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Research article

A New Probability Continuous Distribution with Different Estimation Methods and Application

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Abstract: The inverse Ramos-Louzada distribution (IRLD), a novel one-parameter distribution designed to handle real data with hazard rates shaped like an upside-down bathtub, is presented in this work. The IRLD can be used for many applications and has asymmetric and unimodal density shapes. We derive some of the IRLD's main statistical characteristics, such as moments, incomplete moments, inverse moments, probability-weighted moments, quantile functions, extropy measurements, and stochastic ordering. Focusing on the inference procedures, including the maximum likelihood, the least squares, the weighted least squares, the Cramér-von-Mises, the maximum product of spacing, the Anderson-Darling, the right-tail Anderson-Darling, the left-tailed Anderson-Darling, minimum spacing absolute distance, and minimum spacing absolute log distance, have been used to estimate the parameter of the IRLD. An extensive simulation analysis was performed to compare the performance of various estimates based on some measures of accuracy. Using the partial and general ranks of all estimation methods for various parameter combinations, we show that the maximum likelihood approach is the best estimation strategy, followed by the maximum product of the spacings. We examined real data set to illustrate the potential applications of the proposed distribution. The findings demonstrate that the provided distribution can fit the data more accurately than competing distributions.

Keywords: Ramos–Louzada distribution; inverted distributions; inverse moments; asymmetric; quantile; statistical inference; simulation.
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1. Introduction

In many fields, including engineering, medical, and life testing, lifetime distributions are crucial for modeling data. The exponential and Weibull distributions are the most frequently utilized lifetime distributions in applied sciences. They cannot, however, be implanted in many real-world situations due to their constant and monotone hazard functions. The Lindley distribution of one parameter as an alternative model for data with nonmonotonic hazard functions has been presented by Ghitany et al. [1]. Reference [1] examined the statistical characteristics of the Lindley distribution and demonstrated that, in some cases, the Lindley can be a more accurate model than the exponential distribution. However, the exponential and Lindley distributions might not always be enough in many situations. Recently, Ramos and Louzada [2] introduced a new one-parameter model known as the Ramos-Louzada distribution (RLD) with a probability density function (PDF) that outperforms both the Lindley and exponential distributions. The survival function (SF) and the probability density function (PDF) of the RLD, for $y > 0$, are provided by:

$$\bar{G}(y) = \frac{(\eta + (y/\eta) - 1)}{\eta - 1} e^{-(y/\eta)}, \tag{1.1}$$

and

$$g(y) = \frac{((\eta - 2)\eta + y)}{\eta^2(\eta - 1)} e^{-(y/\eta)}, \tag{1.2}$$

where $\eta \geqslant 2$ is the scale parameter. According to Ref. [2], the RLD can be seen as a reparametrized variation of the Zakerzadeh and Dolati [3] distribution with a specific case of the gamma model and the occurrence of zero value. A two-parameter generalized RLD, with an extra shape parameter, was proposed by Al-Mofleh et al. [4].

On the other hand, many techniques to introduce a new probability distribution have been suggested by several scholars in recent years. One of them, the inverse transformation method of the baseline variable, produces a distribution with few parameters. The inverted distributions are significant in many fields due to their applicability, including biological sciences, life testing, chemistry data, medical sciences, and more. Inverted conformation distributions and non-inverted conformation distributions are different in terms of density and hazard functions. The reader can consult inverse linear exponential distribution [5], inverse power Lindley distribution [6], inverted Kumumaraswamy distribution [7], inverted exponentiated Weibull distribution [8], inverse power Ishita distribution [9], inverted Nadarajah-Haghighi distribution [10], inverse Ishita distribution [11], Bivariate Weibull-g family [12], inverse power Rama distribution [13], unit-exponentiated half-logistic distribution [14], inverted gamma distribution [15], skew product distribution [16], inverse xgamma distribution [17], Marshall-olkin alpha power Lomax distribution [18], inverse Gompertz distribution [19], inverse Hamza distribution [20], inverse exponentiated inverse power Cauchy distribution [21], inverse length biased Maxwell distribution [22], inverse Maxwell distribution [23], inverse unit Teissier distribution [24], inverse power Burr-Hatke distribution (IPBHD) [25], inverse log-logistic distribution (ILLD) [26], and inverse Nakagami-M distribution (INMD) [27], for a list of inverted distributions that have been studied in depth in the literature as well as information on how they can be used in various contexts.

The literature reviewed in this paper showed that the inverse transformation technique could be used to transform distributions that could not model lifetime data with upside-down bathtub shapes. The results showed that the transformed distributions were more flexible and useful than the corresponding baseline distribution for analyzing complex data structures in various fields of life. This work aims to introduce the inverted RLD (IRLD), an extension of the RLD. Since the RLD can only model data with increasing hazard rates, this novel distribution is designed to fill the gap in modeling lifetime data with non-monotone hazard rates or upside-down bathtub shapes. As a result, the IRLD can be applied in medical and engineering disciplines, as well as demography and other applied disciplines. The IRLD is established by applying the inverse transformation $Z = \frac{1}{Y}$ where Y is an RLD. The following are definitions for its cumulative distribution function (CDF), PDF, and hazard function (HF), for $z > 0$, respectively:

$$F(z) = \frac{(\eta + (1/\eta z) - 1)}{\eta - 1} e^{-(1/\eta z)}, \tag{1.3}$$

$$f(z) = \frac{((\eta - 2)\eta z + 1)}{\eta^2 z^3 (\eta - 1)} e^{-(1/\eta z)}, \tag{1.4}$$

and

$$h(z) = \frac{((\eta - 2)\eta z + 1)\, e^{-(1/\eta z)}}{\eta^2 z^3 \left[\eta - 1 - (\eta + (1/\eta z) - 1)\, e^{-(1/\eta z)}\right]},$$

where $\eta \geqslant 2$ is the scale parameter. From now on, Z~ IRLD(η) is a random variable with density (1.4). The behavior of the PDF (1.4), when $z \to 0$ and $z \to \infty$, is given by

$$\underset{z \to 0}{Lim}\, f(z) = 0, \qquad \underset{z \to \infty}{Lim}\, f(z) = 0.$$

The graphs of the IRLD's PDF appear in Figure 1 for some values of η. The unimodal and asymmetrical (right-skewed) densities are the shapes of the IRLD. The HF takes an upside-down shape for various chosen values of η, as shown in Figure 2.

The RLD was developed to overcome the drawbacks of the Lindley and exponential distributions, and it has shown a better fit for a range of real-world lifetime datasets. This achievement sparked additional study, which resulted in the creation of more adaptable distributions. This paper presents the IRLD, a novel one-parameter distribution based on the inverse transformation. The following are some of the special characteristics of the IRLD that make it particularly interesting:

1. The IRLD is a more flexible distribution than RLD, ILLD, INMD, IPBHD, Kumaraswamy distribution, exponential distribution, truncated power Lomax distribution, beta distribution, truncated Weibull distribution, Kumaraswamy Kumaraswamy distribution, especially in modeling real economic data.
2. The IRLD displays increasing, decreasing, and upside-down shaped hazard rates. Therefore, the IRLD is helpful in circumstances where the RLD is irrelevant.
3. There are closed forms in the IRLD's PDF and CDF. Because of this characteristic, it is especially well-suited for censored data analysis, enabling effective modeling and analysis in situations where data points are partially observed or truncated.

Since its creation, the RLD has undergone a number of modifications aimed at increasing flexibility. The discrete RLD [28], the exponentiated generalized RLD [29] distribution, the discrete Poisson RLD [30], in addition to the generated family based on RLD as a generator presented by [31, 32]. This study has followed this pattern by introducing a new one-parameter distribution called the IRLD. Specifically, by introducing the IRLD, we hope to close a significant gap in the statistical literature.

This study stands out in that it compares and systematically evaluates several estimation methods for the IRLD. Numerous studies have examined estimation techniques for a range of distributions. For example, Kundu and Raqab [33] and Gupta and Kundu [34] concentrated on generalized Rayleigh and generalized exponential distributions, respectively. The extended exponential geometric distribution by [35], the Kumaraswamy distribution was studied by Dey et al.[36], and the logistic-exponential distribution was presented by Ali et al.[37]. The weighted Lindley distribution by [38], inverse power xlindley distribution [39], inverse power Zeghdoudi distribution [40], and exponential logarithmic distribution [41]. Evaluating frequentist estimators' performance across a range of sample sizes and parameter values is the main goal. On the basis of intensive comparison and simulation studies, the study contributes to knowledge and practical utilization of estimation procedures of the IRLD, improving statistical analysis instruments in numerous applications. Then, we present a compelling real-world application of our proposed model by analyzing inflation rate data from 45 Asian countries. This analysis serves to empirically demonstrate the practical utility and robustness of our model in accurately characterizing real-world economic phenomena, particularly the diverse and often volatile inflation dynamics across the Asian continent.

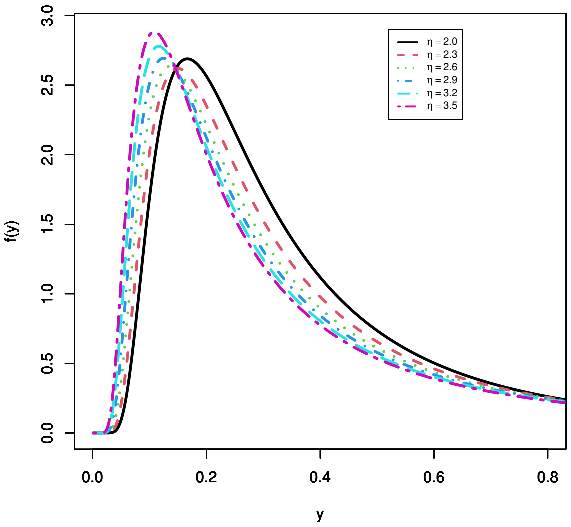

Figure 1. Plots of PDF for the IRLD

This is how the paper is structured. Section 2 derives some statistical properties, such as the quantile function (QF), probability weighted moments (PWM), ordinary and inverse moments, stochastic ordering (SO), and extropy measures for the suggested model. Section 3 describes the various frequent estimation techniques. To examine how well these methods function with the suggested model, a simulation study is done in Section 4. One genuine data set is used to demonstrate the applicability of the

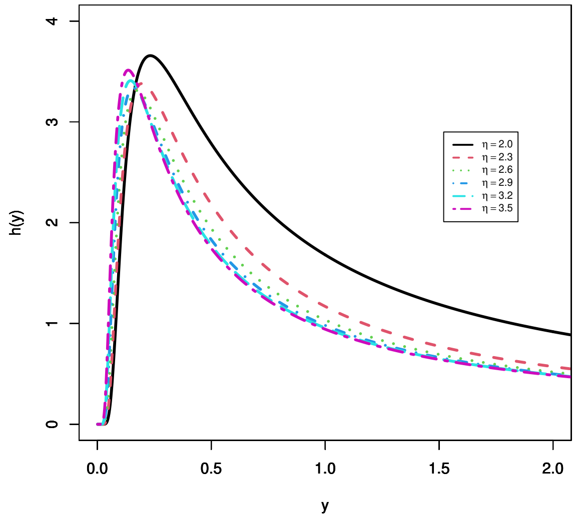

Figure 2. Plots of HF for the IRLD

IRLD in Section 5. Section 6 also includes some final observations.

