead of a , therefore, we have,

$$\sigma(a^{1/\alpha} t) = a\sigma(t)$$

or

$$\sigma(t) = a\sigma\left(\frac{t}{a^{1/\alpha}}\right)$$

or

$$f(t) = \left(f\left(\frac{t}{a^{1/\alpha}}\right)\right)^{a}$$

which proves that,we are dealing with infinitely divisible characteristic functions.

Remark 2 : It is proved in [8] that the liriting distribution for the random sum, is the class of infinitely divisible distributions The last theorem shows that the limit class is a combination of two semi stable laws of class τ_3.

REFERENCES

1. GNEDENKO B., KOLMOGOFOV A. (1954), Limit distribution for sums of independent random variables (trasilated from Russian).
2. LEVY P., (1937 and 1954), Theorie de 1, addition des variables a leatteires.
3. FELLER W. (1966), An introduction to probility theory and its applications, volume II.
4. LOEVE M., (1968) Probability thecry. Third edition.
5. ZNGER A. (1965), On a class of limit distributions for normed sums of independent random variables. Probability theory and its applicati ons volume 10, Page 692
6. HAHN-ROSENTHAL, Set functions. (pages 116...118)
7. GNEDENKO B., FAHIM H., On a transfer theorem. Soviet Math. Doklad vol. 10 (1969) N4.
8. FAHIM H., Determination of the class of limit distributions of normed random sums of independent randent random variables. Proc. Institute of satistical conference and cempuation science, Cairo, (1972) Page 100...108.