Journal of Contemporary Technology and Applied Engineering (JCTAE)
Home Page: https://jctae.journals.ekb.eg/
Published by: Faculty of Engineering, Kafrelsheikh University
Volume: 4 Issue: 1 Year: 2025
Pages: 35 – 40
DOI: 10.21608/jctae.2025.372264.1049
Print ISSN: 2812-5797
Online ISSN: 2812-5800

A FAST TWO-GRID ITERATION FOR INTEGRAL EQUATIONS

Received: 31-03-2025
Accepted: 19-04-2025

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ABSTRACT. In this paper, we introduce a novel two-level iterative approach for Nyström's method, designed for the efficient solution of large equation systems resulting from the discretization of Fredholm integral equations of the second kind. The developed two-level algorithm exhibits a computational cost advantage, demonstrating faster convergence than the Atkinson-Brakhage iterative method and providing convergence for a wider range of parameters, thus enhancing its stability and accuracy. Moreover, a derivation of the convergence of this new method is presented, providing a theoretical foundation for its use and guaranteeing its mathematical soundness. Illustrative numerical examples are included to showcase the method's effectiveness and its superior performance in comparison to existing methods. The practical efficacy of the algorithm is demonstrated through a detailed comparative analysis with the Atkinson-Brakhage method, with a focus on its improved computational efficiency, broader applicability, and reduced sensitivity to parameter selection. This new method leverages a two-stage iterative process where the first level provides a good initial guess for the second, leading to accelerated convergence. The analysis features a rigorous scrutiny of the spectral attributes of the iteration matrix, aiming to establish the wider convergence range. The results indicate that this two-level approach provides a significant advancement in the numerical solution of Fredholm integral equations.

KEYWORDS: Nyström's method; Two-level iteration; Fredholm integral equations of the second kind; Atkinson-Brakhage iteration; Convergence analysis.

1. INTRODUCTION

Direct methods employed to solve linear integral equations of the second kind (see, for instance, [1]) all result in systems of linear equations. These systems, when of limited size, can be solved using Gaussian elimination. However, for larger linear systems, iterative methods are generally more efficient and frequently the only feasible approach to finding a solution. So, there are many papers concerned with the iterative solution of integral equation, see for example ([2]-[5]).

This paper introduces a new two-level iteration of Nyström's method to efficiently solve large systems of equations that result from discretizing Fredholm integral equations of the second kind. The proposed two-level algorithm is roughly half the cost of the Atkinson-Brakhage iteration and demonstrates convergence across a wider parameter range. Furthermore, the convergence of this new method is derived.

The Atkinson-Brakhage algorithm for integral equations ([6]-[9]) is reviewed initially. The core principles of this algorithm are clearly shown through an analysis of the linear equation on $C(D)$,

$$\lambda x(t) = (\mathrm{K}x)(t) + y(t) = \int_D k(t,s)x(s)ds + y(t);\ t \in D \qquad (1)$$

The Atkinson-Brakhage algorithm addresses equation (1), where k and y are continuous functions, and the goal is to determine $x \epsilon C(D)$. We assume $(\lambda - K)$ is a nonsingular map on $C(D)$ for the linear integral operator in (1). The algorithm employs a quadrature rule sequence, with index n, nodal points $\{t_{n,j}\}_{j=1}^{P_n}$ and weights $\{w_{n,j}\}_{j=1}^{P_n}$.

$$\int_D g(t)dt \approx \sum_{j=0}^{P_n} w_{n,j}\, g(t_{n,j}),\ \ n \geq 1 \qquad (2)$$

If (2) is convergent for all $g \in C(D)$; define the numerical integration operator

$$\mathrm{K}_n x = \sum_{j=0}^{P_n} w_{n,j}\, k(t, t_{n,j})\, x(t_{n,j}), \qquad t \epsilon D$$

for $x \in C(D)$ and $n \geq 1$.