2. Characteristics of the IRLD

Here, we go through some of the statistical characteristics of the suggested distribution.

2.1. Quantile Function

The QF, say $Q(u)$, where $0 < u < 1$, of the IRLD given in Equation (1.3), is obtained by solving $F(Q(u)) = u$, as follows:

$$u = \frac{(\eta + (1/\eta Q(u)) - 1)}{\eta - 1} e^{-(1/\eta Q(u))},$$

that produces;

$$u(1-\eta)e^{1-\eta} = -(\eta + (1/\eta Q(u)) - 1)\, e^{-(1/\eta Q(u) + \eta - 1)}. \tag{2.1}$$

From the previous equation, we note that $-(\eta + (1/\eta Q(u)) - 1)$ is Lambert W function of the real argument. The Lambert W function is defined by $W(z)\, e^{W(z)} = z$. Then,

$$-(\eta + (1/\eta Q(u)) - 1) = W_{-1}\left[u(1-\eta)e^{1-\eta}\right].$$

Hence, we have the negative Lambert W function of the real argument

$$Q(u) = \left\{\eta\left[W_{-1}\left(u(1-\eta)\right) e^{1-\eta} + \eta - 1\right]\right\}^{-1}, \tag{2.2}$$

where $0 < u < 1$ and $W_{-1}(.)$ is the negative branch of the Lambert function.

Based on quantiles, MacGillivray skewness (MS) is given by:

$$MS = \frac{Q(u) + Q(1-u) - 2Q(0.5)}{Q(u) - Q(1-u)}.$$

The plots of MS for the IRLD are mentioned in Figure 3.

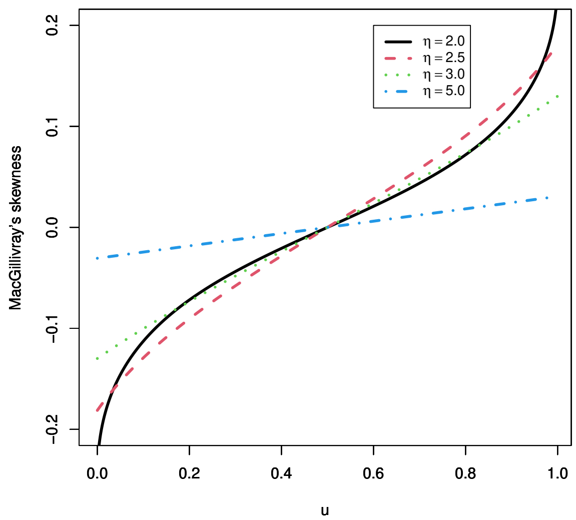

Figure 3. Plots of MS for the IRLD

2.2. *Probability Weighted Moments*

A random variable Z belongs to the class of PWM, represented by the positive integers s and r, and is given by:

$$\aleph_{r,s} = E[Z^r F(z)^s] = \int_{-\infty}^{\infty} z^r f(z)(F(z))^s dz. \tag{2.3}$$

To derive the PWM of the IRLD, we first use binomial expansion to obtain $(F(z))^s$ as next:

$$(F(z))^s = \sum_{q=0}^{s} \binom{s}{q} \frac{1}{(\eta-1)^q \eta^q z^q} e^{-(q/\eta z)}. \tag{2.4}$$

Inserting Equations (1.4) and (2.4) in Equation (2.3), we get

$$\aleph_{r,s} = \sum_{q=0}^{s} \binom{s}{q} \frac{1}{(\eta-1)^q \eta^q} \left[\frac{(\eta-2)}{\eta(\eta-1)} \int_{0}^{\infty} z^{r-q-2} e^{-((q+1)/\eta z)} dz + \frac{1}{\eta^2(\eta-1)} \int_{0}^{\infty} z^{r-q-3} e^{-((q+1)/\eta z)} dz \right]$$
$$= \sum_{q=0}^{s} \binom{s}{q} \frac{1}{(\eta-1)^q \eta^q} \left[\frac{(\eta-2)\Gamma(q-r+1)\eta^{q-r}}{(\eta-1)(q+1)^{r-q+1}} + \frac{\Gamma(q-r+2)\eta^{q-r}}{(\eta-1)(q+1)^{r-q+2}} \right].$$

Hence, after simplification, the PWM of the IRLD takes the following form:

$$\aleph_{r,s} = \sum_{q=0}^{s} \binom{s}{q} \frac{1}{(\eta-1)^{q+1}\eta^{r}} \left[\frac{(\eta-2)\Gamma(q-r+1)}{(q+1)^{r-q+1}} + \frac{\Gamma(q-r+2)}{(s+1)^{r-q+2}} \right],$$

where, $\Gamma(.)$ is the gamma function (GF).

2.3. Moments and Associated Measures

Moments are crucial in statistical theory; thus, we give the mth moment for the IRLD in this subsection.

Proposition 1. The mth moment and mth inverse moment for the random variable Z, which follows the IRLD, are given by

$$\mu'_m = (\eta-1)^{-1}\eta^{-m}\left[(\eta-m-1)\Gamma(1-m)\right], \quad m \in \mathbb{N}, m < 1,$$

and,

$$E(Z^{-m}) = (\eta-1)^{-1}\eta^{m}\left[(\eta+m-1)\Gamma(m+1)\right].$$

Proof. The mth moment of the IRLD is obtained by using PDF (1.4) as follows:

$$\begin{aligned} \mu'_m &= \frac{(\eta-2)}{\eta(\eta-1)} \int_0^\infty z^{m-2} e^{-(1/\eta z)} dz + \frac{1}{\eta^2(\eta-1)} \int_0^\infty z^{m-3} e^{-(1/\eta z)} dz \\ &= (\eta-1)^{-1}\eta^{-m}\left[(\eta-m-1)\Gamma(1-m)\right], m < 1, \end{aligned} \tag{2.5}$$

where, $\Gamma(.)$ is the GF. This indicates that the mth moment for the IRLD does not exist since the expression (2.5) only exists for $m < 1$. Furthermore, the mth inverse moment of the IRLD is as follows:

$$\begin{aligned} E(Z^{-m}) &= \frac{(\eta-2)}{\eta(\eta-1)} \int_0^\infty z^{-m-2} e^{-(1/\eta z)} dz + \frac{1}{\eta^2(\eta-1)} \int_0^\infty z^{-m-3} e^{-(1/\eta z)} dz \\ &= (\eta-1)^{-1}\eta^{m}\left[(\eta+m-1)\Gamma(m+1)\right]. \end{aligned} \tag{2.6}$$

From Equation (2.6), the harmonic mean of the IRLD can be obtained by using the first inverse moment.

Proposition 2. The mth incomplete moment for the random variable Z, which follows the IRLD, is given by

$$\Upsilon_m(t) = (\eta-1)^{-1}\eta^{-m}\left[(\eta-2)\Gamma\left(1-m, \frac{1}{\eta t}\right) + \Gamma\left(2-m, \frac{1}{\eta t}\right)\right], \quad m \in \mathbb{N}.$$

Proof. The mth incomplete moment of the IRLD is obtained by using PDF (1.4) as follows:

$$\begin{aligned} \Upsilon_m(t) &= \frac{(\eta-2)}{\eta(\eta-1)} \int_0^t z^{m-2} e^{-(1/\eta z)} dz + \frac{1}{\eta^2(\eta-1)} \int_0^t z^{m-3} e^{-(1/\eta z)} dz \\ &= (\eta-1)^{-1}\eta^{-m}\left[(\eta-2)\Gamma\left(1-m, \frac{1}{\eta t}\right) + \Gamma\left(2-m, \frac{1}{\eta t}\right)\right], m < 1 \end{aligned}$$

where, $\Gamma(., x)$ is the upper incomplete GF.

2.4. Extropy Measures

A new measurement of uncertainty termed extropy was recently introduced by Lad et al. [42] as the complement dual of entropy (Shannon [43]). Extropy can be used statistically to score forecasting distributions using the total log scoring system. The extropy is defined as follows for a random variable Z that is not negative:

$$\Phi(Z) = \frac{-1}{2} \int_0^\infty f^2(z) dz. \tag{2.7}$$

The IRLD's extropy can thus be expressed as follows using PDF (1.4) in Equation (2.7):

$$\Phi(Z) = \frac{-1}{2} \left[\left(\frac{\eta - 2}{\eta^2(\eta - 1)} \right)^2 \int_0^\infty z^{-4} e^{-(2/\eta z)} dz + \frac{2(\eta - 2)}{\eta^3(\eta - 1)^2} \int_0^\infty z^{-5} e^{-(1/\eta z)} dz + \frac{1}{\eta^4(\eta - 1)^2} \int_0^\infty z^{-6} e^{-(2/\eta z)} dz \right].$$

After simplification, the extropy of the IRLD is given by:

$$\Phi(Z) = \frac{-1}{2} \left[\frac{\eta(\eta + 1)(\eta - 2) + 3\eta}{4(\eta - 1)^2} \right]. \tag{2.8}$$

In a similar manner to residual entropy, Qiu and Jia [44] defined the extropy for residual lifetime Z_t as the residual extropy (REx) at time t as:

$$\Phi(Z_t) = \frac{-1}{2\bar{F}^2(t)} \int_t^\infty f^2(z) dz. \tag{2.9}$$

The IRLD's REx can thus be expressed as follows using PDF (1.4) in Equation (2.9):

$$\begin{aligned} \Phi(Z_t) &= \frac{-1}{2\bar{F}^2(t)} \left[\left(\frac{\eta - 2}{\eta(\eta - 1)} \right)^2 \int_t^\infty z^{-4} e^{-(2/\eta z)} dz + \frac{2(\eta - 2)}{\eta^3(\eta - 1)^2} \int_t^\infty z^{-5} e^{-(1/\eta z)} dz + \frac{1}{\eta^4(\eta - 1)^2} \int_t^\infty z^{-6} e^{-(2/\eta z)} dz \right] \\ &= \frac{-\eta}{64(\eta - 1)^2 \bar{F}^2(t)} \left[4(\eta - 2)^2 \gamma\left[3, (2/\eta t)\right] + 4(\eta - 2) \gamma\left[4, (2/\eta t)\right] + \gamma\left[5, (2/\eta t)\right] \right], \end{aligned}$$

where $\gamma(., x)$ is the lower incomplete GF.

