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GALERKIN METHOD FOR THE NUMERICAL SOLUTION OF SINGULAR BOUNDARY VALUE PROBLEMS USING BERNOULLI WAVELETS

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Abstract. Singular two-point boundary value problems for ordinary differential equations are commonly encountered in various fields of science and engineering. The numerical solution of these singular boundary value problems (SBVPs) is often challenging due to the presence of singularities in the equations. Wavelets are wave-like oscillations with amplitude that begins at zero and it two basic properties: scale and location. Scale defines how “stretched” or “squished” a wavelet is. This property is related to frequency as defined for waves. Location defines where the wavelet is positioned in time. Wavelets enable the decomposition of complex information, such as music, speech, images, and patterns, into simpler components at various positions and scales, which can then be accurately reconstructed. This paper presents a Galerkin method for solving SBVPs numerically using Bernoulli wavelets. It includes numerical examples that illustrate the method’s accuracy, applicability, and usefulness. The findings indicate that the method is highly effective, straightforward, and easy to implement.

1. Introduction

Many problems in the physical and engineering sciences are often modeled using singular boundary value problems (SBVPs), which are a significant class of boundary value problems. The majority of the times, analytical methods are not always able to provide solutions for SBVPs. In reality, this technique is nearly impossible to solve for many real-world physical phenomena; instead, a variety of approximate and numerical methods must be used [13]. An extensive number of authors have contributed to the solution of this class of problems. A Numerical method [9], Legendre wavelet method [10], Hermite wavelet method [14], Laguerre Wavelet based

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Galerkin Method [2], Hermite Wavelet Based Galerkin Method [3], and other novel approaches and methods have enhanced the scientific literature.
Spectral methods, particularly Chebyshev spectral methods, are often used to solve singular boundary value problems, which are differential equations with singularities in the boundary conditions or coefficients. These methods approximate the solution and its derivatives using spectral basis functions, such as Chebyshev polynomials, which are well-suited for handling such singularities [8].
Wavelet-based methods offer a powerful approach to solving singular boundary value problems (SBVPs) by leveraging the properties of wavelets to handle the singularities and discontinuities often present in such problems. These methods, like the Wavelet Galerkin method transform the differential equations into a system of algebraic equations, making them easier to solve numerically [15].
Wavelets have attracted a lot of attention because of their broad mathematical capabilities and strong application in a variety of numerical problems. Furthermore, it can observe that spectral bases have global support but infinite differentiability, while the basis function used in the finite element method (FEM) has a less compact support and an extremely weak continuity property. Similarly, FEM performs well in terms of spatial localization while spectral methods perform poorly in terms of spectral localization. Additionally, the unique benefits of both spectral and FEM bases are fulfilled by wavelet basis. An alternative to conventional piecewise polynomial trial functions in the analysis of differential equations using finite element methods is the investigation of wavelet function bases. Due to its practicality and ease of use, the Galerkin method is well known in the field of applied mathematics [1, 12].
In wavelet frameworks, the Galerkin method is often favoured over collocation or least-squares methods because it excels in managing complex problems and preserving accuracy, particularly with high-order equations or irregular shapes. Although collocation methods are easier to compute, they can face challenges with accuracy and stability in complicated situations. Least-squares methods are flexible but may demand more computational resources and can be less efficient than Galerkin in certain cases. The Galerkin method works by discretizing the differential equation through the projection of the residual onto a test function space, usually employing wavelets as the basis functions. This technique results in a system of equations that can be solved numerically to approximate the solution of the differential equation. Galerkin method are to introduce a trial solution as a linear combination of basis functions, choose weight functions, take the inner product of the residual and weight functions to generate a system of equations for the unknown coefficients, and solve this system to obtain the approximate solution.
The wavelet-Galerkin method is a popular technique in many scientific and engineering domains due to its notable advantages over the finite difference and finite element methods. In certain situations, the wavelet technique offers a convincing substitute for the finite element method, providing a practical way to solve SBVPs. Bernoulli wavelets are well-suited for analyzing functions with discontinuities and sharp edges. These wavelets often possess compact support and orthogonality, making them suitable for various applications.
The study presents the BWGM approach to numerically solving SBVPs. This approach uses Bernoulli wavelets with unknown coefficients to represent the solution. Then, the Galerkin method and the properties of Bernoulli wavelets are used to

compute these coefficients and produce a numerical solution for the SBVPs.
The paper's outline is as follows: Section 2 presents Bernoulli wavelets and their function approximation. Section 3 discusses the Galerkin method for solving SB-VPs, which is based on Bernoulli wavelets. The numerical illustration has showed in Section 4. Lastly, a discussion of the findings from the suggested research is provided in Section 5.