The Nyström approximation to $(\lambda - \mathrm{K})x = y$ is

$$\lambda\, x_N(t) - \sum_{j=0}^{P_N} w_{N,j}\, k\left(t, t_{N,j}\right) x_N\left(t_{N,j}\right) = y(t), \quad t \epsilon D \tag{3}$$

This is denoted abstractly by:

$$(\lambda - \mathrm{K}_N)x_N = y \tag{4}$$

To solve Equation (3), the first step is to solve the finite-dimensional system.

$$\lambda\, x_N\left(t_{N,i}\right) - \sum_{j=0}^{P_N} w_{N,j}\, k\left(t_{N,i}, t_{N,j}\right) x_N\left(t_{N,j}\right) = y\left(t_{N,i}\right), \quad i = 1, P_N \tag{5}$$

to determine the solution values at the nodal points, and then, using the Nyström interpolation formula, to reconstruct the values of x_N at any $x_N \epsilon D$.

$$x_N(t) = \frac{1}{\lambda} \sum_{j=0}^{P_N} w_{N,j}\, k\left(t, t_{N,j}\right) x_N\left(t_{N,j}\right) + \frac{1}{\lambda} y(t), \quad t \epsilon D.$$

Let $x_N^{(0)}$ constitutes a preliminary approximation of the solution x_N for (5); the Atkinson-Brakhage technique is characterized by ([9]-[11]).

$$x_N^{(q+1)} = x_N^{(q)} + \frac{1}{\lambda}\left(\delta_N^{(q)} + r^{(q)}\right) \tag{6}$$

where

$$r^{(q)} = y - (\lambda - \mathrm{K}_N)x_N^{(q)}, \tag{7}$$

$$\delta_N^{(q)} = (\lambda - K_n)^{-1}\, K_N r^{(q)}, \tag{8}$$

$N \gg n$; in which N is commonly designated as the fine mesh or fine level, while n is designated as the coarse mesh or level.

This definition reveals that each iteration necessitates two calculations of K_N. Initially, $K_N x_N^{(q)}$ is calculated to determine the residual $r^{(q)}$. Subsequently, $K_N r^{(q)}$ is computed. The method proposed here substitutes the second calculation of K_N with a calculation of K_μ, where $n \leq \mu \ll N$. This alteration approximately halves the computational cost of the proposed technique's algorithm compared to the Atkinson-Brakhage iteration and expands the range of parameters for which our method achieves convergence beyond that of the Atkinson-Brakhage method.

2. Main Results

2.1. The General Form

To develop the new iteration method, we will utilize the operator realization (4) derived from (3) and then examine its implications for solving (5). We use (4) because it allows for a straightforward application of the collectively compact operator theory to demonstrate the method's convergence. A philosophy throughout this paper is that we iteratively solve (4) by presuming the direct solvability of the equation

$$z = (\lambda - \mathrm{K}_n)\omega$$

for some much smaller parameterization variable n. Rewrite (4) as

$$0 = y - (\lambda - \mathrm{K}_N)x_N \tag{9}$$

Assume $x_N^{(0)}$ represents an initial approximation of x_N solution for (3). Establish the residual

$$\begin{aligned} r^{(0)} &= y - (\lambda - \mathrm{K}_N)x_N^{(0)} \\ &= (\lambda - \mathrm{K}_N)\left((\lambda - \mathrm{K}_N)^{-1}y - x_N^{(0)}\right) \\ &= (\lambda - \mathrm{K}_N)\left(x_N - x_N^{(0)}\right) \end{aligned}$$

$$\begin{aligned} x_N &= x_N^{(0)} + (\lambda - \mathrm{K}_N)^{-1}\, r^{(0)} \\ &= x_N^{(0)} + \frac{1}{\lambda}\left(I - \frac{\mathrm{K}_N}{\lambda}\right)^{-1} r^{(0)} \\ &= x_N^{(0)} + \frac{1}{\lambda}\left(I + \frac{\mathrm{K}_N}{\lambda} + \frac{\mathrm{K}_N^2}{\lambda^2} + \cdots\right) r^{(0)} \\ &= x_N^{(0)} + \frac{1}{\lambda}\left(I + \left[I + \frac{\mathrm{K}_N}{\lambda} + \frac{\mathrm{K}_N^2}{\lambda^2} + \cdots\right]\frac{\mathrm{K}_N}{\lambda}\right) r^{(0)} \end{aligned}$$

$$x_N = x_N^{(0)} + \frac{1}{\lambda}(I + (\lambda - \mathrm{K}_N)^{-1}\mathrm{K}_N)\, r^{(0)} \tag{10}$$