2.5. Stochastic Ordering

In reliability theory and other areas, the SO is a well-researched concept in probability distributions that is used to assess how well random variables behave. Assume that Z_i, $i = 1, 2$ have the IRLD with parameters (η_i). Assume that $F_i(z)$ and $f_i(z)$ indicate, respectively, Zi's CDF and PDF.

If $f_{Z_1}(z)/f_{Z_2}(z)$ is a decreasing function $\forall z$, then, in terms of likelihood ratio order; Z_1 is said to be stochastically less than Z_2 (represented by $Z_1 \leqslant_{lr} Z_2$). Let $Z_1 \sim$ IRLD (η_1) and $Z_2 \sim$ IRLD (η_2) then the likelihood ratio ordering is as follows

$$\frac{f_{Z_1}(z)}{f_{Z_2}(z)} = \frac{((\eta_1 - 2)\eta_1 z + 1) e^{-(1/\eta_1 z)} \eta_2^2 (\eta_2 - 1)}{\eta_1^2 (\eta_1 - 1) \left[((\eta_2 - 2)\eta_2 z + 1) e^{-(1/\eta_2 z)} \right]},$$

and

$$\frac{d}{dz}\log\left\{\frac{f_{Z_1}(z)}{f_{Z_2}(z)}\right\}=\frac{(\eta_1-2)\eta_1}{(\eta_1-2)\eta_1 z+1}+\frac{1}{\eta_1 z^2}-\frac{(\eta_2-2)\eta_2}{(\eta_2-2)\eta_2 z+1}-\frac{1}{\eta_2 z^2}.$$

For $\eta_1 < \eta_2$ we get $\frac{d}{dz}\log\{f_{Z_1}(z)/f_{Z_2}(z)\}$ $\forall z \geqslant 0$, hence $f_{Z_1}(z)/f_{Z_2}(z)$ is decreasing in z and hence $Z_1 \leqslant_{lr} Z_2$. Moreover, Z_1 is said to be smaller than Z_2 in other different orderings as SO (denoted by $Z_1 \leqslant_{SO} Z_2$), HF order (denoted by $Z_1 \leqslant_{ho} Z_2$), and reversed HF (denoted by $Z_1 \leqslant_{rho} Z_2$).

3. Estimation Methods

In this section, we want to find an estimator for the proposed model parameter ($\hat{\eta}$) that we propose by employing several estimation strategies that are defined by maximizing or minimizing an objective function, as seen in the next steps.

The maximum likelihood estimation approach (EM_1) is used to construct our recommended model estimator ($\hat{\eta}$) by optimising the following equation:

$$\log L=-\frac{1}{\eta}\sum_{i=1}^{n}\frac{1}{z_i}+\sum_{i=1}^{n}\log\left((\eta-2)\eta z_i+1\right)-3\sum_{i=1}^{n}\log\left(z_i\right)+n\log\left(\frac{1}{(\eta-1)\eta^2}\right).$$

The following equation is minimized using the Anderson-Darling estimation approach (EM_2) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Anderson and Darling [45].

$$\begin{aligned}A&=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(z_{(i)})+\log \bar{F}(z_{(n-i+1)})]\\&=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[-\frac{1}{\eta z_{(i)}}+\log\frac{\left(\eta+\frac{1}{\eta z_{(i)}}-1\right)}{\eta-1}+\log\left(1-\frac{e^{-\frac{1}{\eta z_{(n-i+1)}}}\left(\eta+\frac{1}{\eta z_{(n-i+1)}}-1\right)}{\eta-1}\right)\right].\end{aligned}$$

The following equation is minimized using the Cramér-von Mises estimation technique (EM_3) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Choi and Bulgren [46].

$$C=-\frac{1}{12n}+\sum_{i=1}^{n}\left[F(z_{(i)})-\frac{2i-1}{2n}\right]^2=-\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{e^{-\frac{1}{\eta z_{(i)}}}\left(\eta+\frac{1}{\eta z_{(i)}}-1\right)}{\eta-1}-\frac{2i-1}{2n}\right]^2.$$

The following equation is maximized using the maximum product of the spacings estimation technique (EM_4) to yield our recommended model estimator ($\hat{\eta}$). See Kao [47] for more details about this method.

$$\Lambda=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\Delta_i,\ \ \Delta_i=F\left(z_{(i)}\right)-F\left(z_{(i-1)}\right).$$

The following equation is minimized using the least-squares estimation technique (EM_5) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Swain et al. [48].

$$LS = \sum_{i=1}^{n}\left[F(z_{(i)}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[\frac{e^{-\frac{1}{\eta z_{(i)}}}\left(\eta + \frac{1}{\eta z_{(i)}} - 1\right)}{\eta - 1} - \frac{i}{n+1}\right]^2.$$

The following equation is minimized using the right-tail Anderson-Darling estimation technique (EM_6) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Anderson and Darling [45].

$$\begin{aligned} R &= \frac{n}{2} - 2\sum_{i=1}^{n} F\left(z_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \bar{F}\left(z_{(n+1-i)}\right) \\ &= \frac{n}{2} - 2\sum_{i=1}^{n}\frac{e^{-\frac{1}{\eta z_{(i)}}}\left(\eta + \frac{1}{\eta z_{(i)}} - 1\right)}{\eta - 1} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1 - \frac{e^{-\frac{1}{\eta z_{(-i+n+1)}}}\left(\eta + \frac{1}{\eta z_{(-i+n+1)}} - 1\right)}{\eta - 1}\right). \end{aligned}$$

The following equation is minimized using the weighted least-squares estimation technique (EM_7) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Swain et al. [48].

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(z_{(i)}) - \frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{e^{-\frac{1}{\eta z_{(i)}}}\left(\eta + \frac{1}{\eta z_{(i)}} - 1\right)}{\eta - 1} - \frac{i}{n+1}\right]^2.$$

The following equation is minimized using the left-tailed Anderson–Darling estimation (EM_8) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Anderson and Darling [45].

$$\begin{aligned} LT &= -\frac{3}{2}n + 2\sum_{i=1}^{n} F(z_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(z_{(i)}) \\ &= -\frac{3}{2}n + 2\sum_{i=1}^{n}\frac{e^{-\frac{1}{\eta z_{(i)}}}\left(\eta + \frac{1}{\eta z_{(i)}} - 1\right)}{\eta - 1} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(-\frac{1}{\eta z_{(i)}} + \log\frac{\left(\eta + \frac{1}{\eta z_{(i)}} - 1\right)}{\eta - 1}\right). \end{aligned}$$

The following equation is minimized using the minimum spacing absolute distance estimation (EM_9) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Torabi [49].

$$M_1 = \sum_{i=1}^{n+1}|\Delta_i - \frac{1}{n+1}|.$$

The following equation is minimized using the minimum spacing absolute-log distance estimation (EM_{10}) to yield our recommended model estimator ($\hat{\eta}$). For more details about this method, see Torabi [49].

$$M_2 = \sum_{i=1}^{n+1}|\log\Delta_i - \log\frac{1}{n+1}|.$$

4. Numerical Simulation

To generate data sets randomly using our recommended model, this part analyzes the behavior of all estimate strategies indicated in Section 3 and identifies suggested model estimators using these estimation approaches. Many measures were used in this investigation, including an average of absolute bias (BIAS), $|Bias(\widehat{\boldsymbol{\eta}})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})^2$, and mean absolute relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}|/\boldsymbol{\eta}$. The simulation's second purpose is to discover the best estimating strategy to employ when producing our model estimators. To conduct this simulation, we generate random samples of varying sizes from our model, calculate the measures we'll use, and then repeat those operations numerous times.

Tables 1-5 reflect the outcomes of our simulation. The ranking of any value is defined by where it ranks among all estimating approaches. The partial and total ranks of our estimates are provided in Table 6. We obtain the following findings from the simulation and ranking tables:

- All of the estimates share the consistency property.
- For all estimating procedures, the BIAS of $\hat{\eta}$ reduces as n increases.
- For all estimating procedures, the MSE of $\hat{\eta}$ reduces as n increases.
- For all estimating procedures, the MRE of $\hat{\eta}$ reduces as n increases.
- The best estimation technique is the maximum likelihood estimation method, and it's the most popularly used method in the literature. Therefore, if researchers have data sets from our suggested model, we urge them to apply this method.

Table 1. Simulation values of BIAS, MSE and MRE for $\eta = 2.0$.

