Journal of Fractional Calculus and Applications
Vol. 16(2) July 2025, No.7.
ISSN: 2090-5858.
ISSN 2090-584X (print)
http://jfca.journals.ekb.eg/

ANALYTICAL PROPERTIES OF FRACTIONAL CALCULUS AND TRANSFORMS ASSOCIATED WITH EXTENDED MITTAG-LEFFLER FUNCTION

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Abstract. The main object of the present paper is to introduce a new extension of the generalized Mittag-Leffler function utilizing the extended beta function. Among the many properties we evaluated for the extended Mittag-Leffler function are derivative formulas, Mellin transform, Laplace transform, Euler-Beta transform, and Whittaker transform. Further, we establish some results based on the consequences of Riemann-Liouville fractional integral and differential operators on the extended Mittag-Leffler function.

1. Introduction and Preliminaries

Special functions are essential across scientific fields, serving as indispensable tools for mathematicians, engineers, and scientists in various applied and computational mathematics branches. In recent decades, several authors have created compelling and useful extensions for various special functions.([1]-[5], [7]-[15],[24],[25],[31]).

In 2012, Srivastava et al. [26] defined $\Theta\left(\{j_n\}_{n\in\mathbf{N}_0};\ x\right)$ that is an appropriate function of sequence $\{j_n\}_{n\in\mathbf{N}_0}$ of arbitrary numbers (real or complex) as

$$\Theta\left(\{j_n\}_{n\in\mathbf{N}_0};\ x\right) := \left\{\begin{array}{ll} \sum_{m=0}^{\infty} j_n \frac{x^n}{n!} & (|\ x\ |< R;\ 0<R<\infty;\ j_0:=1) \\ \\ w_0\ x^{\alpha}\ \exp(x)\left[1+O\left(\frac{1}{x}\right)\right] & (R(x)\to\infty : w_0>0;\ \alpha\in\mathbf{C}) \end{array}\right\} \tag{1}$$

here w_0 and α are some suitable constants essentially dependent on the sequence $\{j_n\}_{n\in\mathbf{N}_0}$. With the help of this sequence, they introduced the following extended

2020 *Mathematics Subject Classification.* 33B15, 33E12, 26A33.
Key words and phrases. Extended beta function, generalized Mittag-Leffler function, integral transforms and Riemann-Liouville fractional integrals and derivatives.
Submitted May 21, 2024. Accepted Jan. 28, 2025.

generalized gamma function, beta function, and the Gauss hypergeometric function.

$$\Gamma_b^{(\{j_n\})_{n\in\mathbf{N}_0})}(x) = \int_0^\infty y^{x-1}\Theta\left(\{j_n\}_{n\in\mathbf{N}_0};\ -y-\frac{b}{y}\right)dy, \quad (R(x)>0; R(b)\geq 0) \tag{2}$$

$$B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\alpha,\beta;b) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\Theta\left(\{j_n\}_{n\in\mathbf{N}_0};\ -\frac{b}{y(1-y)}\right)dy, \tag{3}$$
$$(\min\{R(\alpha),R(\beta)\}>0; R(b)\geq 0)$$

and

$$F_b^{(\{j_n\}_{n\in\mathbf{N}_0})}(a,\theta_1;\theta_2;\ x) = \sum_{m=0}^\infty (a)_m \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\frac{x^m}{m!}, \tag{4}$$
$$(R(b)\geq 0;\ |x|<1;\ R(\theta_2)>R(\theta_1)>0)$$

respectively. Equations (2)-(4) produce extensions for different specific cases of the sequence $\{j_n\}_{n\in\mathbf{N}_0}$. For example, if we put

$$j_n = \frac{(\omega_1)_n}{(\omega_2)_n} \quad (n\in\mathbf{N}_0),$$

Equations (2)-(4) yields the following extensions by Özergin et al. [20]

$$\Gamma_b^{(\omega_1,\omega_2)}(x) = \int_0^\infty y^{x-1}\ {}_1F_1\left(\omega_1;\omega_2;\ -y-\frac{b}{y}\right)dy, \tag{5}$$
$$(R(b)\geq 0;\ \min\{R(x),R(\omega_1),R(\omega_2)\}>0).$$

$$B^{(\omega_1,\omega_2)}(\alpha,\beta;b) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\ {}_1F_1\left(\omega_1;\omega_2;-\frac{b}{y(1-y)}\right)dy, \tag{6}$$
$$(R(b)\geq 0;\ \min\{R(\alpha),R(\beta),R(\omega_1),R(\omega_2)\}>0)$$

and

$$F_b^{(\omega_1,\omega_2)}(a,\theta_1;\theta_2;x) = \frac{1}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^\infty (a)_m B^{(\omega_1,\omega_2)}(\theta_1+m,\theta_2-\theta_1;b)\frac{x^m}{m!}, \tag{7}$$
$$(|x|<1;\ \min\{R(\omega_1),R(\omega_2)\}>0; R(\theta_2)>R(\theta_1)>0;\ R(b)\geq 0)$$