Then consider the approximation

$$I + (\lambda - \mathrm{K}_N)^{-1}\mathrm{K}_N \approx I + (\lambda - \mathrm{K}_n)^{-1}\mathrm{K}_\mu$$

where $n \leq \mu \ll N$. Using it in (10), define

$$x_N^{(1)} = x_N^{(0)} + \frac{1}{\lambda}\left(I + (\lambda - \mathrm{K}_n)^{-1}\mathrm{K}_\mu\right) r^{(0)}$$

Define $(\lambda - \mathrm{K}_n)^{-1}\mathrm{K}_\mu\, r^{(0)} = \delta_N^{(0)}$,

$$x_N^{(1)} = x_N^{(0)} + \frac{1}{\lambda}\left(r^{(0)} + \delta_N^{(0)}\right)$$

Finally, the generalized iteration is given by

$$x_N^{(q+1)} = x_N^{(q)} + \frac{1}{\lambda}\left(\delta_N^{(q)} + r^{(q)}\right) \tag{11}$$

$$r^{(q)} = y - (\lambda - \mathrm{K}_N)x_N^{(q)} \tag{12}$$

$$\delta_N^{(q)} = (\lambda - \mathrm{K}_n)^{-1}\, \mathrm{K}_\mu\, r^{(q)} \tag{13}$$

where $q \geq 0$, and $x_N^{(0)}$ is properly selected.

2.2. Convergence

Here we prove the convergence of the proposed method. In the foregoing analysis we need the following assumptions [2]

(A1) K and K_n, $n \geq 1$; are linear operator $C(D) \to C(D)$.

(A2) $\|(K_n - K)g\| \to 0\ as\ n \to \infty, \forall g \in C(D)$.

(A3) The operators' family $\{K_n\}$ is collectively compact exhibits collective compactness; that is, the set

$\Psi = \{K_n\, g : n \geq 1\ and\ \|g\| \leq 1\}$ possesses a compact closure within $C(D)$:

LEMMA 1 From the assumptions (A1)-(A3) follows:

(i) K is necessarily compact.

(ii) The sequence $\{K_n\}$ demonstrates uniform

boundedness; in other words,

$$C_1 = \sup_{n\geq 1}\|K_n\| < \infty.$$

(iii) If $(\lambda - K)^{-1}$ exists, then for all n large enough, specifically for $n \geq N(\lambda)$, $(\lambda - K_n)^{-1}$ also exists and its norm is bounded by $C_2(\lambda)$,

$$C_2(\lambda) = \sup_{n\geq N(\lambda)} \|(\lambda - K_n)^{-1}\|.$$

(iv) $\|(K - K_n)K_n\|$ and $\|(K - K_n)K\|$ converge to zero as $n \to \infty$.

(v) $\alpha_n = \sup_{m\geq n}\sup_{\mu\geq 1}\|(K - K_m)K_\mu\| \to 0 \; as \; n \to \infty$.

(vi) $\|(K - K_p)(\lambda - K_n)^{-1}K_q x\| \leq \frac{a_p}{|\lambda|}(1 + C_1C_2(\lambda))$, $with\ p = \mu\ or\ N, q = \mu\ or\ N\ and \|x\| < 1$

PROOF. see ([9], pp. 96 and 138).

THEOREM 2 Assume the integral equation (1) is uniquely solvable for all $y \in C(D)$, also, let $k(t,s)$ be continuous for $t,s \in D$. Assume that the numerical integration scheme (2) converges for all $g \in C(D)$. Provided that n is adequately large, the iterative procedure (12-11) becomes convergent; specifically,