2. Bernoulli wavelets and Function approximation

Wavelets: Wavelets constitute a family of functions constructed from dialation and translation of a single function $\psi(x)$ called mother wavelet [5, 11]. When the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\psi(\frac{x-b}{a})\ \forall a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$,$b = nb_0a_0^{-k}$,$a_0 > 1$,$b_0 > 1$.We have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{-1}{2}}\psi(a_0^k x - nb_0)\ \forall a, b \in R, a \neq 0.$$

$\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis.
Bernoulli wavelets: The Bernoulli wavelets $\psi_{n,m}(x) = \psi_{n,m}(k, \hat{n}, m, x)$ have four arguments $\hat{n} = n-1$, $n = 1, 2, 3, ...2^{k-1}$, k can be any positive integer, m is the degree of the Bernoulli polynomials and x is the normalized time. They are defined on the interval by [6]

$$\psi_{n,m}(x) = \begin{cases} 2^{k-1}\tilde{B}_m(2^{k-1}x - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \leq x \leq \frac{\hat{n}+1}{2^{k-1}} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

where

$$\tilde{B}_m(x) == \begin{cases} \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2\alpha_{2m}}{(2m!)}}}B_m(x), & m > 0 \\ 1, & m = 0 \end{cases}$$

also, $m = 0, 1, 2, 3, ..., M-1$ and the coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2B_{2m}}{2m!}}}$ is used for orthonormal condition. Here $B_m(x)$ are the Bernoulli polynomials of order m, which are defined on the interval as

$$B_m(x) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right)\alpha_{m-j}x^j \tag{2}$$

where $B_j = B_j(0), j = 0, 1, 2, ..., m$ are Bernoulli numbers.
The first few Bernoulli polynomials are: $B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{2}, ...$
Bernoulli polynomials form a complete basis over the interval with the following condition:

$$\int_0^1 B_m(x)B_n(x)dx = (-1)^{n-1}\frac{(m!)(n!)}{(m+n)!}B_{m+n}, m, n \geq 1 \tag{3}$$

For instance, $k = 1$ and $M = 3$, we get the Bernoulli wavelet bases as follows:

$$\psi_{1,0}(x) = 1,$$
$$\psi_{1,1}(x) = \sqrt{3}(2x-1),$$
$$\psi_{1,2}(x) = \sqrt{5}(6x^2-6x+1) \text{ and so on.}$$

Function approximation:
Suppose $y(x) \in L^2[0,1)$ is expanded in terms of Bernoulli wavelets as:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \tag{4}$$

Truncating the above infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{5}$$

3. Method of solution

Consider the SBVP in the following form,

$$y" + P(x)y' + Q(x)y = f(x) \tag{6}$$

with boundary conditions

$$y(a) = \alpha, y(b) = \beta \tag{7}$$

where the functions $P(x), Q(x)$ and $f(x)$ are analytic in $x \in (0,1]$ and the functions $P(x)$ and $Q(x)$ are not analytic for $x = 0$ i.e. Singularity at $x = 0$. Rewrite the Eq. 6 when $R(x) = 0$, for the exact solution, $y(x)$ only which satisfied the given boundary conditions. The trial series solution of Eq. 6, within the range of (0,1], meets the specified boundary conditions and can be expanded to a modified Bernoulli wavelet by introducing unknown parameters in the process as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{8}$$