n	Measure	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
20	BIAS	$0.12893^{\{1\}}$	$0.36434^{\{5\}}$	$0.29945^{\{3\}}$	$0.39556^{\{7\}}$	$0.45162^{\{9\}}$	$0.23339^{\{2\}}$	$0.34073^{\{4\}}$	$0.5135^{\{10\}}$	$0.38758^{\{6\}}$	$0.43571^{\{8\}}$
	MSE	$0.11155^{\{1\}}$	$0.5871^{\{7\}}$	$0.41125^{\{3\}}$	$0.46874^{\{5\}}$	$0.69185^{\{10\}}$	$0.2743^{\{2\}}$	$0.45344^{\{4\}}$	$0.63689^{\{9\}}$	$0.46976^{\{6\}}$	$0.59495^{\{8\}}$
	MRE	$0.06446^{\{1\}}$	$0.18217^{\{5\}}$	$0.14972^{\{3\}}$	$0.19778^{\{7\}}$	$0.22581^{\{9\}}$	$0.1167^{\{2\}}$	$0.17036^{\{4\}}$	$0.25675^{\{10\}}$	$0.19379^{\{6\}}$	$0.21786^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$17^{\{5\}}$	$9^{\{3\}}$	$19^{\{7\}}$	$28^{\{9\}}$	$6^{\{2\}}$	$12^{\{4\}}$	$29^{\{10\}}$	$18^{\{6\}}$	$24^{\{8\}}$
70	BIAS	$0.0402^{\{1\}}$	$0.12861^{\{7\}}$	$0.11177^{\{5\}}$	$0.10632^{\{4\}}$	$0.14987^{\{9\}}$	$0.0841^{\{2\}}$	$0.09508^{\{3\}}$	$0.24098^{\{10\}}$	$0.13237^{\{8\}}$	$0.11341^{\{6\}}$
	MSE	$0.0092^{\{1\}}$	$0.09524^{\{8\}}$	$0.0606^{\{7\}}$	$0.03485^{\{2\}}$	$0.10403^{\{9\}}$	$0.03593^{\{3\}}$	$0.04858^{\{4\}}$	$0.14796^{\{10\}}$	$0.05931^{\{6\}}$	$0.05335^{\{5\}}$
	MRE	$0.0201^{\{1\}}$	$0.0643^{\{7\}}$	$0.05588^{\{5\}}$	$0.05316^{\{4\}}$	$0.07493^{\{9\}}$	$0.04205^{\{2\}}$	$0.04754^{\{3\}}$	$0.12049^{\{10\}}$	$0.06619^{\{8\}}$	$0.0567^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$22^{\{7.5\}}$	$17^{\{5.5\}}$	$10^{\{3.5\}}$	$27^{\{9\}}$	$7^{\{2\}}$	$10^{\{3.5\}}$	$30^{\{10\}}$	$22^{\{7.5\}}$	$17^{\{5.5\}}$
150	BIAS	$0.02044^{\{1\}}$	$0.05457^{\{5\}}$	$0.05785^{\{6\}}$	$0.05175^{\{4\}}$	$0.06626^{\{9\}}$	$0.04609^{\{3\}}$	$0.04069^{\{2\}}$	$0.1332^{\{10\}}$	$0.06572^{\{8\}}$	$0.05824^{\{7\}}$
	MSE	$0.00203^{\{1\}}$	$0.01387^{\{7\}}$	$0.01287^{\{6\}}$	$0.00661^{\{3\}}$	$0.0143^{\{9\}}$	$0.00714^{\{4\}}$	$0.00548^{\{2\}}$	$0.04269^{\{10\}}$	$0.01422^{\{8\}}$	$0.01118^{\{5\}}$
	MRE	$0.01022^{\{1\}}$	$0.02729^{\{5\}}$	$0.02892^{\{6\}}$	$0.02588^{\{4\}}$	$0.03313^{\{9\}}$	$0.02305^{\{3\}}$	$0.02034^{\{2\}}$	$0.0666^{\{10\}}$	$0.03286^{\{8\}}$	$0.02912^{\{7\}}$
	$\sum Ranks$	$3^{\{1\}}$	$17^{\{5\}}$	$18^{\{6\}}$	$11^{\{4\}}$	$27^{\{9\}}$	$10^{\{3\}}$	$6^{\{2\}}$	$30^{\{10\}}$	$24^{\{8\}}$	$19^{\{7\}}$
200	BIAS	$0.01882^{\{1\}}$	$0.04107^{\{6\}}$	$0.04563^{\{7\}}$	$0.03697^{\{2\}}$	$0.05765^{\{9\}}$	$0.03757^{\{3\}}$	$0.03759^{\{4\}}$	$0.10301^{\{10\}}$	$0.049^{\{8\}}$	$0.04053^{\{5\}}$
	MSE	$0.00162^{\{1\}}$	$0.00734^{\{7\}}$	$0.00742^{\{8\}}$	$0.00342^{\{2\}}$	$0.01369^{\{9\}}$	$0.00531^{\{5\}}$	$0.00448^{\{3\}}$	$0.02351^{\{10\}}$	$0.00669^{\{6\}}$	$0.0049^{\{4\}}$
	MRE	$0.00941^{\{1\}}$	$0.02054^{\{6\}}$	$0.02282^{\{7\}}$	$0.01849^{\{2\}}$	$0.02882^{\{9\}}$	$0.01878^{\{3\}}$	$0.0188^{\{4\}}$	$0.05151^{\{10\}}$	$0.0245^{\{8\}}$	$0.02026^{\{5\}}$
	$\sum Ranks$	$3^{\{1\}}$	$19^{\{6\}}$	$22^{\{7.5\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$11^{\{3.5\}}$	$11^{\{3.5\}}$	$30^{\{10\}}$	$22^{\{7.5\}}$	$14^{\{5\}}$
300	BIAS	$0.01478^{\{1\}}$	$0.02924^{\{5\}}$	$0.03841^{\{8\}}$	$0.02777^{\{2\}}$	$0.04386^{\{9\}}$	$0.02807^{\{3\}}$	$0.0289^{\{4\}}$	$0.07942^{\{10\}}$	$0.03297^{\{7\}}$	$0.03164^{\{6\}}$
	MSE	$0.00095^{\{1\}}$	$0.00253^{\{3\}}$	$0.0045^{\{8\}}$	$0.00195^{\{2\}}$	$0.00648^{\{9\}}$	$0.00265^{\{5\}}$	$0.00254^{\{4\}}$	$0.01255^{\{10\}}$	$0.00286^{\{6\}}$	$0.00316^{\{7\}}$
	MRE	$0.00739^{\{1\}}$	$0.01462^{\{5\}}$	$0.01921^{\{8\}}$	$0.01389^{\{2\}}$	$0.02193^{\{9\}}$	$0.01403^{\{3\}}$	$0.01445^{\{4\}}$	$0.03971^{\{10\}}$	$0.01649^{\{7\}}$	$0.01582^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$13^{\{5\}}$	$24^{\{8\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$11^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$20^{\{7\}}$	$19^{\{6\}}$
400	BIAS	$0.01235^{\{1\}}$	$0.02291^{\{3\}}$	$0.03354^{\{8\}}$	$0.02387^{\{4\}}$	$0.03518^{\{9\}}$	$0.02594^{\{5\}}$	$0.02214^{\{2\}}$	$0.06946^{\{10\}}$	$0.02761^{\{7\}}$	$0.02606^{\{6\}}$
	MSE	$0.00067^{\{1\}}$	$0.00172^{\{4\}}$	$0.00341^{\{8\}}$	$0.00132^{\{2\}}$	$0.00368^{\{9\}}$	$0.00215^{\{7\}}$	$0.00151^{\{3\}}$	$0.00916^{\{10\}}$	$0.00192^{\{5\}}$	$0.00207^{\{6\}}$
	MRE	$0.00617^{\{1\}}$	$0.01145^{\{3\}}$	$0.01677^{\{8\}}$	$0.01193^{\{4\}}$	$0.01759^{\{9\}}$	$0.01297^{\{5\}}$	$0.01107^{\{2\}}$	$0.03473^{\{10\}}$	$0.0138^{\{7\}}$	$0.01303^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$10^{\{3.5\}}$	$24^{\{8\}}$	$10^{\{3.5\}}$	$27^{\{9\}}$	$17^{\{5\}}$	$7^{\{2\}}$	$30^{\{10\}}$	$19^{\{7\}}$	$18^{\{6\}}$

Table 2. Simulation values of BIAS, MSE and MRE for $\eta = 2.25$.

n	Measure	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
20	BIAS	$0.34929^{\{1\}}$	$0.58417^{\{7\}}$	$0.57903^{\{6\}}$	$0.59212^{\{8\}}$	$0.71246^{\{10\}}$	$0.4418^{\{2\}}$	$0.54318^{\{4\}}$	$0.54655^{\{5\}}$	$0.52681^{\{3\}}$	$0.67996^{\{9\}}$
	MSE	$0.26059^{\{1\}}$	$0.84711^{\{7\}}$	$0.92703^{\{8\}}$	$0.68005^{\{4\}}$	$1.15262^{\{10\}}$	$0.49146^{\{2\}}$	$0.73068^{\{6\}}$	$0.68597^{\{5\}}$	$0.60075^{\{3\}}$	$1.00211^{\{9\}}$
	MRE	$0.15524^{\{1\}}$	$0.25963^{\{7\}}$	$0.25735^{\{6\}}$	$0.26316^{\{8\}}$	$0.31665^{\{10\}}$	$0.19636^{\{2\}}$	$0.24141^{\{4\}}$	$0.24291^{\{5\}}$	$0.23414^{\{3\}}$	$0.3022^{\{9\}}$
	$\sum Ranks$	$3^{\{1\}}$	$21^{\{8\}}$	$20^{\{6.5\}}$	$20^{\{6.5\}}$	$30^{\{10\}}$	$6^{\{2\}}$	$14^{\{4\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$27^{\{9\}}$
70	BIAS	$0.18329^{\{1\}}$	$0.27414^{\{5\}}$	$0.31373^{\{8\}}$	$0.25211^{\{3\}}$	$0.35608^{\{10\}}$	$0.24499^{\{2\}}$	$0.26642^{\{4\}}$	$0.32391^{\{9\}}$	$0.28598^{\{7\}}$	$0.27822^{\{6\}}$
	MSE	$0.06493^{\{1\}}$	$0.18926^{\{7\}}$	$0.2521^{\{9\}}$	$0.12611^{\{2\}}$	$0.29814^{\{10\}}$	$0.13291^{\{3\}}$	$0.16569^{\{4\}}$	$0.21117^{\{8\}}$	$0.1734^{\{6\}}$	$0.16925^{\{5\}}$
	MRE	$0.08146^{\{1\}}$	$0.12184^{\{5\}}$	$0.13943^{\{8\}}$	$0.11205^{\{3\}}$	$0.15826^{\{10\}}$	$0.10888^{\{2\}}$	$0.11841^{\{4\}}$	$0.14396^{\{9\}}$	$0.1271^{\{7\}}$	$0.12365^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$17^{\{5.5\}}$	$25^{\{8\}}$	$8^{\{3\}}$	$30^{\{10\}}$	$7^{\{2\}}$	$12^{\{4\}}$	$26^{\{9\}}$	$20^{\{7\}}$	$17^{\{5.5\}}$
150	BIAS	$0.13201^{\{1\}}$	$0.17814^{\{5\}}$	$0.20624^{\{9\}}$	$0.15127^{\{2\}}$	$0.21856^{\{10\}}$	$0.16604^{\{4\}}$	$0.15489^{\{3\}}$	$0.19508^{\{8\}}$	$0.18941^{\{7\}}$	$0.18134^{\{6\}}$
	MSE	$0.03059^{\{1\}}$	$0.07746^{\{7\}}$	$0.10889^{\{9\}}$	$0.04235^{\{2\}}$	$0.10974^{\{10\}}$	$0.05377^{\{3\}}$	$0.05546^{\{4\}}$	$0.07859^{\{8\}}$	$0.07155^{\{6\}}$	$0.0634^{\{5\}}$
	MRE	$0.05867^{\{1\}}$	$0.07917^{\{5\}}$	$0.09166^{\{9\}}$	$0.06723^{\{2\}}$	$0.09714^{\{10\}}$	$0.0738^{\{4\}}$	$0.06884^{\{3\}}$	$0.0867^{\{8\}}$	$0.08418^{\{7\}}$	$0.08059^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$17^{\{5.5\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$30^{\{10\}}$	$11^{\{4\}}$	$10^{\{3\}}$	$24^{\{8\}}$	$20^{\{7\}}$	$17^{\{5.5\}}$
200	BIAS	$0.10397^{\{1\}}$	$0.13297^{\{4\}}$	$0.16642^{\{8\}}$	$0.12504^{\{2\}}$	$0.18026^{\{10\}}$	$0.14188^{\{6\}}$	$0.12541^{\{3\}}$	$0.16738^{\{9\}}$	$0.15472^{\{7\}}$	$0.13876^{\{5\}}$
	MSE	$0.01814^{\{1\}}$	$0.03841^{\{5\}}$	$0.06833^{\{9\}}$	$0.02904^{\{2\}}$	$0.07728^{\{10\}}$	$0.03874^{\{6\}}$	$0.03005^{\{3\}}$	$0.05302^{\{8\}}$	$0.04418^{\{7\}}$	$0.03562^{\{4\}}$
	MRE	$0.04621^{\{1\}}$	$0.0591^{\{4\}}$	$0.07396^{\{8\}}$	$0.05557^{\{2\}}$	$0.08011^{\{10\}}$	$0.06306^{\{6\}}$	$0.05574^{\{3\}}$	$0.07439^{\{9\}}$	$0.06876^{\{7\}}$	$0.06167^{\{5\}}$
	$\sum Ranks$	$3^{\{1\}}$	$13^{\{4\}}$	$25^{\{8\}}$	$6^{\{2\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$26^{\{9\}}$	$21^{\{7\}}$	$14^{\{5\}}$
300	BIAS	$0.08552^{\{1\}}$	$0.11042^{\{4\}}$	$0.12983^{\{8\}}$	$0.09689^{\{2\}}$	$0.13128^{\{9\}}$	$0.11321^{\{6\}}$	$0.09888^{\{3\}}$	$0.1362^{\{10\}}$	$0.1241^{\{7\}}$	$0.11317^{\{5\}}$
	MSE	$0.01228^{\{1\}}$	$0.02534^{\{6\}}$	$0.03657^{\{9\}}$	$0.01654^{\{2\}}$	$0.04091^{\{10\}}$	$0.02133^{\{4\}}$	$0.01882^{\{3\}}$	$0.03309^{\{8\}}$	$0.02787^{\{7\}}$	$0.02296^{\{5\}}$
	MRE	$0.03801^{\{1\}}$	$0.04907^{\{4\}}$	$0.0577^{\{8\}}$	$0.04306^{\{2\}}$	$0.05835^{\{9\}}$	$0.05032^{\{6\}}$	$0.04395^{\{3\}}$	$0.06053^{\{10\}}$	$0.05515^{\{7\}}$	$0.0503^{\{5\}}$
	$\sum Ranks$	$3^{\{1\}}$	$14^{\{4\}}$	$25^{\{8\}}$	$6^{\{2\}}$	$28^{\{9.5\}}$	$16^{\{6\}}$	$9^{\{3\}}$	$28^{\{9.5\}}$	$21^{\{7\}}$	$15^{\{5\}}$
400	BIAS	$0.07919^{\{1\}}$	$0.0881^{\{4\}}$	$0.10022^{\{7\}}$	$0.08427^{\{3\}}$	$0.11034^{\{10\}}$	$0.09964^{\{6\}}$	$0.08306^{\{2\}}$	$0.10766^{\{9\}}$	$0.10327^{\{8\}}$	$0.09427^{\{5\}}$
	MSE	$0.01006^{\{1\}}$	$0.01509^{\{4\}}$	$0.01981^{\{8\}}$	$0.01304^{\{3\}}$	$0.02911^{\{10\}}$	$0.01628^{\{6\}}$	$0.01248^{\{2\}}$	$0.02204^{\{9\}}$	$0.01916^{\{7\}}$	$0.0155^{\{5\}}$
	MRE	$0.03519^{\{1\}}$	$0.03916^{\{4\}}$	$0.04454^{\{7\}}$	$0.03746^{\{3\}}$	$0.04904^{\{10\}}$	$0.04428^{\{6\}}$	$0.03692^{\{2\}}$	$0.04785^{\{9\}}$	$0.0459^{\{8\}}$	$0.0419^{\{5\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$22^{\{7\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$23^{\{8\}}$	$15^{\{5\}}$