$$x_N^{(q)} \to x_N \; as \; q \to \infty \; for \; all \; N > \mu \geq n.$$

PROOF. Using (11),

$$\begin{aligned} x_N - x_N^{(q+1)} &= x_N - x_N^{(q)} - \frac{1}{\lambda}\left(r^{(q)} + \delta_N^{(q)}\right) \\ &= x_N - x_N^{(q)} - \frac{1}{\lambda}\left(I + (\lambda - K_n)^{-1}K_\mu\right)r^{(q)} \\ &= x_N - x_N^{(q)} - \tfrac{1}{\lambda}(\lambda - K_n)^{-1}\left(\lambda - K_n + K_\mu\right) r^{(q)} \\ &= x_N - x_N^{(q)} - \tfrac{1}{\lambda}(\lambda - K_n)^{-1}\left(\lambda - K_n + K_\mu\right) \\ &\quad \times (\lambda - \mathrm{K}_N)\left(x_N - x_N^{(q)}\right) \\ &= \left(I - \tfrac{1}{\lambda}(\lambda - K_n)^{-1}\left(\lambda - K_n + K_\mu\right)(\lambda - \mathrm{K}_N)\right) \\ &\quad \times \left(x_N - x_N^{(q)}\right) \\ &= \tfrac{1}{\lambda}(\lambda - K_n)^{-1}(\lambda(\lambda - \mathrm{K}_n) \\ &\quad -\left(\lambda - K_n + K_\mu\right)(\lambda - \mathrm{K}_N))\left(x_N - x_N^{(q)}\right) \end{aligned}$$

Finally,

$$\begin{aligned} x_N - x_N^{(q+1)} &= \frac{(\lambda - K_n)^{-1}}{\lambda}\Big(\left(K_\mu - K_n\right)K_N \\ &\quad + \lambda\left(\mathrm{K}_N - K_\mu\right)\Big)\left(x_N - x_N^{(q)}\right) \\ &= (R_1 + R_2)\left(x_N - x_N^{(q)}\right) \qquad (14) \end{aligned}$$

where

$$R_1 = \tfrac{1}{\lambda}(\lambda - K_n)^{-1}\left(\left(K_\mu - K_n\right)K_N\right) \qquad (15)$$

$$R_2 = (\lambda - K_n)^{-1}\left(\mathrm{K}_N - K_\mu\right) \qquad (16)$$

However, since

$\|(K - K_n)\| \geq \|(K)\| \nrightarrow 0 \; as \; n \to \infty$.

(see [9]) ; we can't show

$$\|(R_2)\| \to 0 \; for \; N \geq \mu \; and \; \mu \to \infty.$$

Instead, we will show

$$\sup_{N\geq\mu}\|(R_2)^2\| \to 0 \; as \; \mu \to \infty \qquad (17).$$

Also, we can show that $\|(R_2)\|$ is bounded and $\|R_1\| \to 0 \; as \; n \to \infty$. These turn to be sufficient, since we can use (14) to write

$$x_N - x_N^{(q+2)} = (R_1 + R_2)^2\left(x_N - x_N^{(q)}\right)$$

Then,

$$\begin{aligned} \left\|x_N - x_N^{(q+2)}\right\| &\leq \|(R_1 + R_2)^2\|\left\|x_N - x_N^{(q)}\right\| \\ &\leq (\|R_1\|^2 + 2\|R_1\|\|R_2\| + \|(R_2)^2\|)\left\|x_N \right. \\ &\quad \left. - x_N^{(q)}\right\| \qquad (18) \end{aligned}$$

To prove that $\|(R_1)\| \to 0$; using (15) and Lemma 1 we get

$$\begin{aligned} \|R_1\| &\leq \tfrac{1}{|\lambda|}\|(\lambda - K_n)^{-1}\|\left(\|(K - K_n)K_N\| \right. \\ &\quad \left. + \left\|\left(K - K_\mu\right)K_N\right\|\right) \\ &\leq \tfrac{1}{|\lambda|}C_2(\lambda)\left(a_n + a_\mu\right) \leq \tfrac{1}{|\lambda|}C_2(\lambda)(2a_n), \qquad \mu \geq n. \end{aligned}$$

Since $a_n \to 0 \; as \; n \to \infty$ and $a_\mu \leq a_n$, hence, $\|(R_1)\| \to 0 \; as \; n \to \infty$.