The unknown coefficients $c_{n,m}$'s, which are to be determined, The precision of the solution is improved by choosing higher degree Bernoulli wavelet polynomials. Compute the second derivative, w.r.t. x from Eq. 8 to determine the values $y, y', y"$, then enter these values into Eq. 7. Use weight functions as the assumed basis elements to solve for the unknown coefficients, then integrate the residual and boundary values to get zero [7]

$$i.e. \int_0^1 \psi_{1,m}(x)R(x)dx = 0, \quad m = 0,1,2,...$$

From the above equation, a system of linear algebraic equations can be derived involving unknown coefficients and can be found by solving them. After finding these unknowns and substitute these in Eq. 8, to determine the numerical solution for Eq. 6. To evaluate the BWGM's accuracy on the test cases, we use the maximum absolute error as an error metric. Here are the formulas for calculating the (i) maximum absolute error, (ii) L_2-norm, (iii) L_∞-norm.

(1) Maximum absolute error = $E_{max} = max|y(x)_e - y(x)_n|$, where $y(x)_e$ and $y(x)_n$ are exact and numerical solution.

(2) $L_2 - norm = ||\sum_{m=1}^{n} E_m^2||^{\frac{1}{2}}$.

(3) L_∞-norm = $||Max(E_m)||, m = 1, 2, ..., 9$

4. Numerical Illustration

Problem 4.1 First, consider the SBVP [4],

$$y" + \frac{1}{x}y' + y = x^2 - x^3 - 9x + 4, 0 \leq x \leq 1 \tag{9}$$

and boundary conditions:

$$y(0) = 0, y(1) = 0 \tag{10}$$

Here, $P(x) = \frac{1}{x}$, $Q(x) = 1$, and $f(x) = x^2 - x^3 - 9x + 4$. At $x = 0$, $P(x)$ is not analytic. Therefore, the given equation is SBVP. The Eq. 9 is implemented according to the procedure outlined in section 3 in the following manner: The residual of Eq. 9 can be written as:

$$R(x) = xy" + y' + xy - (x^3 - x^4 - 9x^2 + 4x) \tag{11}$$

Subsequently, the appropriate weight function $w(x) = x(x-1)$ must be chosen for Bernoulli wavelet bases to satisfy the prescribed boundary conditions Eq. 10.

$$\begin{aligned}\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)\\ \psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \sqrt{3}(2x-1)x(1-x)\\ \psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \sqrt{5}(6x^2-6x+1)x(1-x)\end{aligned}$$

Assuming the trial solution of Eq. 9 for $k = 1$ and $m = 2$ is given by

$$y(x) = c_{1,0}\psi_{1,0}(x) + c_{1,1}\psi_{1,1}(x) + c_{1,2}\psi_{1,2}(x) \tag{12}$$

Then the Eq. 12 becomes

$$y(x) = c_{1,0}x(1-x) + c_{1,1}\sqrt{3}(2x-1)x(1-x) + c_{1,2}\sqrt{5}(6x^2-6x+1)x(1-x) \tag{13}$$

By differentiating Eq. 13 twice with respect to the variable and substituting the corresponding values into Eq. 11 then obtain the residual of Eq.9. The "weight functions" are the same to the basis functions.
Subsequently, employing the weighted Galerkin method, we examine the following

$$\int_0^1 \psi_{1,j}(x)R(x)dx = 0,\ j = 0, 1, 2 \tag{14}$$

For $j = 0, 1, 2$ in Eq.14,

$$\int_0^1 \psi_{1,0}(x)R(x)dx = 0 \tag{15}$$

$$\int_0^1 \psi_{1,1}(x)R(x)dx = 0 \tag{16}$$

$$\int_0^1 \psi_{1,2}(x)R(x)dx = 0 \tag{17}$$

Using Eq. 14, it is possible to derive a system of algebraic equations with unknown coefficients, specifically $c_{1,0}$, $c_{1,1}$, and $c_{1,2}$. The values of $c_{1,0} = 0.4995$, $c_{1,1} = 0.2889$, and $c_{1,2} = -0.0002$ can be obtained by solving this using Gaussian elimination or any other technique. These values are entered into Eq. 13 then the numerical solution for Eq. 9 is obtained. Table 1 compares the BWGM and the absolute errors and Table 2 compare for error norms L_2, L_∞ to compare with exact solutions, while the BWGM and the exact solution of Eq. 9 $y(x) = x^2 - x^3$ are shown in Figure 1.