Table 3. Simulation values of BIAS, MSE, and MRE for $\eta = 3.0$.

n	Measure	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
20	BIAS	$0.66951^{\{1\}}$	$0.80537^{\{5\}}$	$0.79123^{\{3\}}$	$0.80014^{\{4\}}$	$0.8349^{\{8\}}$	$0.81875^{\{7\}}$	$0.80849^{\{6\}}$	$0.76467^{\{2\}}$	$0.87128^{\{10\}}$	$0.86893^{\{9\}}$
	MSE	$0.68607^{\{1\}}$	$1.23141^{\{6\}}$	$1.11917^{\{4\}}$	$1.09542^{\{3\}}$	$1.34271^{\{7\}}$	$1.35803^{\{8\}}$	$1.20791^{\{5\}}$	$0.99157^{\{2\}}$	$1.37048^{\{9\}}$	$1.42288^{\{10\}}$
	MRE	$0.22317^{\{1\}}$	$0.26846^{\{5\}}$	$0.26374^{\{3\}}$	$0.26671^{\{4\}}$	$0.2783^{\{8\}}$	$0.27292^{\{7\}}$	$0.2695^{\{6\}}$	$0.25489^{\{2\}}$	$0.29043^{\{10\}}$	$0.28964^{\{9\}}$
	$\sum Ranks$	$3^{\{1\}}$	$16^{\{5\}}$	$10^{\{3\}}$	$11^{\{4\}}$	$23^{\{8\}}$	$22^{\{7\}}$	$17^{\{6\}}$	$6^{\{2\}}$	$29^{\{10\}}$	$28^{\{9\}}$
70	BIAS	$0.37418^{\{1\}}$	$0.44358^{\{3\}}$	$0.49937^{\{8\}}$	$0.43165^{\{2\}}$	$0.5003^{\{9\}}$	$0.55061^{\{10\}}$	$0.44753^{\{4\}}$	$0.45581^{\{5\}}$	$0.49102^{\{7\}}$	$0.48326^{\{6\}}$
	MSE	$0.22408^{\{1\}}$	$0.31955^{\{4\}}$	$0.40629^{\{8\}}$	$0.29282^{\{2\}}$	$0.42491^{\{9\}}$	$0.57008^{\{10\}}$	$0.32561^{\{5\}}$	$0.31637^{\{3\}}$	$0.39917^{\{7\}}$	$0.39255^{\{6\}}$
	MRE	$0.12473^{\{1\}}$	$0.14786^{\{3\}}$	$0.16646^{\{8\}}$	$0.14388^{\{2\}}$	$0.16677^{\{9\}}$	$0.18354^{\{10\}}$	$0.14918^{\{4\}}$	$0.15194^{\{5\}}$	$0.16367^{\{7\}}$	$0.16109^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$10^{\{3\}}$	$24^{\{8\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$13^{\{4.5\}}$	$13^{\{4.5\}}$	$21^{\{7\}}$	$18^{\{6\}}$
150	BIAS	$0.27787^{\{1\}}$	$0.3172^{\{4\}}$	$0.35666^{\{8\}}$	$0.27842^{\{2\}}$	$0.37287^{\{9\}}$	$0.39859^{\{10\}}$	$0.32013^{\{5\}}$	$0.30179^{\{3\}}$	$0.34734^{\{7\}}$	$0.34043^{\{6\}}$
	MSE	$0.11783^{\{1\}}$	$0.15956^{\{4\}}$	$0.20105^{\{8\}}$	$0.12734^{\{2\}}$	$0.21561^{\{9\}}$	$0.26931^{\{10\}}$	$0.15979^{\{5\}}$	$0.1411^{\{3\}}$	$0.19713^{\{7\}}$	$0.18469^{\{6\}}$
	MRE	$0.09262^{\{1\}}$	$0.10573^{\{4\}}$	$0.11889^{\{8\}}$	$0.09281^{\{2\}}$	$0.12429^{\{9\}}$	$0.13286^{\{10\}}$	$0.10671^{\{5\}}$	$0.1006^{\{3\}}$	$0.11578^{\{7\}}$	$0.11348^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$24^{\{8\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$18^{\{6\}}$
200	BIAS	$0.23298^{\{1\}}$	$0.28572^{\{5\}}$	$0.32753^{\{9\}}$	$0.23742^{\{2\}}$	$0.31745^{\{8\}}$	$0.35804^{\{10\}}$	$0.28294^{\{4\}}$	$0.26467^{\{3\}}$	$0.29533^{\{6\}}$	$0.29701^{\{7\}}$
	MSE	$0.08476^{\{1\}}$	$0.12963^{\{5\}}$	$0.16827^{\{9\}}$	$0.09546^{\{2\}}$	$0.16396^{\{8\}}$	$0.22006^{\{10\}}$	$0.12808^{\{4\}}$	$0.11143^{\{3\}}$	$0.14062^{\{6\}}$	$0.14348^{\{7\}}$
	MRE	$0.07766^{\{1\}}$	$0.09524^{\{5\}}$	$0.10918^{\{9\}}$	$0.07914^{\{2\}}$	$0.10582^{\{8\}}$	$0.11935^{\{10\}}$	$0.09431^{\{4\}}$	$0.08822^{\{3\}}$	$0.09844^{\{6\}}$	$0.099^{\{7\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$18^{\{6\}}$	$21^{\{7\}}$
300	BIAS	$0.19401^{\{2\}}$	$0.22872^{\{4\}}$	$0.26964^{\{9\}}$	$0.18854^{\{1\}}$	$0.26503^{\{8\}}$	$0.29074^{\{10\}}$	$0.23288^{\{6\}}$	$0.2213^{\{3\}}$	$0.25265^{\{7\}}$	$0.23111^{\{5\}}$
	MSE	$0.05951^{\{2\}}$	$0.08047^{\{4\}}$	$0.11323^{\{9\}}$	$0.05912^{\{1\}}$	$0.11018^{\{8\}}$	$0.14804^{\{10\}}$	$0.08333^{\{5\}}$	$0.07584^{\{3\}}$	$0.10358^{\{7\}}$	$0.08476^{\{6\}}$
	MRE	$0.06467^{\{2\}}$	$0.07624^{\{4\}}$	$0.08988^{\{9\}}$	$0.06285^{\{1\}}$	$0.08834^{\{8\}}$	$0.09691^{\{10\}}$	$0.07763^{\{6\}}$	$0.07377^{\{3\}}$	$0.08422^{\{7\}}$	$0.07704^{\{5\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$3^{\{1\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$17^{\{6\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$16^{\{5\}}$
400	BIAS	$0.16201^{\{1\}}$	$0.19245^{\{5\}}$	$0.23181^{\{9\}}$	$0.16623^{\{2\}}$	$0.22345^{\{8\}}$	$0.25133^{\{10\}}$	$0.20346^{\{6\}}$	$0.19056^{\{3\}}$	$0.21936^{\{7\}}$	$0.19222^{\{4\}}$
	MSE	$0.04128^{\{1\}}$	$0.05809^{\{4\}}$	$0.08317^{\{9\}}$	$0.04847^{\{2\}}$	$0.07692^{\{7\}}$	$0.11015^{\{10\}}$	$0.06392^{\{6\}}$	$0.05699^{\{3\}}$	$0.07868^{\{8\}}$	$0.05936^{\{5\}}$
	MRE	$0.054^{\{1\}}$	$0.06415^{\{5\}}$	$0.07727^{\{9\}}$	$0.05541^{\{2\}}$	$0.07448^{\{8\}}$	$0.08378^{\{10\}}$	$0.06782^{\{6\}}$	$0.06352^{\{3\}}$	$0.07312^{\{7\}}$	$0.06407^{\{4\}}$
	$\sum Ranks$	$3^{\{1\}}$	$14^{\{5\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$23^{\{8\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$22^{\{7\}}$	$13^{\{4\}}$