To show (17), write

$$\begin{aligned} \|(R_2)^2\| &= \left\|\left((\lambda - K_n)^{-1}\left(\mathrm{K}_N - K_\mu\right)\right)^2\right\| \\ &= \|(\lambda - K_n)^{-1}\| \times \\ &\quad \left\|\left(\mathrm{K}_N - K_\mu\right)(\lambda - K_n)^{-1}\left(\mathrm{K}_N - K_\mu\right)\right\| \\ &\leq \|(\lambda - K_n)^{-1}\| \times \\ &\quad \left(\left\|(\mathrm{K} - \mathrm{K}_N)(\lambda - K_n)^{-1}\left(\mathrm{K}_N - K_\mu\right)\right\|\right. \\ &\quad \left. + \left\|(\mathrm{K} - \mathrm{K}_\mu)(\lambda - K_n)^{-1}\left(\mathrm{K}_N - K_\mu\right)\right\|\right) \end{aligned}$$

$$\begin{aligned} &\leq \|(\lambda - K_n)^{-1}\|\left(\|(\mathrm{K} - \mathrm{K}_N)(\lambda - K_n)^{-1}\mathrm{K}_N\|\right. \\ &\quad + \left\|(\mathrm{K} - \mathrm{K}_N)(\lambda - K_n)^{-1}K_\mu\right\| \\ &\quad + \left\|\left(K - K_\mu\right)(\lambda - K_n)^{-1}\mathrm{K}_N\right\| \\ &\quad \left. + \left\|\left(K - K_\mu\right)(\lambda - K_n)^{-1}K_\mu\right\|\right) \end{aligned}$$

Using Lemma (1) we get

$$\begin{aligned} \|(R_2)^2\| &\leq C_2(\lambda)\,\frac{2(a_\mu + a_N)}{|\lambda|}\left(1 + C_1C_2(\lambda)\right) \\ &\leq C_2(\lambda)\,\frac{4a_\mu}{|\lambda|}\left(1 + C_1C_2(\lambda)\right) \end{aligned}$$

Since $a_n \to 0 \; as \; n \to \infty$ and $N \leq a_\mu \leq a_n$, hence, $\|(R_2)^2\| \to 0 \; as \; n \to \infty$. From (16) we get

$$\begin{aligned} \|R_2\| &\leq \|(\lambda - K_n)^{-1}\|\left(\|\mathrm{K}_N\| + \left\|K_\mu\right\|\right) \\ &\leq 2C_1C_2(\lambda) \end{aligned}$$

Since $\|R_2\|$ is bounded and

$$\|(R_1)\| \to 0 \; as \; n \to \infty,$$

then $\|(R_1)\|\|(R_2)\| \to 0 \; as \; n \to \infty$.

This completes the proof.

Based on (18), provided that n is sufficiently large, there exists

$$\begin{aligned} \varepsilon &= (\|R_1\|^2 + 2\|R_1\|\|R_2\| + \|(R_2)^2\|) \\ &= \varepsilon(n) < 1, \; so \; that \end{aligned}$$

$$\begin{aligned} \left\|x_N - x_N^{(q+2)}\right\| &\leq \varepsilon\left\|x_N - x_N^{(q)}\right\| \\ &\leq \cdots \leq \varepsilon^{0.5q}\,\varepsilon\left\|x_N - x_N^{(0)}\right\| \to 0 \; as \; q \to \infty. \end{aligned}$$

2.3. IMPLEMENTATION

The following subsection outlines the implementation of the iterative scheme defined by (11). The linear system requiring solution is (5), with the unknown quantities being

$$X_N = \left(X_N(t_{N,1}), \dots, X_N(t_{N,P_N})\right)^T$$

Considering the iteration formula (12-11), suppose that $\left\{x_N^{(q)}(t_{N,i})\right\}$ is known.

First, calculate the residual $r^{(q)}$ at $t \in \{t_{n,i}\} \cup \{t_{N,i}\}$:

$$r^{(q)}(t) = y(t) - \lambda\, x_N^{(q)}(t) + \sum_{j=1}^{P_N} w_{N,j} x_N^{(q)}(t_{N,j})\, k(t, t_{N,j}) \quad (19)$$

Second, determine $K_\mu\, r^{(q)}$ at the nodal points of both the coarse and fine mesh:

$$K_\mu\, r^{(q)}(t) = \sum_{j=1}^{P_\mu} w_{\mu,j}\, r^{(q)}(t_{\mu,j}) k(t, t_{\mu,j}), \quad t \in \{t_{n,i}\} \cup \{t_{N,i}\} \quad (20)$$