respectively. When $j_n=1$, eqs. (2)-(4) become the following functions studied by Chaudhry et al. [3, 4] as follows

$$\Gamma_b(x) = \int_0^\infty y^{x-1}\exp\left(-y-\frac{b}{y}\right)dy, \ (R(b)>0;\ x\in\mathbf{C}) \tag{8}$$

$$B(\alpha,\beta;b) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\exp\left(-\frac{b}{y(1-y)}\right)dy, \ (R(b)>0) \tag{9}$$

and

$$F_b(a,\theta_1;\theta_2;x) = \sum_{m=0}^\infty (a)_m\frac{B(\theta_1+m,\theta_2-\theta_1;\ b)}{B(\theta_1,\theta_2-\theta_1)}\frac{x^m}{m!}, \tag{10}$$
$$(R(\theta_2)>R(\theta_1)>0;\ b\geq 0,\ |x|<1)$$

respectively. For $j_n=0$ $(n\in\mathbf{N}_0)$ or for $b=0$, eqs. (2)-(4) become to their classical forms [16, 19].

In 1903, Mittag -Leffler [17, 18] introduced and studied the Mittag-Leffler function $E_\alpha(z)$ defined as,

$$E_{\alpha}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+1)}\ , \quad (z,\alpha\in\mathbf{C};R(\alpha)>0) \tag{11}$$

In 1905,Wiman [28] introduced $E_{\alpha,\beta}(z)$ defined as,

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}\ , \quad (z,\alpha,\beta\in\mathbf{C};R(\alpha)>0,R(\beta)>0) \tag{12}$$

Further, the generalized Mittag-Leffler function was introduced and studied by Salim [22] in 2009 and defined as,

$$E_{\alpha,\beta}^{\theta_1,\theta_2}(z)=\sum_{m=0}^{\infty}\frac{(\theta_1)_m\ z^m}{\Gamma(\alpha m+\beta)\ (\theta_2)_m}\ , \tag{13}$$

$$(\alpha,\beta,z,\theta_1,\theta_2\in\mathbf{C};\ \min\{R(\alpha),R(\beta),R(\theta_2)\}>0;\ R(\theta_2)>R(\theta_1)>0)$$

where, $(\theta_1)_m$ is the classical pochhammer symbol.

The Fox H-function is known as a generalization of the Fox-Wright function [6] defined by a Mellin-Barnes integral

$$\begin{aligned}&H_{p,q}^{m,n}\left[t\ \middle|\ \begin{matrix}(g_1,\acute{g_1}),(g_2,\acute{g_2}),...,(g_p,\acute{g_p})\\(h_1,\acute{h_1}),(h_2,\acute{h_2}),...,(h_q,\acute{h_q})\end{matrix}\right]\\&=\frac{1}{2\pi i}\int_L\frac{\prod_{k=1}^{m}\Gamma(h_k+\acute{h}_kx)\prod_{k=1}^{n}\Gamma(1-g_k+\acute{g}_kx)}{\prod_{k=m+1}^{q}\Gamma(1-h_k+\acute{h}_kx)\prod_{k=n+1}^{p}\Gamma(g_k+\acute{g}_kx)}t^{-x}dx\end{aligned} \tag{14}$$

where L is a contour separating the poles of the two factors in the numerator.

The generalized Wright hypergeometric function ${}_p\Psi_q[t]$, known as the Fox-Wright function, is a particular case of Fox H-function [29, 30] defined as

$${}_p\Psi_q[t]={}_p\Psi_q\left[\begin{matrix}(g_1,\acute{g}_1),...,(g_p,\acute{g}_p);\\(h_1,\acute{h}_1),...,(h_q,\acute{h}_q);\end{matrix}\ t\right] \tag{15}$$

$$=\sum_{m=0}^{\infty}\frac{\Gamma(g_1+\acute{g}_1m)...\Gamma(g_p+\acute{g}_pm)}{\Gamma(h_1+\acute{h}_1m)...\Gamma(h_q+\acute{h}_qm)}\frac{t^m}{m!} \tag{16}$$

$$=H_{p,q+1}^{1,p}\left[-t\middle|\begin{matrix}(1-g_1,\acute{g_1}),...,(1-g_p,\acute{g_p})\\(0,1),(1-h_1,\acute{h_1}),...,(1-h_q,\acute{h_q})\end{matrix}\right] \tag{17}$$

where, $H_{p,q+1}^{1,p}[t]$ denotes the Fox-H function [6].

For $\acute{g}_1=...=\acute{g}_p=1,\acute{h}_1=...=\acute{h_q}=1$ Eq.(14) becomes the generalized hypergeometric function ${}_pF_q$ [31]

$${}_p\Psi_q\left[\begin{matrix}(g_1,1),...,(g_p,1);\\(h_1,1),...,(h_q,1);\end{matrix}\ t\right]=\frac{\Gamma(g_1)...\Gamma(g_p)}{\Gamma(h_1)...\Gamma(h_q)}\ {}_pF_q(g_1,...,g_p;h_1,...,h_q;t) \tag{18}$$

Primarily due to the significant applications of these extended hypergeometric functions, we extend the generalized Mittag-Leffler function (13) employing the extended beta function $B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\alpha,\beta;b)$ as defined in (3) and consider the fundamental properties such as differentiation and derivative formulas alongside various integral transforms such as Euler-Beta transform, Laplace transform, Mellin transform, and Whittaker transform. Furthermore, we explore some results involving Riemann-Liouville fractional integrals and derivatives of the extended Mittag-Leffler function.