Table 4. Simulation values of BIAS, MSE, and MRE for $\eta = 3.5$.

n	Measure	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
20	BIAS	$0.79563^{\{1\}}$	$0.89736^{\{2\}}$	$0.97899^{\{7\}}$	$0.9835^{\{8\}}$	$0.93906^{\{4\}}$	$0.98806^{\{10\}}$	$0.94402^{\{5\}}$	$0.92314^{\{3\}}$	$0.95981^{\{6\}}$	$0.98421^{\{9\}}$
	MSE	$0.9446^{\{1\}}$	$1.3399^{\{3\}}$	$1.53657^{\{6\}}$	$1.57034^{\{7\}}$	$1.48803^{\{5\}}$	$1.72119^{\{10\}}$	$1.43236^{\{4\}}$	$1.32543^{\{2\}}$	$1.69991^{\{9\}}$	$1.63148^{\{8\}}$
	MRE	$0.22732^{\{1\}}$	$0.25639^{\{2\}}$	$0.27971^{\{7\}}$	$0.281^{\{8\}}$	$0.2683^{\{4\}}$	$0.2823^{\{10\}}$	$0.26972^{\{5\}}$	$0.26375^{\{3\}}$	$0.27423^{\{6\}}$	$0.2812^{\{9\}}$
	$\sum Ranks$	$3^{\{1\}}$	$7^{\{2\}}$	$20^{\{6\}}$	$23^{\{8\}}$	$13^{\{4\}}$	$30^{\{10\}}$	$14^{\{5\}}$	$8^{\{3\}}$	$21^{\{7\}}$	$26^{\{9\}}$
70	BIAS	$0.44713^{\{1\}}$	$0.5348^{\{6\}}$	$0.60221^{\{9\}}$	$0.47639^{\{2\}}$	$0.57703^{\{8\}}$	$0.62804^{\{10\}}$	$0.53063^{\{4\}}$	$0.524^{\{3\}}$	$0.56093^{\{7\}}$	$0.53102^{\{5\}}$
	MSE	$0.32058^{\{1\}}$	$0.45334^{\{5\}}$	$0.55543^{\{9\}}$	$0.3791^{\{2\}}$	$0.50879^{\{7\}}$	$0.60945^{\{10\}}$	$0.43548^{\{4\}}$	$0.4204^{\{3\}}$	$0.51877^{\{8\}}$	$0.45532^{\{6\}}$
	MRE	$0.12775^{\{1\}}$	$0.1528^{\{6\}}$	$0.17206^{\{9\}}$	$0.13611^{\{2\}}$	$0.16487^{\{8\}}$	$0.17944^{\{10\}}$	$0.15161^{\{4\}}$	$0.14971^{\{3\}}$	$0.16027^{\{7\}}$	$0.15172^{\{5\}}$
	$\sum Ranks$	$3^{\{1\}}$	$17^{\{6\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$23^{\{8\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$22^{\{7\}}$	$16^{\{5\}}$
150	BIAS	$0.31249^{\{1\}}$	$0.36266^{\{4\}}$	$0.42588^{\{9\}}$	$0.31594^{\{2\}}$	$0.40412^{\{7\}}$	$0.47479^{\{10\}}$	$0.37519^{\{5\}}$	$0.34122^{\{3\}}$	$0.41033^{\{8\}}$	$0.38026^{\{6\}}$
	MSE	$0.14886^{\{1\}}$	$0.20864^{\{4\}}$	$0.2884^{\{9\}}$	$0.16449^{\{2\}}$	$0.25442^{\{7\}}$	$0.3397^{\{10\}}$	$0.21191^{\{5\}}$	$0.18984^{\{3\}}$	$0.2662^{\{8\}}$	$0.23594^{\{6\}}$
	MRE	$0.08928^{\{1\}}$	$0.10362^{\{4\}}$	$0.12168^{\{9\}}$	$0.09027^{\{2\}}$	$0.11546^{\{7\}}$	$0.13565^{\{10\}}$	$0.1072^{\{5\}}$	$0.09749^{\{3\}}$	$0.11724^{\{8\}}$	$0.10865^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$24^{\{8\}}$	$18^{\{6\}}$
200	BIAS	$0.27659^{\{2\}}$	$0.33925^{\{6\}}$	$0.35511^{\{7\}}$	$0.27179^{\{1\}}$	$0.35614^{\{8\}}$	$0.40657^{\{10\}}$	$0.32995^{\{5\}}$	$0.3115^{\{3\}}$	$0.35617^{\{9\}}$	$0.32854^{\{4\}}$
	MSE	$0.11982^{\{1\}}$	$0.17904^{\{6\}}$	$0.1949^{\{7\}}$	$0.1249^{\{2\}}$	$0.19605^{\{8\}}$	$0.24347^{\{10\}}$	$0.17036^{\{4\}}$	$0.15135^{\{3\}}$	$0.20635^{\{9\}}$	$0.17183^{\{5\}}$
	MRE	$0.07903^{\{2\}}$	$0.09693^{\{6\}}$	$0.10146^{\{7\}}$	$0.07766^{\{1\}}$	$0.10175^{\{8\}}$	$0.11616^{\{10\}}$	$0.09427^{\{5\}}$	$0.089^{\{3\}}$	$0.10176^{\{9\}}$	$0.09387^{\{4\}}$
	$\sum Ranks$	$5^{\{2\}}$	$18^{\{6\}}$	$21^{\{7\}}$	$4^{\{1\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$14^{\{5\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$13^{\{4\}}$
300	BIAS	$0.22377^{\{1\}}$	$0.27391^{\{7\}}$	$0.3072^{\{9\}}$	$0.22619^{\{2\}}$	$0.30358^{\{8\}}$	$0.33532^{\{10\}}$	$0.27176^{\{6\}}$	$0.24605^{\{3\}}$	$0.26598^{\{5\}}$	$0.25681^{\{4\}}$
	MSE	$0.07856^{\{1\}}$	$0.11455^{\{6\}}$	$0.142^{\{9\}}$	$0.08508^{\{2\}}$	$0.13939^{\{8\}}$	$0.17056^{\{10\}}$	$0.11374^{\{5\}}$	$0.09298^{\{3\}}$	$0.1161^{\{7\}}$	$0.1046^{\{4\}}$
	MRE	$0.06393^{\{1\}}$	$0.07826^{\{7\}}$	$0.08777^{\{9\}}$	$0.06462^{\{2\}}$	$0.08674^{\{8\}}$	$0.0958^{\{10\}}$	$0.07765^{\{6\}}$	$0.0703^{\{3\}}$	$0.07599^{\{5\}}$	$0.07337^{\{4\}}$
	$\sum Ranks$	$3^{\{1\}}$	$20^{\{7\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$17^{\{5.5\}}$	$9^{\{3\}}$	$17^{\{5.5\}}$	$12^{\{4\}}$
400	BIAS	$0.19701^{\{2\}}$	$0.22228^{\{4\}}$	$0.26463^{\{9\}}$	$0.18514^{\{1\}}$	$0.25464^{\{8\}}$	$0.29927^{\{10\}}$	$0.23905^{\{6\}}$	$0.20887^{\{3\}}$	$0.24473^{\{7\}}$	$0.2322^{\{5\}}$
	MSE	$0.06046^{\{2\}}$	$0.07873^{\{4\}}$	$0.10773^{\{9\}}$	$0.05952^{\{1\}}$	$0.10267^{\{8\}}$	$0.13583^{\{10\}}$	$0.08981^{\{6\}}$	$0.07034^{\{3\}}$	$0.09846^{\{7\}}$	$0.08658^{\{5\}}$
	MRE	$0.05629^{\{2\}}$	$0.06351^{\{4\}}$	$0.07561^{\{9\}}$	$0.0529^{\{1\}}$	$0.07275^{\{8\}}$	$0.08551^{\{10\}}$	$0.0683^{\{6\}}$	$0.05968^{\{3\}}$	$0.06992^{\{7\}}$	$0.06634^{\{5\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$3^{\{1\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$15^{\{5\}}$

Table 5. Simulation values of BIAS, MSE, and MRE for $\eta = 4.0$.