Third, Compute the correction factor $\delta_N^{(q)}$ on the coarse mesh through the solution of the system.

$$\lambda\, \delta_N^{(q)}(t_{n,i}) - \sum_{j=1}^{P_n} w_{n,j}\; k(t_{n,i}, t_{n,j})\, \delta_N^{(q)}(t_{n,j}) = K_\mu\, r^{(q)}(t_{n,i}), \qquad i = 1,2, \dots, P_n \quad (21)$$

Fourth, extending this correction to the fine mesh

$$\delta_N^{(q)}(t_{N,i}) = \frac{1}{\lambda}\left[K_\mu\, r^{(q)}(t_{N,i}) + \sum_{j=1}^{P_n} w_{n,j}\, \delta_N^{(q)}(t_{n,j})\, k(t_{N,i}, t_{n,j})\right], \quad i = 1,2, \dots, P_N \quad (22)$$

Finally, define the new iterate $x_N^{(q+1)}$ on the fine mesh by

$$x_N^{(q+1)}(t_{N,i}) = x_N^{(q)}(t_{N,i}) + \frac{1}{\lambda}\left(r^{(q)}(t_{N,i}) + \delta_N^{(q)}(t_{N,i})\right), \quad i = 1,2, \dots, P_N \quad (23)$$

We stop the iteration when

$$\left\|x_N^{(q+1)}(t_{N,i}) - x_N^{(q)}(t_{N,i})\right\| = \frac{1}{|\lambda|}\left\|r^{(q)}(t_{N,i}) + \delta_N^{(q)}(t_{N,i})\right\| \le \sigma, \qquad i = 1,2, \dots, P_N$$

2.4. Operations Count

We will analyze arithmetic operations' number required to compute a single iteration of (19-23). For this analysis, we assume that quantities like $\{y(t_{N,i})\}$ and $\{w_{N,j}\, k(t_{N,i}, t_{N,j})\}$ are determined in advance and retained for later employment during the iterative steps.

1. To compute the residuals $\{r^{(q)}(t_{N,i})\}$ and $\{r^{(q)}(t_{\mu,i})\}$ of (19), approximately $2P_N(P_N + P_\mu)$ arithmetic operations are needed (including additions, subtractions, multiplications, and divisions).

2. Calculating $\{K_\mu\, r^{(q)}(t_{N,i})\}$ and $\{K_\mu\, r^{(q)}(t_{n,i})\}$ of (20) requires approximately $2P_\mu(P_N + P_n)$ arithmetic operations.

3. To obtain the solution $\left\{\delta_N^{(q)}(t_{n,i})\right\}$ from the linear system (21) necessitates approximately $2P_n^2$ number of arithmetic operations, provided that an LU decomposition of the linear system's matrix has been previously computed and stored.

4. Calculating $\left\{\delta_N^{(q)}(t_{N,i})\right\}$ with the Nyström interpolation formula (22) necessitates roughly $2P_N P_n$ arithmetic operations.

5. The last step, (23), requires just $3P_N$ arithmetic operations.

Combining these, we have approximately

$$2P_N^2 + 4P_\mu P_N + 2P_n P_N + 2P_\mu P_n + 2P_n^2 + 3P_N$$

arithmetic operations. The cost of a single iteration of the Atkinson-Brakhage's method was approximately [10]

$$4P_N^2 + 4P_n P_N + 2P_n^2 + 3P_N$$

Thus, for $N \gg \mu \ge n$ the proposed iteration method (12-11) is approximately *half as costly* as the Atkinson-Brakhage's method (6-8) per iteration.

3. Numerical Computation

This section demonstrates the theoretical convergence findings from the preceding section. The following example was used in [10] to illustrate the Atkinson Brakhage's method.

Example. Consider solving the equation

$$\lambda x(t) - \int_0^1 k_\gamma(s+t)\, x(s)\, ds = y(t), \quad 0 \le t \le 1. \quad (24)$$

with

$$k_\gamma(\tau) = \frac{1-\gamma^2}{1-\gamma^2-2\gamma\cos(2\pi\tau)} = 1 + 2\sum_{j=1}^{\infty} \gamma^j \cos(2\pi j\tau)$$

and $0 \le \gamma < 1$. The function $y(t)$ was so chosen that the true solution was $x(t) = 1$.

