Computational Journal of Mathematical and Statistical Sciences
x(x), 0–0
DOI:10.21608/cjmss.2025.373673.1148
https://cjmss.journals.ekb.eg/

Research article

Statistical Inference for the Marshall-Olkin Extended Modified Inverse Rayleigh Distribution Under Adaptive Type-II Progressive Censoring

K. Hisham[1,*], **N. A. Mokhlis**[2], **M. M. M. Mansour**[3]

[1] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt; karimhesham@sci.asu.edu.eg.

[2] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo 11566, Egypt; nahed@sci.asu.edu.eg.

[3] Department of Basic Science, Faculty of Engineering, The British University in Egypt, Cairo 11837, Egypt; mahmoud.mansour@bue.edu.eg.

* **Correspondence:** karimhesham@sci.asu.edu.eg

Abstract: The *Marshall-Olkin extended modified inverse Rayleigh* (MOEMIR) distribution, a new extension of the *modified inverse Rayleigh* (MIR) distribution, is introduced as a member of a proposed *Marshall-Olkin extended general inverse exponential* (MOEGIE) family. This extension offers enhanced flexibility for modeling lifetime data. Statistical properties of the MOEGIE family are presented, and hence those of the MOEMIR distribution. Parameter estimation for the MOEMIR distribution is discussed under an adaptive Type-II progressive censoring scheme involving discrete uniformly distributed random removals. The parameters of the MOEMIR distribution are estimated using both maximum likelihood and Bayesian methods. The Bayesian estimation is refined under symmetric *squared error loss* (SEL) and asymmetric *linear exponential loss* (LINEX) functions, using a *Metropolis-Hastings* (M-H) sampling method of the *Markov chain Monte Carlo* (MCMC) technique. A simulation study is performed to highlight the obtained theoretical results. Finally, the utility of the MOEMIR distribution is demonstrated using a real-world dataset involving the remission times of patients with bladder cancer.

Keywords: Marshall-Olkin extended distribution; Modified inverse Rayleigh distribution; Maximum likelihood estimation; Bayesian estimation.

Mathematics Subject Classification: 60E05, 62E10, 62F10.

Received: 9 April 2025; Revised: 8 July 2025; Accepted: 11 July 2025; Online: 23 July 2025.

1. Introduction

Marshall and Olkin [11] proposed a versatile model that enhances conventional models by including extra parameters, thereby increasing the flexibility and accuracy of data fitting. This model is valuable for analyzing lifetime data with either increasing or decreasing hazard rates, making it well-suited for applications in reliability engineering, biostatistics, and survival analysis. For further details, see Ahmed et al. [1], Jose [8], and Santos et al. [15]. The *cumulative distribution function* (CDF) of a Marshall-Olkin extended distribution takes the form,

$$F(x;\delta) = \frac{G(x)}{\delta + \bar{\delta}G(x)}; -\infty < x < \infty, \delta > 0, \bar{\delta} = 1 - \delta, \tag{1.1}$$

where $G(x)$ is the CDF of the parent distribution. However, the effectiveness of such extensions depends on the choice of the baseline distribution. Among these, the *Inverse Rayleigh* (IR) distribution is a common statistical model applied in reliability engineering, life testing, and survival analysis. It is characterized by a decreasing hazard rate, which makes it suitable for systems that are most likely to fail early in their lifetimes due to some manufacturing errors—for example, mechanical components and systems in pumps and electric motors. Despite its applicability, the IR distribution may not always provide an adequate fit for real-world data, especially when the hazard rate does not follow a strictly decreasing pattern. Khan [9] introduced the *modified inverse Rayleigh* (MIR) distribution as a modification of the IR distribution to overcome its limitations. The CDF of the MIR distribution is expressed as:

$$G(x;\alpha,\zeta) = \exp(-\alpha(\frac{1}{x} + \zeta\frac{1}{x^2})), x > 0, \alpha, \zeta > 0. \tag{1.2}$$