2. Extended Mittag-Leffler Function.

Here by using (3) with the approach

$$\frac{(\theta_1)_m}{(\theta_2)_m}=\frac{B(\theta_1+m,\theta_2-\theta_1)}{B(\theta_1,\theta_2-\theta_1)}\rightarrow\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ , \tag{19}$$

We will define the new extended Mittag-Leffler function as follows:

$$\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{z^m}{\Gamma(\alpha m+\beta)}\ , \tag{20}$$

$(\alpha,\beta,z,\theta_1,\theta_2\in\mathbf{C};\ b\geq 0,\ \min\{R(\alpha),R(\beta),R(\theta_2)\}>0;R(\theta_2)>R(\theta_1)>0)$.

Remark 1. The special case of (20) when we confirmed the sequence $j_n=\frac{(\omega_1)_n}{(\omega_2)_n}$ $(n\in\mathbf{N}_0)$, yields another form

$$\mathbb{E}_{\alpha,\beta}^{(\omega_1,\omega_2);\theta_1,\theta_2}(z;b)=\sum_{m=0}^{\infty}\frac{B^{(\omega_1,\omega_2)}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{z^m}{\Gamma(\alpha m+\beta)} \tag{21}$$

$(\alpha,\beta,z,\theta_1,\theta_2\in\mathbf{C};\ b\geq 0,\ \min\{R(\alpha),R(\beta),R(\theta_2)\}>0;R(\theta_2)>R(\theta_1)>0,R(\omega_1)>0,R(\omega_2)>0)$.

For $b=0$ or $j_n=0$ $(n\in\mathbf{N}_0)$, eq (20) reduces to (13).

Remark 2. If we put $\alpha=0$, $\beta=1$ in (20) and (21) we get

$$\mathbb{E}_{0,1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)=\frac{\Gamma(\theta_2)}{\Gamma(\theta_1)}{}_2\Psi_1^{(\{j_n\}_{n\in\mathbf{N}_0})}\left[\begin{matrix}(\theta_1,1),(1,1);\\(\theta_2,1);\end{matrix}(z;b)\right]. \tag{22}$$

and

$$\mathbb{E}_{0,1}^{(\omega_1,\omega_2);\theta_1,\theta_2}(z;b)=\frac{\Gamma(\theta_2)}{\Gamma(\theta_1)}{}_2\Psi_1^{(\omega_1,\omega_2)}\left[\begin{matrix}(\theta_1,1),(1,1);\\(\theta_2,1);\end{matrix}(z;b)\right]. \tag{23}$$

Further if we put, $j_n=0$, $b=0$, $\alpha=1$ and z is replaced by $-z$, we get

$$\mathbb{E}_{1,\beta}^{\theta_1,\theta_2}(-z)=\frac{\Gamma(\theta_2)}{\Gamma(\theta_1)}{}_2\Psi_2\left[\begin{matrix}(\theta_1,1),(1,1);\\(\theta_2,1),(\beta,1);\end{matrix}-z\right]. \tag{24}$$

3. Derivative Properties of $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$

Here, we consider some derivative properties of the extended Mittag-Leffler function (20).

Theorem 3.1. We have the following differential formula for the extended Mittag-Leffler function

$$\left(\frac{d}{dz}\right)^k \left[z^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(\lambda z^\alpha;b)\right] = z^{\beta-k-1}\mathbb{E}_{\alpha,\beta-k}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(\lambda z^\alpha;b) \tag{25}$$

$(\alpha,\beta,z,\theta_1,\theta_2 \in \mathbf{C};\ R(\beta-k)>0, k\in\mathbf{N};\ R(\theta_2)>R(\theta_1)>0)$.

Proof. Using (20) and applying term-wise k times differentiations on (25), we get

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^k \left[z^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(\lambda z^\alpha;b)\right]\\
&=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{\lambda^m}{\Gamma(\alpha m+\beta)}\left(\frac{d}{dz}\right)^k(z^{\alpha m+\beta-1})\\
&=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{\lambda^m}{\Gamma(\alpha m+\beta-k)}(z^{\alpha m+\beta-k-1})\\
&=(z^{\beta-k-1})\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{\lambda^m z^{\alpha m}}{\Gamma(\alpha m+\beta-k)}\\
&=z^{\beta-k-1}\mathbb{E}_{\alpha,\beta-k}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(\lambda z^\alpha;b).
\end{aligned}$$

Theorem 3.2. The following derivative formula for the extended Mittag-Leffler function holds

$$\begin{aligned}
&\left(\frac{d^k}{dz^k}\right)\left[\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)\right]\\
&=\frac{(\theta_1)_k}{(\theta_2)_k}\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m+k,\theta_2-\theta_1;b)}{B(\theta_1+k,\theta_2-\theta_1)}\ \frac{(m+1)_k z^m}{\Gamma(\alpha m+\beta+\alpha k)}
\end{aligned} \tag{26}$$

$(\alpha,\beta,\theta_1,\theta_2\in\mathbf{C};\ k\in\mathbf{N};\ R(b)>0, R(\theta_2)>R(\theta_1)>0)$.