Table 1. Comparison BWGM and absolute error with the exact solution for problem 4.1

x	FDM Sol.	Ref. [2] Sol.	BWGM Sol.	Exact Sol.	FDM error	Ref. [2] error	BWGM error
0.1	-0.014709	0.009677	0.008908	0.009000	2.37e-02	6.77e-04	9.20e-05
0.2	-0.013726	0.032675	0.031880	0.032000	4.57e-02	6.75e-04	1.20e-04
0.3	-0.002584	0.063354	0.062887	0.063000	6.56e-02	3.54e-04	1.10e-04
0.4	0.015387	0.095981	0.095987	0.096000	8.06e-02	1.90e-05	1.13e-05
0.5	0.036564	0.124731	0.124931	0.125000	8.84e-02	2.69e-04	6.90e-05
0.6	0.056572	0.143688	0.143946	0.144000	8.74e-02	3.12e-04	5.40e-05
0.7	0.070066	0.146841	0.146952	0.147000	7.69e-02	1.59e-04	4.80e-05
0.8	0.070568	0.128089	0.127955	0.128000	5.74e-02	8.90e-05	4.50e-05
0.9	0.050294	0.080862	0.080965	0.081000	3.07e-02	1.38e-04	3.50e-05

Problem 4.2 Next, consider another SBVP [4],

$$y" + \frac{8}{x}y' + xy = x^5 - x^4 + 44x - 30x, 0 \le x \le 1 \tag{18}$$

and boundary conditions:

$$y(0) = 0, y(1) = 0. \tag{19}$$

Here, $P(x) = \frac{8}{x}$, $Q(x) = x$, and $f(x) = x^5 - x^4 + 44x^2 - 30x$. At $x = 0$, $P(x)$ is not analytic. Therefore, the given equation is SBVP. Both in the previous problem and in section 3, the values of $c_{1,0} = -0.3334$, $c_{1,1} = -0.2886$, and $c_{1,2} = -0.0746$ are determined. By putting these numbers in Eq. 12 then reach the numerical solution for Eq. 18. Table 3 presents the comparison between BWGM and the

Table 2. Comparison for error norms L_2, L_∞ to compare with exact solutions for problem 4.1

Method	$L_2 norm$	$L_\infty norm$
FDM	1.98e-01	8.84e-02
Ref. [2]	1.10e-03	6.77e-04
BWGM	1.98e-04	1.20e-04

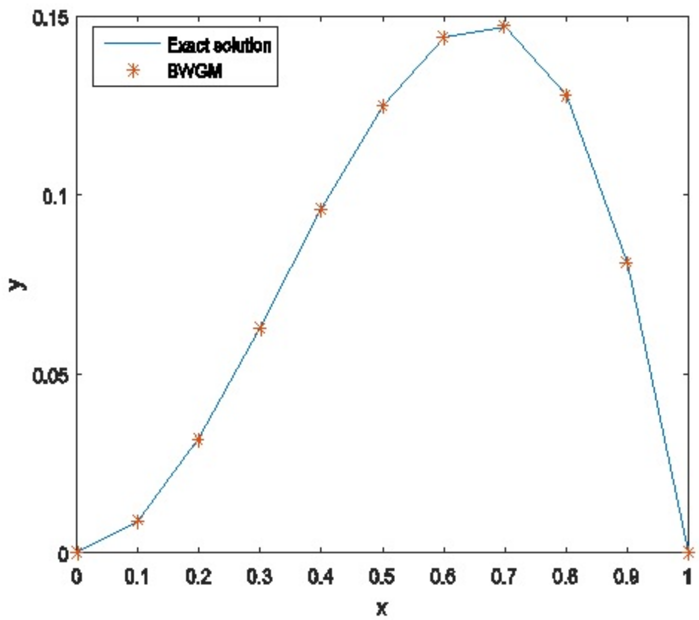

Figure 1. Comparison of BWGM with exact solution of the problem 4.1.

absolute errors and Table 4 compare for error norms L_2, L_∞ to compare with exact solutions, Figure 2 shows the contrast between BWGM with the exact solution of Eq. 18 $y(x) = x^4 - x^3$.