n	Measure	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
20	BIAS	$0.95837^{\{1\}}$	$1.02435^{\{3\}}$	$1.17078^{\{9\}}$	$1.00857^{\{2\}}$	$1.10171^{\{6\}}$	$1.14313^{\{7\}}$	$1.10117^{\{5\}}$	$1.02527^{\{4\}}$	$1.20278^{\{10\}}$	$1.17017^{\{8\}}$
	MSE	$1.37605^{\{1\}}$	$1.62139^{\{3\}}$	$2.10705^{\{7\}}$	$1.72714^{\{4\}}$	$1.89583^{\{5\}}$	$2.14532^{\{8\}}$	$1.93017^{\{6\}}$	$1.61082^{\{2\}}$	$2.53192^{\{10\}}$	$2.36237^{\{9\}}$
	MRE	$0.23959^{\{1\}}$	$0.25609^{\{3\}}$	$0.2927^{\{9\}}$	$0.25214^{\{2\}}$	$0.27543^{\{6\}}$	$0.28578^{\{7\}}$	$0.27529^{\{5\}}$	$0.25632^{\{4\}}$	$0.30069^{\{10\}}$	$0.29254^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$9^{\{3\}}$	$25^{\{8.5\}}$	$8^{\{2\}}$	$17^{\{6\}}$	$22^{\{7\}}$	$16^{\{5\}}$	$10^{\{4\}}$	$30^{\{10\}}$	$25^{\{8.5\}}$
70	BIAS	$0.50183^{\{1\}}$	$0.57467^{\{4\}}$	$0.6548^{\{9\}}$	$0.54002^{\{2\}}$	$0.63939^{\{8\}}$	$0.7269^{\{10\}}$	$0.5925^{\{5\}}$	$0.55294^{\{3\}}$	$0.63168^{\{6\}}$	$0.63802^{\{7\}}$
	MSE	$0.39741^{\{1\}}$	$0.50457^{\{4\}}$	$0.66991^{\{8\}}$	$0.47371^{\{2\}}$	$0.62869^{\{6\}}$	$0.79212^{\{10\}}$	$0.53999^{\{5\}}$	$0.48329^{\{3\}}$	$0.66949^{\{7\}}$	$0.68414^{\{9\}}$
	MRE	$0.12546^{\{1\}}$	$0.14367^{\{4\}}$	$0.1637^{\{9\}}$	$0.135^{\{2\}}$	$0.15985^{\{8\}}$	$0.18172^{\{10\}}$	$0.14813^{\{5\}}$	$0.13824^{\{3\}}$	$0.15792^{\{6\}}$	$0.1595^{\{7\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$26^{\{9\}}$	$6^{\{2\}}$	$22^{\{7\}}$	$30^{\{10\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$19^{\{6\}}$	$23^{\{8\}}$
150	BIAS	$0.36304^{\{2\}}$	$0.41612^{\{6\}}$	$0.44694^{\{8\}}$	$0.36232^{\{1\}}$	$0.43923^{\{7\}}$	$0.50824^{\{10\}}$	$0.40425^{\{4\}}$	$0.37026^{\{3\}}$	$0.44805^{\{9\}}$	$0.40831^{\{5\}}$
	MSE	$0.20382^{\{1\}}$	$0.26875^{\{5\}}$	$0.315^{\{8\}}$	$0.21101^{\{3\}}$	$0.29819^{\{7\}}$	$0.38364^{\{10\}}$	$0.24926^{\{4\}}$	$0.20789^{\{2\}}$	$0.33345^{\{9\}}$	$0.26892^{\{6\}}$
	MRE	$0.09076^{\{2\}}$	$0.10403^{\{6\}}$	$0.11174^{\{8\}}$	$0.09058^{\{1\}}$	$0.10981^{\{7\}}$	$0.12706^{\{10\}}$	$0.10106^{\{4\}}$	$0.09256^{\{3\}}$	$0.11201^{\{9\}}$	$0.10208^{\{5\}}$
	$\sum Ranks$	$5^{\{1.5\}}$	$17^{\{6\}}$	$24^{\{8\}}$	$5^{\{1.5\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$8^{\{3\}}$	$27^{\{9\}}$	$16^{\{5\}}$
200	BIAS	$0.28778^{\{1\}}$	$0.35274^{\{5\}}$	$0.40102^{\{9\}}$	$0.31639^{\{2\}}$	$0.394^{\{8\}}$	$0.45105^{\{10\}}$	$0.36679^{\{6\}}$	$0.32999^{\{3\}}$	$0.38749^{\{7\}}$	$0.34935^{\{4\}}$
	MSE	$0.13421^{\{1\}}$	$0.19048^{\{4\}}$	$0.2442^{\{8\}}$	$0.16414^{\{2\}}$	$0.23847^{\{7\}}$	$0.31526^{\{10\}}$	$0.21108^{\{6\}}$	$0.17187^{\{3\}}$	$0.24861^{\{9\}}$	$0.2003^{\{5\}}$
	MRE	$0.07195^{\{1\}}$	$0.08818^{\{5\}}$	$0.10026^{\{9\}}$	$0.0791^{\{2\}}$	$0.0985^{\{8\}}$	$0.11276^{\{10\}}$	$0.0917^{\{6\}}$	$0.0825^{\{3\}}$	$0.09687^{\{7\}}$	$0.08734^{\{4\}}$
	$\sum Ranks$	$3^{\{1\}}$	$14^{\{5\}}$	$26^{\{9\}}$	$6^{\{2\}}$	$23^{\{7.5\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$23^{\{7.5\}}$	$13^{\{4\}}$
300	BIAS	$0.24313^{\{2\}}$	$0.29051^{\{5\}}$	$0.32115^{\{8\}}$	$0.23399^{\{1\}}$	$0.32247^{\{9\}}$	$0.37573^{\{10\}}$	$0.28792^{\{4\}}$	$0.26948^{\{3\}}$	$0.31353^{\{7\}}$	$0.29549^{\{6\}}$
	MSE	$0.09374^{\{1\}}$	$0.13195^{\{5\}}$	$0.16327^{\{9\}}$	$0.09434^{\{2\}}$	$0.16074^{\{8\}}$	$0.21918^{\{10\}}$	$0.12961^{\{4\}}$	$0.1155^{\{3\}}$	$0.15947^{\{7\}}$	$0.1413^{\{6\}}$
	MRE	$0.06078^{\{2\}}$	$0.07263^{\{5\}}$	$0.08029^{\{8\}}$	$0.0585^{\{1\}}$	$0.08062^{\{9\}}$	$0.09393^{\{10\}}$	$0.07198^{\{4\}}$	$0.06737^{\{3\}}$	$0.07838^{\{7\}}$	$0.07387^{\{6\}}$
	$\sum Ranks$	$5^{\{2\}}$	$15^{\{5\}}$	$25^{\{8\}}$	$4^{\{1\}}$	$26^{\{9\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$18^{\{6\}}$
400	BIAS	$0.20825^{\{2\}}$	$0.24548^{\{4\}}$	$0.28852^{\{9\}}$	$0.19866^{\{1\}}$	$0.27985^{\{8\}}$	$0.31854^{\{10\}}$	$0.2513^{\{6\}}$	$0.23592^{\{3\}}$	$0.27006^{\{7\}}$	$0.24794^{\{5\}}$
	MSE	$0.07052^{\{2\}}$	$0.09508^{\{4\}}$	$0.12961^{\{9\}}$	$0.06978^{\{1\}}$	$0.1198^{\{8\}}$	$0.15861^{\{10\}}$	$0.09743^{\{5\}}$	$0.08573^{\{3\}}$	$0.11859^{\{7\}}$	$0.09873^{\{6\}}$
	MRE	$0.05206^{\{2\}}$	$0.06137^{\{4\}}$	$0.07213^{\{9\}}$	$0.04966^{\{1\}}$	$0.06996^{\{8\}}$	$0.07963^{\{10\}}$	$0.06282^{\{6\}}$	$0.05898^{\{3\}}$	$0.06752^{\{7\}}$	$0.06199^{\{5\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$3^{\{1\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$17^{\{6\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$16^{\{5\}}$

Table 6. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of model parameters.

Parameter	n	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}
$\eta = 2.0$	20	1.0	5.0	3.0	7.0	9.0	2.0	4.0	10.0	6.0	8.0
	70	1.0	7.5	5.5	3.5	9.0	2.0	3.5	10.0	7.5	5.5
	150	1.0	5.0	6.0	4.0	9.0	3.0	2.0	10.0	8.0	7.0
	200	1.0	6.0	7.5	2.0	9.0	3.5	3.5	10.0	7.5	5.0
	300	1.0	5.0	8.0	2.0	9.0	3.0	4.0	10.0	7.0	6.0
	400	1.0	3.5	8.0	3.5	9.0	5.0	2.0	10.0	7.0	6.0
$\eta = 2.25$	20	1.0	8.0	6.5	6.5	10.0	2.0	4.0	5.0	3.0	9.0
	70	1.0	5.5	8.0	3.0	10.0	2.0	4.0	9.0	7.0	5.5
	150	1.0	5.5	9.0	2.0	10.0	4.0	3.0	8.0	7.0	5.5
	200	1.0	4.0	8.0	2.0	10.0	6.0	3.0	9.0	7.0	5.0
	300	1.0	4.0	8.0	2.0	9.5	6.0	3.0	9.5	7.0	5.0
	400	1.0	4.0	7.0	3.0	10.0	6.0	2.0	9.0	8.0	5.0
$\eta = 3$	20	1.0	5.0	3.0	4.0	8.0	7.0	6.0	2.0	10.0	9.0
	70	1.0	3.0	8.0	2.0	9.0	10.0	4.5	4.5	7.0	6.0
	150	1.0	4.0	8.0	2.0	9.0	10.0	5.0	3.0	7.0	6.0
	200	1.0	5.0	9.0	2.0	8.0	10.0	4.0	3.0	6.0	7.0
	300	2.0	4.0	9.0	1.0	8.0	10.0	6.0	3.0	7.0	5.0
	400	1.0	5.0	9.0	2.0	8.0	10.0	6.0	3.0	7.0	4.0
$\eta = 3.5$	20	1.0	2.0	6.0	8.0	4.0	10.0	5.0	3.0	7.0	9.0
	70	1.0	6.0	9.0	2.0	8.0	10.0	4.0	3.0	7.0	5.0
	150	1.0	4.0	9.0	2.0	7.0	10.0	5.0	3.0	8.0	6.0
	200	2.0	6.0	7.0	1.0	8.0	10.0	5.0	3.0	9.0	4.0
	300	1.0	7.0	9.0	2.0	8.0	10.0	5.5	3.0	5.5	4.0
	400	2.0	4.0	9.0	1.0	8.0	10.0	6.0	3.0	7.0	5.0
$\eta = 4.0$	20	1.0	3.0	8.5	2.0	6.0	7.0	5.0	4.0	10.0	8.5
	70	1.0	4.0	9.0	2.0	7.0	10.0	5.0	3.0	6.0	8.0
	150	1.5	6.0	8.0	1.5	7.0	10.0	4.0	3.0	9.0	5.0
	200	1.0	5.0	9.0	2.0	7.5	10.0	6.0	3.0	7.5	4.0
	300	2.0	5.0	8.0	1.0	9.0	10.0	4.0	3.0	7.0	6.0
	400	2.0	4.0	9.0	1.0	8.0	10.0	6.0	3.0	7.0	5.0
$\sum$ Ranks		35.5	145.0	231.0	79.0	251.0	218.5	130.0	165.0	216.0	179.0
Overall Rank		1	4	9	2	10	8	3	5	7	6

5. Modeling to Inflation Data

In this part, we analyze one real data set to show the importance and validity of our suggested model in modeling real data. The real data set represents the 45 countries' rate of inflation in Asia (31-Jan-2023) studied by Elgarhy et al. [50]. Table 7 presents the values of the real data set. Some descriptive analysis of the real data set is presented in Table 8. We show that our suggested model outperforms the other comparative distributions examined for fitting the real data set. The compared models are RLD [2], Kumaraswamy distribution (KwD) [52], truncated power Lomax distribution (TPLD) [51], beta distribution (BD) [53], truncated Weibull distribution (TWD) [54], Kumaraswamy Kumaraswamy distribution (KwKwD) [55], exponential distribution (ED), IPBHD [25], ILLD [26], and INMD [27].