Proof. Differentiating eq. (20) with respect to the variable z we get,

$$\begin{aligned}
\left(\frac{d}{dz}\right)\left[\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)\right] &=\frac{d}{dz}\left[\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{z^m}{\Gamma(\alpha m+\beta)}\right]\\
&=\sum_{m=1}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{z^{m-1}m}{\Gamma(\alpha m+\beta)}
\end{aligned}$$

Now replacing $m\to m+1$ and applying a known result ([21], p.46)

$$B(\theta_1,\theta_2-\theta_1)=\frac{\theta_2}{\theta_1}B(\theta_1+1,\theta_2-\theta_1)$$

we obtain,

$$= \frac{\theta_1}{\theta_2} \sum_{m=0}^{\infty} \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m+1, \theta_2-\theta_1; b)}{B(\theta_1+1, \theta_2-\theta_1)} \frac{(m+1)z^m}{\Gamma(\alpha m+\beta+\alpha)}$$

After repeating this process as term-wise kth time differentiation, we obtain

$$\left(\frac{d^k}{dz^k}\right)\left[\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)\right]$$
$$= \frac{(\theta_1)_k}{(\theta_2)_k} \sum_{m=0}^{\infty} \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m+k, \theta_2-\theta_1; b)}{B(\theta_1+k, \theta_2-\theta_1)} \frac{(m+1)_k\, z^m}{\Gamma(\alpha m+\beta+\alpha k)}.$$

Remark 3. If we take $b=0$ or $j_n=0$, we get a known result [22] as a special case of (26) that is

$$\left(\frac{d^k}{dz^k}\right)\left[\mathbb{E}_{\alpha,\beta}^{\theta_1,\theta_2}(z)\right] = \frac{(\theta_1)_k}{(\theta_2)_k} \sum_{m=0}^{\infty} \frac{(\theta_1+k)_m}{(\theta_2+k)_m} \frac{(m+1)_k\, z^m}{\Gamma(\alpha m+\beta+\alpha k)}.$$

Theorem 3.3. Extended Mittag-Leffler function satisfies the following differentiation formula:

$$\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) = \beta\,\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) + \alpha z\frac{d}{dz}\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) \quad (27)$$

$(\alpha, \beta, \theta_1, \theta_2 \in \mathbf{C};\ R(b)>0, R(\theta_2)>R(\theta_1)>0)$.

In particular,

$$\mathbb{E}_{\alpha,\beta}^{\theta_1,\theta_2}(z) = \beta\,\mathbb{E}_{\alpha,\beta+1}^{\theta_1,\theta_2}(z) + \alpha z\frac{d}{dz}\mathbb{E}_{\alpha,\beta+1}^{\theta_1,\theta_2}(z). \quad (28)$$

Proof. Applying (20) in left hand side of (27) of we obtain,

$$\beta\,\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) + \alpha z\frac{d}{dz}\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$$
$$=\beta\,\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) + \alpha z\frac{d}{dz}\sum_{m=0}^{\infty} \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m, \theta_2-\theta_1; b)}{B(\theta_1, \theta_2-\theta_1)} \frac{z^m}{\Gamma(\alpha m+\beta+1)}$$
$$=\beta\,\mathbb{E}_{\alpha,\beta+1}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b) + \sum_{m=0}^{\infty} \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m, \theta_2-\theta_1; b)}{B(\theta_1, \theta_2-\theta_1)} \frac{z^m \alpha m}{\Gamma(\alpha m+\beta+1)}$$

Now using $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$, above equation becomes

$$= \sum_{m=0}^{\infty} \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m, \theta_2-\theta_1; b)}{B(\theta_1, \theta_2-\theta_1)} \frac{z^m}{\Gamma(\alpha m+\beta)}$$
$$=\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b).$$

When $b=0$ or $j_n=0$ $(n\in\mathbf{N}_0)$, we get relation (28) from (27).

4. Integral Transform of Extended Mittag-Leffler function

Here, we will discuss certain integral transform for $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$.