5. Conclusions

In this work, the Galerkin method based on Bernoulli wavelets is presented for the numerical solution of SBVPs. As can be seen from the tables and figures above,

- The suggested approach produces numerical solutions that are closer to the exact solution than those produced by the finite difference method (FDM) and other current methods (Ref [2]: Laguerre wavelets and Ref [4]-Fibonacci wavelets).

Table 3. Comparison of BWGM and absolute with the exact solution for problem 4.2.

x	Ref. [2] Sol.	Ref. [4] Sol.	BWGM	Exact Sol.	Ref. [2] error	Ref.[4] error	BWGM error
0.1	-0.000823	-0.000937	-0.000921	-0.000900	7.70e-05	3.70e-05	2.10e-05
0.2	-0.004844	-0.006426	-0.006424	-0.006400	1.56e-03	2.60e-05	2.40e-05
0.3	-0.016861	-0.018899	-0.018901	-0.018900	2.04e-03	1.00e-06	1.00e-06
0.4	-0.037304	-0.038381	-0.038407	-0.038400	1.10e-03	1.90e-05	7.00e-06
0.5	-0.062986	-0.062482	-0.062499	-0.062500	4.86e-04	1.80e-05	1.00e-06
0.6	-0.087854	-0.086406	-0.086395	-0.086400	1.45e-03	6.00e-06	5.00e-06
0.7	-0.103744	-0.102944	-0.102895	-0.102900	8.44e-04	4.40e-05	5.00e-06
0.8	-0.101131	-0.102477	-0.102399	-0.102400	1.27e-03	7.70e-05	1.00e-06
0.9	-0.069880	-0.072976	-0.072903	-0.072900	3.02e-03	7.60e-05	3.00e-06

Table 4. Comparison for error norms L_2, L_∞ to compare with exact solutions for problem 4.2

Method	$L_2 norm$	$L_\infty norm$
Ref [2]	4.60e-03	3.00e-03
Ref [4]	1.28e-04	7.70e-05
BWGM	3.36e-05	2.40e-05

- In contrast to FDM and the existing methods (Ref [2]: Laguerre wavelets and Ref. [4]- Fibonacci wavelets), the margin of error that results from this approach is significantly smaller.
- Also, error norms L_∞, L_2 of the proposed method is smaller as compared to FDM and the existing methods (Ref. [2]: Laguerre wavelets and Ref. [4]- Fibonacci wavelets).

Thus, the Galerkin method has shown great success in solving singular boundary value problems (SBVPs) through the use of Bernoulli wavelets.

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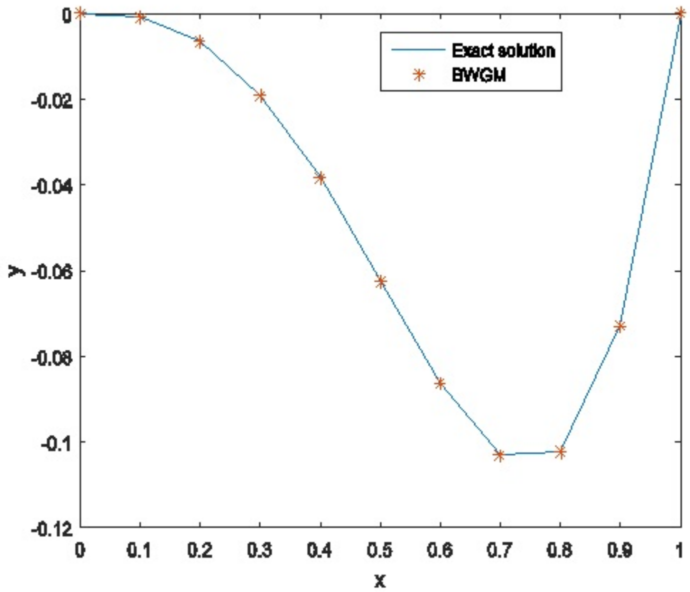

FIGURE 2. Comparison of BWGM with exact solution of the problem 4.2.

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