We apply some of the analytical information criteria (IC) such as Akaike-IC (IC_1), the corrected-AIC (IC_2), Bayesian-IC (IC_3), Hannan-Quinn IC (IC_4), and some of the goodness-of-fit measures, such as Anderson-Darling (G_1), Cramer-von Mises (G_2), and Kolmogorov-Smirnov (G_3) with its p-value ($G_3(p)$) to determine which model is most suited for usage with the real data set. Table 9 reports estimates of the parameters by the maximum likelihood method, standard errors (SEs), and the eight discrimination measures for the real data set. Consequently, we can conclude that the suggested model outperforms the other models that are equivalent to it in modeling the real data set.

Figure 4 shows the box plot, total time on test (TTT) plot, and estimated HF for the data set. Figure 5 depicts the behavior of the log-likelihood function with an estimator $\hat{\eta}$ for the real data set, which provides an unimodal function and indicates that the calculated parameter is a global maximum point. This means that they optimize the log-likelihood function and provide the best estimator. The estimated PDF of the IRLD and other compared models with histograms for the real data set are presented in Figure 6. The probability-probability (P-P) plots for our proposed model with all compared models for the real data set are presented in Figure 7.

Table 7. The real data set values.

0.0077	0.018	0.0198	0.02	0.0271	0.0284	0.031	0.0315	0.032
0.033	0.036	0.038	0.04	0.0409	0.042	0.0435	0.045	0.0459
0.0489	0.05	0.053	0.0551	0.0572	0.0589	0.0593	0.065	0.067
0.0677	0.0738	0.081	0.083	0.0871	0.091	0.098	0.123	0.132
0.139	0.147	0.175	0.1955	0.203	0.245	0.3927	0.522	0.542

Table 8. Some descriptive analysis of the real data set.

n	Mean	Median	Skewness	Kurtosis	Range	Minimum	Maximum	Sum
45	0.0998	0.0572	2.58413	9.34416	0.5343	0.0077	0.542	4.492

6. Summary and Conclusion

The inverse Ramos-Louzada distribution is a novel inverted probability distribution that we introduce and examine in this research. The IRLD can generate asymmetrical and unimodal densities, and increasing, decreasing, and upside-down hazard rates form. Moments and inverse moments, PWM,

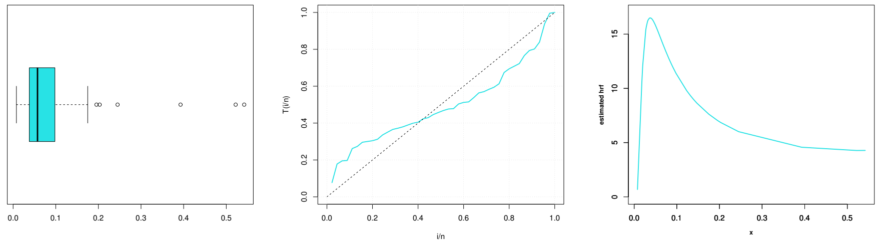

Figure 4. Box plot, TTT plot and estimated HF for the data set

Table 9. Numerical values for analyzing the rate of inflation dataset.

Model	IC_1	IC_2	IC_3	IC_4	G_1	G_2	G_3	$G_3(p)$	Est. parameters (SEs)
IRLD	-122.974	-122.881	-121.168	-122.301	1.37998	0.227008	0.129811	0.434282	$\hat{\eta}$ = 20.673 (3.2433)
IPBHD	-14.8259	-14.5402	-11.2126	-13.4789	33.493	6.9242	0.634119	< 0.00001	$\hat{\alpha}$ = 0.032695 (0.0327681)
									$\hat{\eta}$ = 0.533722 (0.0654248)
ED	-115.393	-115.3	-113.586	-114.719	1.79841	0.290478	0.148862	0.271506	$\hat{a}$ = 10.0178 (1.49336)
ILLD	-15.5269	-15.4339	-13.7202	-14.8534	30.4644	6.43107	0.617739	< 0.00001	$\hat{a}$ = 0.550369 (0.0641869)
INMD	-116.682	-116.396	-113.069	-115.335	1.24534	0.203576	0.146496	0.288982	$\hat{\alpha}$ = 0.478805 (0.0829952)
									$\hat{\lambda}$ = 900.691 194.039()
RLD	161.436	161.529	163.243	162.11	136.368	13.8885	0.909207	< 0.00001	$\hat{\lambda}$ = 3.36538 (0.303801)
KwD	-109.9738	-109.6881	-106.3605	-108.6268	2.1523	0.3636	0.1696	0.1337	$\hat{\psi}$ = 0.9771 (0.1189)
									$\hat{\xi}$ = 8.2079 (2.2611)
TPLD	-115.4946	-115.2089	-111.8813	-114.1476	1.5730	0.2679	0.1409	0.3044	$\hat{\psi}$ = 13.5572 (3.2506)
									$\hat{\xi}$ = 1.1154 (0.1087)
BD	-110.1222	-109.8365	-106.5089	-108.7752	2.1750	0.3676	0.1834	0.0851	$\hat{\psi}$ = 1.0848 (0.2030)
									$\hat{\xi}$ = 9.2659 (2.1154)
TWD	-113.6829	-113.3972	-110.0696	-112.3359	1.7549	0.2979	0.1477	0.2537	$\hat{\psi}$ = 11.1457 (2.7729)
									$\hat{\xi}$ = 1.0588 (0.1111)
KwKwD	-120.1412	-119.1412	-112.9146	-117.4472	0.8628	0.1391	0.1398	0.3129	$\hat{\psi}$ = 0.6216 (0.0034)
									$\hat{\xi}$ = 31.1470 (0.0034)
									$\hat{a}$ = 18.2273 (0.0315)
									$\hat{b}$ = 0.1840 (0.0276)

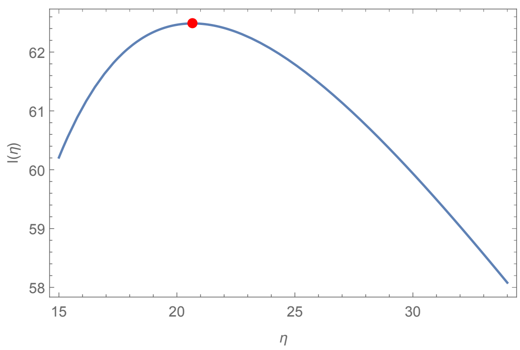

Figure 5. The profile of the log-likelihood functions for $\hat{\eta}$ of the real data set.

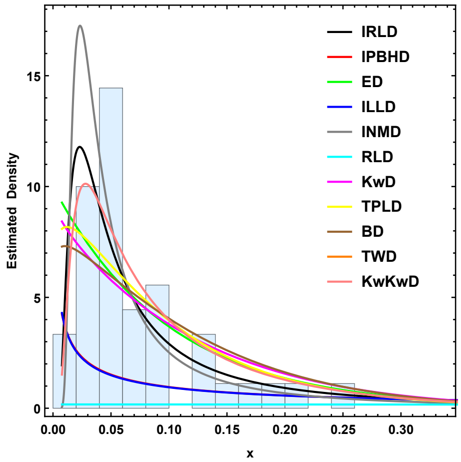

Figure 6. Histogram of real data set with the fitted PDFs of all compared models.

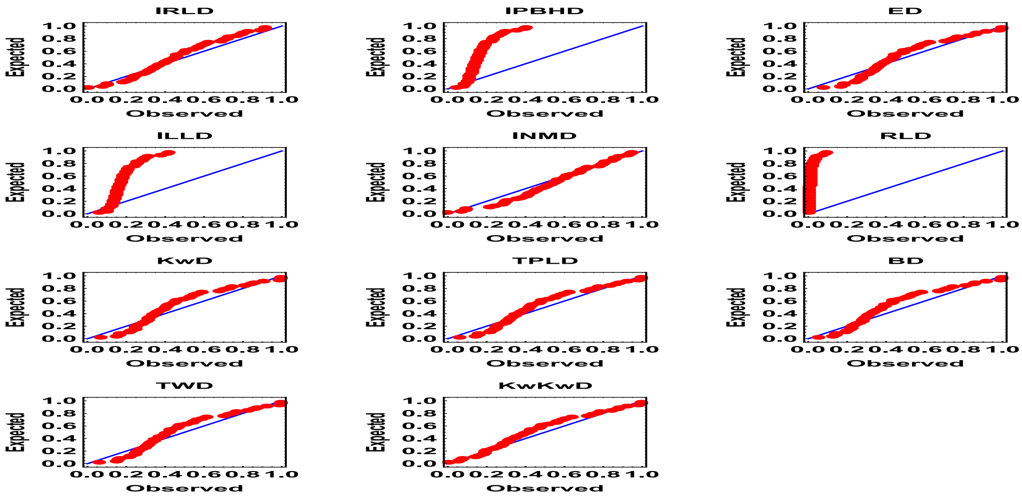

Figure 7. The P-P plots of the fitted models for the real data set.

QF, incomplete moments, stochastic ordering, and extropy measures are a few statistical features of the proposed distribution that have been calculated and analyzed. The maximum likelihood, least squares, weighted least squares, Cramér-von-Mises, Anderson-Darling, maximum product of spacing, left-tailed Anderson-Darling, minimum spacing absolute distance, right-tail Anderson-Darling, and minimum spacing absolute log distance are just a few of the methods that have been used to estimate the parameter of the IRLD. The effectiveness of these estimates in terms of biases, mean squared errors, and mean absolute relative errors has been compared through a simulation study. The optimal estimation method and ordering performance of the estimates were determined using the partial and general ranks of all estimate methods for different parameter combinations. Our quantitative results show that, in both general and partial rankings, the maximum likelihood approach outperforms other estimation techniques, with a cite score of 35.5. This shows that it has a strong and very good estimation potential. Having a cite score of 79, the maximum product spacing method seems to be the next best method. In this sense, maximum likelihood's lower cite score suggests greater impact and possibly more accuracy or credibility. Cite score for maximum likelihood estimates indicates a higher impact in this context, along with possibly higher accuracy or reliability. An inflation data set is examined to demonstrate how the suggested distribution might be used. The findings demonstrate that the given distribution can fit the data more precisely when compared to other competing distributions. Lastly, we hope the new model will offer a versatile framework for establishing the lifetime data phenomena in practical areas like reliability, medicine, and other applied disciplines. The proposed IRLD may lack the necessary flexibility to accurately model all complex real-world datasets, particularly those with wide variations in skewness or tail characteristics, despite the parsimony provided by its single-parameter structure. Future studies should look into generalized formulations to increase their adaptability, possibly utilizing different transformation techniques. These extensions seek to improve the distribution's shape flexibility so that it can support a greater variety of data structures while maintaining its fundamental theoretical benefits. Another suggestion for future research will focus on developing IRLD-based regression models for covariate analysis, expanding their applicability.

Data availability: The corresponding author can provide the datasets created and/or studied during the current work upon reasonable request.
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