1. Euler-Beta Transform.

For the function $f(z)$, the Euler-Beta transform [29] is defined as

$$B\left\{f(z);p,q\right\}=\int_0^1 z^{p-1}(1-z)^{q-1}f(z)\,dz \tag{29}$$

Theorem 4.1. The following Euler-Beta transform for $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$ holds:

$$\begin{aligned} &B\left\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b);p,q\right\}\\ =&\Gamma(q)\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\;{}_2\Psi_2\left[\begin{matrix}(p,\omega_2),(1,1);\\(\beta,\alpha),(p+q,\omega_2);\end{matrix}\;t\right]. \end{aligned}\tag{30}$$

$(\alpha,\beta,\theta_1,\theta_2\in\mathbf{C};\;R(b)>0,R(p)>0,R(q)>0,R(\theta_2)>R(\theta_1)>0).$

Proof. Using (29) in (20), we obtain

$$\begin{aligned} &B\left\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b);p,q\right\}\\ &=\int_0^1 z^{p-1}(1-z)^{q-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b)\,dz\\ &=\int_0^1 z^{p-1}(1-z)^{q-1}\left(\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\;\frac{t^m z^{\omega_2 m}}{\Gamma(\alpha m+\beta)}\right)dz\\ &=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\;\frac{t^m}{\Gamma(\alpha m+\beta)}\int_0^1 z^{p+\omega_2 m-1}(1-z)^{q-1}\,dz\\ &=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\;\frac{t^m}{\Gamma(\alpha m+\beta)}\;\frac{\Gamma(p+\omega_2 m)\Gamma(q)}{\Gamma(p+q+\omega_2 m)} \end{aligned}\tag{31}$$

$$=\Gamma(q)\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^{\infty}\frac{\Gamma(p+\omega_2 m)\Gamma(m+1)\,t^m}{\Gamma(\alpha m+\beta)\Gamma(p+q+\omega_2 m)\,m!} \tag{32}$$

Using eq (15) and (16) in (32), we obtain the desired result.

Remark 4. Replace z by $(1-z)$ in $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$ we get,

$$\begin{aligned} &B\left\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(t(1-z)^{\omega_2};b);p,q\right\}\\ =&\int_0^1 z^{p-1}(1-z)^{q-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(t(1-z)^{\omega_2};b)\,dz\\ =&\Gamma(p)\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\;{}_2\Psi_2\left[\begin{matrix}(q,\omega_2),(1,1);\\(\beta,\alpha),(p+q,\omega_2);\end{matrix}\;t\right]. \end{aligned}\tag{33}$$

Remark 5. If we put $b=0$ or $j_n=0$ in (30), we get the known result for Euler-Beta transform of generalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}^{\theta_1,\theta_2}(z)$ [22].

2. Laplace Transform.

For the function $f(z)$, the Laplace transform [23] is defined as:

$$\mathbb{L}\left\{f(z)\right\} = \int_0^\infty e^{-sz} f(z)\, dz \tag{34}$$

Theorem 4.2. The following Laplace Transform for $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$ holds:

$$\begin{aligned}&\int_0^\infty z^{p-1}e^{-sz}\ \mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b)\, dz\\ =&\frac{1}{s^p}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ {}_2\Psi_1\left[\begin{matrix}(p,\omega_2),(1,1); & \dfrac{t}{s^{\omega_2}}\\ (\beta,\alpha); & \end{matrix}\right].\end{aligned} \tag{35}$$

$(\alpha,\beta,\theta_2,\theta_1 \in \mathbf{C};\ R(s)>0, R(b)>0, R(\theta_2)>R(\theta_1)>0)$.

Proof. Using (20) in the left hand side of (35), we obtain

$$\begin{aligned}&\int_0^\infty z^{p-1}e^{-sz}\ \mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b)\, dz\\ =&\int_0^\infty z^{p-1}e^{-sz}\ \sum_{m=0}^\infty \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\frac{t^m z^{\omega_2 m}}{\Gamma(\alpha m+\beta)}\, dz\\ =&\sum_{m=0}^\infty \frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\frac{t^m}{\Gamma(\alpha m+\beta)}\int_0^\infty z^{p+\omega_2 m-1}e^{-sz}\, dz\\ =&\frac{1}{s^p}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^\infty\frac{\Gamma(p+\omega_2 m)\Gamma(m+1)}{\Gamma(\alpha m+\beta)\, m!}\left(\frac{t}{s^{\omega_2}}\right)^m\end{aligned} \tag{36}$$

Now, using eq (15) and (16) in (36), we get the desired result.

Remark 6. If we put $b=0$ or $j_n=0$ in (35), we get the known result for Euler-Beta transform for the generalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}^{\theta_1,\theta_2}(z)$ [22] as

$$\begin{aligned}&\int_0^\infty z^{p-1}e^{-sz}\ \mathbb{E}_{\alpha,\beta}^{\theta_1,\theta_2}(tz^{\omega_2})\, dz\\ =&\frac{1}{s^p}\frac{\Gamma(\theta_2)}{\Gamma(\theta_1)}\ {}_3\Psi_2\left[\begin{matrix}(\theta_1,1),(p,\omega_2),(1,1); & \dfrac{t}{s^{\omega_2}}\\ (\theta_2,1),(\beta,\alpha); & \end{matrix}\right].\end{aligned}$$

3. Mellin Transform.

A properly integrable function $f(y)$ with index s has the following definition for the Mellin transform [26]:

$$M\{f(z);z\to s\} := \int_0^\infty z^{s-1}f(z)\, dz \tag{37}$$

whenever the improper integral in (37) exists.