The eigenvalues and eigenfunctions for the corresponding integral operator (K) are $\gamma^j, \cos(2\pi jt)$, $j = 0,1,2, \dots$ and $-\gamma^j, \sin(2\pi jt)$, $j = 1,2, \dots$

The reformulation of the Dirichlet problem for Laplace's equation $(\Delta u = 0)$ on an elliptical region within the plane yields this integral equation, as detailed in Kanwal ([11], p. 119). To define the numerical integration operator (K_n), the midpoint rule [12] is utilized as the integration rule:

$$\int_0^1 g(s)\, ds \approx \frac{1}{n}\sum_{j=1}^{n} g\left(\frac{2j-1}{2n}\right), g \in C[0,1] \quad (25)$$

In the following tables the initial guess is $\underline{x}_N^{(0)} = 0$, and the iterations was performed until $E_1 = \left\|\underline{x}_N^{(q)} - \underline{x}_N^{(q-1)}\right\|_\infty$ was less than 10^{-13}. The column *It* gives the number of iterates that were calculated, the column E_2 gives the error $\left\|\underline{x} - \underline{x}_N^{(q)}\right\|_\infty$ in the .nal

computed iterate when compared with the true solution $\underline{x} = \left[x(t_{N,1}), x(t_{N,2}), \dots, x(t_{N,N})^T\right]$; with $\underline{x}_N^{(q)}$ denoting the final iterate.

As can be seen in Table 1, the iteration's convergence rate is enhanced by increasing the coarse mesh parameter n, with this rate being independent of N. Additionally, it is noteworthy that a mesh independence principle is in effect. That is, an increase in N does not substantially alter the number of iterates needed to attain a specified decrease in the initial error. The results from the last three rows of this table and the subsequent table indicate that $N \gg \mu \geq n$ constitutes the most effective and economical parameter choice.

*Table 1. The effect of **n**, **μ**; and **N** on the convergence.*

n	μ	N	γ	λ	E_1	E_2	It
14	**100**	**500**	**0.8**	**0.9**	**diverges**	**diverges**	
18	**100**	**500**	**0.8**	**0.9**	**8.4E-14**	**2.2E-14**	**27**
36	**100**	**500**	**0.8**	**0.9**	**7.7E-16**	**4.0E-15**	**08**
18	**100**	**800**	**0.8**	**0.9**	**8.3E-14**	**2.5E-14**	**27**
18	**16**	**500**	**0.8**	**0.9**	**diverges**	**diverges**	
18	**20**	**500**	**0.8**	**0.9**	**9.0E-14**	**2.6E-14**	**29**
18	**500**	**500**	**0.8**	**0.9**	**8.4E-14**	**2.2E-14**	**27**

Table 2. A comparison between the proposed technique and Atkinson-Brakhage technique

					Atk.-Brakh. tech.		**Proposed tech.**		
λ	γ	n	N	τ	E_2	It	μ	E_2	It
0.999	0.8	32	5000	0.51	4.46E-07	30	48	4.47E-7	30
0.001	0.5	16	500		diverges		16	2.95e-07	21
2	0.5	2	3000		7.66E-015	100	4	6.66E-16	19
5	0.95	8	800	0.1	diverges		16	6.88E-15	55

Column τ gives $\tau = \frac{\text{time of calculations of the proposed algorithm}}{\text{time of calculations of Atkinson}-\text{Brakhage algorithm}}$

There are many observations to make when comparing the two methods.

First, from the first case, if the two methods have the same rate of convergence and $N \gg \mu \geq n$, the time consumed by the proposed technique equals 0.51 of the time consumed by Atkinson-Brakhage technique (this observation coincides with the results of the last subsection)

Second, when the value of λ ($\lambda = 0.999$) tends to an eigenvalue, take $\mu > n$.

Third, when the value of λ ($\lambda = 0.001$) tends to zero, the proposed technique converges and Atkinson-Brakhage technique diverges. In this case we must take $\mu = n$.

Finally, from the third and fourth case, the proposed technique is much faster in its convergence than Atkinson-Brakhage technique and it gives a good convergence in some cases in which Atkinson-Brakhage technique gives no convergence.

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