Theorem 4.3. The following Mellin Transform for $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$ holds:

$$\begin{aligned}&M\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b);b\to s\}\\=&\frac{\Gamma(\theta_2)\Gamma_0^{(\{j_n\})_{n\in\mathbf{N}_0})}(s)\Gamma(\theta_2-\theta_1+s)}{\Gamma(\theta_1)\Gamma(\theta_2-\theta_1)}\ {}_2\Psi_2\left[\begin{matrix}(\theta_1+s,1),(1,1);\\(\theta_1+2s,1),(\beta,\alpha);\end{matrix}\ z\right].\end{aligned} \tag{38}$$

$(R(s)>0$, and $R(\omega_2-\omega_1+s)>0)$,

where $\Gamma_0^{(\{j_n\})_{n\in\mathbf{N}_0})}$ is a particular case of (2) for $b=0$.

Proof. Using (37) in (20), we obtain

$$\begin{aligned}&M\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b);b\to s\}\\=&\int_0^\infty b^{s-1}\,\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)\,db\\=&\int_0^\infty b^{s-1}\ \left(\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\ \frac{z^m}{\Gamma(\alpha m+\beta)}\right)\\=&\frac{1}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}\ \int_0^\infty b^{s-1}B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,db\end{aligned}$$

Which on further solving gives,

$$=\frac{1}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\beta)}\ \int_0^\infty b^{s-1}B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,db \tag{39}$$

Applying a known result [26]

$$\int_0^\infty b^{s-1}B^{(\{j_n\}_{n\in\mathbf{N}_0})}(t,z;b)db=\Gamma_0^{(\{j_n\})_{n\in\mathbf{N}_0})}(s)\ B(t+s,z+s),\ R(s)>0,$$

in (39) we get,

$$\begin{aligned}&M\{\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b);b\to s\}\\=&\frac{\Gamma_0^{(\{j_n\})_{n\in\mathbf{N}_0})}(s)}{B(\theta_1,\theta_2-\theta_1)}\sum_{m=0}^{\infty}B(\theta_1+m+s,\theta_2-\theta_1+s)\,\frac{z^m}{\Gamma(\alpha m+\beta)}\end{aligned}$$

$$=\frac{\Gamma_0^{(\{j_n\})_{n\in\mathbf{N}_0})}(s)\,\Gamma(\theta_2)\,\Gamma(\theta_2-\theta_1+s)}{\Gamma(\theta_1)\,\Gamma(\theta_2-\theta_1)}\sum_{m=0}^{\infty}\frac{\Gamma(\theta_1+s+m)\,\Gamma(m+1)\,z^m}{\Gamma(\theta_2+m+2s)\,\Gamma(\alpha m+\beta)\,m!} \tag{40}$$

Now using, (15) and (16) in (40), we get the required result.

4. Whittaker Transform.

Theorem 4.4. The following Whittaker Transform for $\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(z;b)$ holds:

$$\int_0^\infty z^{\nu-1}e^{\frac{-pz}{2}}\,W_{\lambda,\eta}(pz)\,\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(tz^{\omega_2};b)\,dz$$

$$=p^{-\nu}\,\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}\,{}_3\Psi_2\left[\begin{matrix}(\frac{1}{2}+\eta+\nu,\omega_2),(\frac{1}{2}-\eta+\nu,\omega_2),(1,1); & \dfrac{t}{p^{\omega_2}}\\ (\beta,\alpha),(1-\lambda+\nu,\omega_2); & \end{matrix}\right] \tag{41}$$

Proof. Substitute $pz=u$ in L.H.S of (41), we obtain

$$\int_0^\infty\left(\frac{u}{p}\right)^{\nu-1}e^{\frac{-u}{2}}\,W_{\lambda,\eta}(u)\,\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,t^m}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)}\,\left(\frac{u}{p}\right)^{\omega_2 n}\frac{1}{p}\,du$$

$$=p^{-\nu}\,\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,t^m}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)\,p^{\omega_2 m}}\,\int_0^\infty u^{\omega_2 m+\nu-1}e^{\frac{-u}{2}}\,W_{\lambda,\eta}(u)\,du \tag{42}$$

Applying the integral formula

$$\int_0^\infty x^{\nu-1}e^{\frac{-x}{2}}\,W_{\lambda,\eta}(x)\,dx=\frac{\Gamma(\frac{1}{2}+\eta+\nu)\;\Gamma(\frac{1}{2}-\eta+\nu)}{\Gamma(1-\lambda+\nu)},\;\left(R(\nu\pm\eta)>\frac{-1}{2}\right)$$

where $W_{\lambda,\eta}(x)$ is the whittaker function [27],

we get,

$$=p^{-\nu}\,\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)}$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\eta+\nu+\omega_2 m)\;\Gamma(\frac{1}{2}-\eta+\nu+\omega_2 m)\;\Gamma(m+1)}{\Gamma(1-\lambda+\nu+\omega_2 m)\;\Gamma(\alpha m+\beta)\;m!}\left(\frac{t}{p^{\omega_2}}\right)^m \tag{43}$$

Using (15) and (16) in (42), we get the desired result.

5. Fractional Property of Extended Mittag-Leffler function

In this section, we will discuss the results involving the Riemann-Liouville right-sided fractional integral operator I_{0+}^{η} and the derivative operator D_{0+}^{η}, which are defined respectively as [15, 23]

$$(I_{o+}^{\eta}\phi)(t)=\frac{1}{\Gamma(\eta)}\int_0^t\frac{\phi(z)}{(t-z)^{1-\eta}}dz,\;(R(\eta)>0,\;\eta\in\mathbf{C}) \tag{44}$$

and

$$(D_{0+}^{\eta}\phi)(t)=\left(\frac{d}{dt}\right)^m(I_{0+}^{m-\eta}\phi)(t),\quad(m=[R(\eta)]+1;\;R(\eta)>0,\eta\in\mathbf{C}) \tag{45}$$

where, $[t]$ is the greatest integer.

A generalized Riemann-Liouville right-sided fractional derivative operator $D_{0+}^{\eta,\gamma}$ of order $0<\eta<1$ and $0\leq\gamma\leq 1$ with respect to t by Hilfer [9] is defined as follows:

$$(D_{0+}^{\eta,\gamma}\phi)(t)=\left(I_{0+}^{\gamma(1-\eta)}\frac{d}{dt}\right)\left(I_{0+}^{(1-\gamma)(1-\eta)}\phi\right)(t),\quad (m=[R(\eta)]+1;\ R(\eta)>0,\eta\in\mathbf{C}). \tag{46}$$

Theorem 5.1. For $t>r$, the following results hold:

$$\begin{aligned}&\left(I_{0+}^{\eta}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\&=(t-r)^{\beta+\eta-1}\mathbb{E}_{\alpha,\beta+\eta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(t-r)^{\alpha};\ b).\end{aligned} \tag{47}$$

and

$$\begin{aligned}&\left(D_{0+}^{\eta}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\&=(t-r)^{\beta-\eta-1}\mathbb{E}_{\alpha,\beta-\eta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(t-r)^{\alpha};\ b).\end{aligned} \tag{48}$$

where,
$r\in R_+;\ \alpha,\beta,\theta_1,\theta_2,\eta,q\in\mathbf{C};\ R(\alpha)>0,R(\beta)>0,R(\eta)>0.$

Proof. Using (20) and (44), term by term fractional integration and using the relation

$$\left(I_{0+}^{\alpha}\left[(z-r)^{\beta-1}\right]\right)(t)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(t-r)^{\alpha+\beta-1},\ (\alpha,\beta\in\mathbf{C},\ R(\alpha)>0,\ R(\beta)>0) \tag{49}$$

yields for $t>r$:

$$\begin{aligned}&\left(I_{0+}^{\eta}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\&=\left(I_{0+}^{\eta}\left[\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,q^m}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)}\,(z-r)^{\alpha m+\beta-1}\right]\right)(t)\\&=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)}\left(I_{0+}^{\eta}\left[(z-r)^{\alpha m+\beta-1}\right]\right)(t)\\&=(t-r)^{\beta+\eta-1}\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta+\eta)}\,(q(t-r)^{\alpha})^m\\&=(t-r)^{\beta+\eta-1}\mathbb{E}_{\alpha,\beta+\eta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(t-r)^{\alpha};\ b).\end{aligned}$$

Hence, we get the desired result (47).

Next, by using (20) and (45), we get,

$$\begin{aligned}&\left(D_{0+}^{\eta}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\&=\left(\frac{d}{dt}\right)^m\left(I_{0+}^{m-\eta}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\&=\left(\frac{d}{dt}\right)^m\left[(t-r)^{\beta+m-\eta-1}\mathbb{E}_{\alpha,\beta+m-\eta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(t-r)^{\alpha};b)\right]\end{aligned}$$

Now applying (25), we get the desired result (48).

Finally, from eqs. (20) and (46), we obtain

$$\begin{aligned}
&\left(D_{0+}^{\eta,\gamma}\left[(z-r)^{\beta-1}\mathbb{E}_{\alpha,\beta}^{(\{j_n\}_{n\in\mathbf{N}_0});\theta_1,\theta_2}(q(z-r)^{\alpha};b)\right]\right)(t)\\
&=\left(D_{0+}^{\eta,\gamma}\left[\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)\,q^m}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)}\,(z-r)^{\alpha m+\beta-1}\right]\right)(t)\\
&=\sum_{m=0}^{\infty}\frac{B^{(\{j_n\}_{n\in\mathbf{N}_0})}(\theta_1+m,\theta_2-\theta_1;b)}{B(\theta_1,\theta_2-\theta_1)\Gamma(\alpha m+\beta)}\left(D_{0+}^{\eta,\gamma}\left[(z-r)^{\alpha m+\beta-1}\right]\right)(t) \qquad (50)
\end{aligned}$$

Using the well-known result of Srivastava and Tomovski [23], in (50), we are keeping the desired result (46).

References

[1] D. Baleanu and B. Shiri, Nonlinear higher order fractional terminal value problems, AIMS Mathematics, 7(5), 7489-7506, 2022.

[2] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, Extension of Euler's beta function, Journal of computational and applied mathematics, 78(1), 19-32, 1997.

[3] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, Applied Mathematics and Computation, 159(2), 589-602, 2004.

[4] M. A. Chaudhry and S. M. Zubair, Generalized incomplete gamma functions with applications, Journal of Computational and Applied Mathematics, 55(1), 99-123, 1994.

[5] M. A. Chaudhry and S. M. Zubair, On a class of incomplete gamma functions with applications, Chapman and Hall/CRC, 2001.

[6] C. Fox, The G and H functions as symmetrical Fourier kernels, Transactions of the American Mathematical Society, 98(3), 395-429, 1961.

[7] C. Y . Gu, G. C. Wu and B. Shiri, An inverse problem approach to determine possible memory length of fractional differential equations, ctional Calculus and Applied Analysis, 24(6), 1919-1936, 2021.

[8] C. Y. Gu, F. X. Zheng and B. Shiri, Mittag-Leffler stability analysis of tempered fractional neural networks with short memory and variable-order, Fractals, 2140029, 2021.

[9] R. Hilfer, Application of fractional calculus in physics, World Scientific Publishing Co. Pte. Ltd, Singapore, 2000.

[10] M. Kamarujjama, N. U. Khan and Owais Khan, Fractional calculus of generalized p-k-Mittag-Leffler function using Marichev-Saigo-Maeda operators, Arab Journal of Mathematical Sciences, 25(2), 156-168, 2019.

[11] N. U. Khan, T. Usman and Mohd. Aman, Some properties concerning the analysis of generalized Wright function, Journal of Computational and Applied Mathematics, 376, 112840, 2020.

[12] N. U. Khan, M. I. Khan and Owais Khan, Evaluation of Transforms and Fractional Calculus of New Extended Wright Function, International Journal of Applied and Computational Mathematics 8.4, 163, 2022.

[13] N. U. Khan, T. Usman and Mohd. Aman, Extended beta, hypergeometric and confluent hypergeometric functions, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.Mathematics, 39(1), 83-97, 2019.

[14] N. U. Khan, T. Usman, M. Kamarujjama, M. Aman and M. I. Khan, A new extension of beta function and its properties, JMI International journal of Mathematical Sciences, Vol. 9, 51-63, 2018.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, Vol. 204, elsevier, 2006.

[16] M. J. Luo and R. K. Raina, Extended generalized hypergeometric functions and their applications, Bull. Math. Anal. Appl, 5(4), 65-77, 2013.

[17] G. M. Mittag-Leffler, Une généralisation de l'intégrale de Laplace-Abel, CR Acad. Sci. Paris (Ser. II) 137, 537-539, 1903.
[18] G. M. Mittag-Leffler, Sur la nouvelle fonction $E\alpha(x)$, CR Acad. Sci. Paris 137.2, 554-558, 1903.
[19] M. A. Özarslan and E. Özergine, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Mathematical and Computer Modelling, 52(9-10), 1825-1833, 2010.
[20] E. Özergine, M.A. Özarslan and A. Altin, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math., 235(16), 4601-4610, 2011.
[21] R. K. Parmar, A new generalization of gamma, beta hypergeometric and confluent hypergeometric functions, Le Matematiche, 68(2), 33-52, 2013.
[22] Tariq O. Salim, Some properties relating to the generalized Mittag-Leffler function, Adv. Appl. Math. Anal 4.1, 21-30, 2009.
[23] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Vol. 1. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon, 1993.
[24] B. Shiri, O. I. Wu and D. Baleanu, Terminal value problems for the nonlinear systems of fractional differential equations, Applied Numerical Mathematics, 170, 162-178, 2021.
[25] D. Shiri and D. Baleanu, A General Fractional Pollution Model for Lakes, Communications on Applied Mathematics and Computation, 1-26, 2021.
[26] H. M. Srivastava, R. K. Parmar and Purnima Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms 1.3, 238-258, 2018.
[27] E.T. Whittaker and G.N. Watson, A course of modern analysis, Cambridge Univ. Press, Cambridge, 1962.
[28] Adders. Wiman, Über den Fundamentalsatz in der Teorie der Funktionen $E_\alpha(x)$, 191-201, 1905.
[29] E.M. Wright, The asymptotic expansion the generalized hypergeometric functions, J.Lond. Math. Soc., 10(4), 286-293, 1935.
[30] E.M. Wright, The asymptotic expansion the generalized hypergeometric functions, Proc. Lond. Math. Soc., 46(1), 389-408, 1940.
[31] G. Yang, B. Shiri, H. Kong and G. C. Wu, Intermediate value problems for fractional differential equations, Computational and Applied Mathematics, 40(6), 1-20, 2021.

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