However, the flexibility of the MIR distribution may be further improved by incorporating an additional parameter $\delta > 0$, via the Marshall-Olkin extended distribution in (1.1), obtaining the proposed distribution, which we call the MOEMIR distribution, with CDF

$$F(x;\alpha,\zeta,\delta) = \frac{\exp[-\alpha(\frac{1}{x} + \frac{\zeta}{x^2})]}{\delta + \bar{\delta}\exp[-\alpha(\frac{1}{x} + \frac{\zeta}{x^2})]}; x > 0,\ \alpha,\zeta,\delta > 0, \bar{\delta} = 1 - \delta. \tag{1.3}$$

Thus, our motivation behind proposing the MOEMIR distribution is enhancing the flexibility of the MIR distribution. The MOEMIR distribution is a member of the proposed general family of distributions, the MOEGIE family, which will be introduced in Section 2. Some important statistical properties of the MOEGIE family are studied and hence are applied to the MOEMIR distribution.
In real-world applications, particularly medical studies, data collection is often hindered by patient dropouts, or limited follow-up periods, or censoring. Traditional estimation methods may perform poorly under such conditions. To address this problem, we employ the adaptive Type-II progressive censoring scheme proposed by Ng et al. [14], which combines the advantages of Type-I censoring and Type-II progressive censoring, offering greater efficiency in parameter estimation. . Using this scheme, we derive *maximum likelihood estimates* (MLEs) and Bayes estimates for the MOEMIR parameters. The Bayesian approach is particularly valuable in medical studies, where prior knowledge from historical data can enhance inference, especially with small sample sizes. To validate the proposed model, we conduct a simulation study to evaluate its performance and demonstrate its effectiveness through a real-world application analyzing bladder cancer remission times. The results highlight the superiority

of the MOEMIR distribution over existing models.
The organization of the present paper is as follows: The MOEGIE family, of which MOEMIR is a member, is introduced in Section 2 along with some of its important statistical properties. In Section 3, the results obtained in Section 2 are applied to the MOEMIR distribution. Additionally, the MLEs and Bayes estimates of the MOEMIR parameters are examined under an adaptive Type-II progressive censoring scheme with random removals. A simulation study is performed to evaluate the performance of these estimation methods. Furthermore, the proposed distribution is demonstrated to provide a good fit to a real medical dataset, enabling the application of the derived theoretical results. Finally, the conclusions of the study are summarized in Section 4.

2. MOEGIE Family of Distributions

2.1. Definition of the MOEGIE family

A non-negative *random variable* (r.v.) X is considered a member of the MOEGIE(α, ζ, δ) family if its CDF is defined by

$$F(x;\alpha,\zeta,\delta) = \frac{\exp[-\alpha H(x;\zeta)]}{\delta + \bar{\delta}\exp[-\alpha H(x;\zeta)]}; x > 0,\ \alpha,\zeta,\delta > 0, \bar{\delta} = 1-\delta, \tag{2.1}$$

where $H(x;\boldsymbol{\zeta})$ is a continuous, monotone decreasing and differentiable function with $H(x;\boldsymbol{\zeta}) \to 0$ as $x \to \infty$ and $H(x;\boldsymbol{\zeta}) \to \infty$ as $x \to 0$.
Comparing Equations (1.1) and (2.1), we observe that $F(x;\alpha,\zeta,\delta)$ represents the Marshall-Olkin extended distribution of the parent variable $Y(\alpha,\zeta)$ with the corresponding CDF and *probability density function* (PDF) given by

$$G(x;\alpha,\zeta) = \exp\left[-\alpha H(x;\boldsymbol{\zeta})\right], x > 0, \tag{2.2}$$

and

$$g(x;\alpha,\zeta) = \alpha\exp[-\alpha H(x;\zeta)]h^*(x;\zeta), \tag{2.3}$$

respectively, where $h^*(x;\zeta) > 0$ is the derivative of $-H(x;\zeta)$. The distribution in (2.2) is said to have an inverse exponential form (see Mohie El-Din et al. [12]).
Clearly from Equation (1.3) that the MOEMIR distribution is a member of the MOEGIE family with $H(X;\zeta) = \frac{1}{x} + \zeta\frac{1}{x^2}$.

2.2. PDF of the MOEGIE family

The following theorem shows that the PDF of a MOEGIE(α,ζ,δ) is a countable linear combination of parent densities.

Theorem 2.1. *Let X be a member of MOEGIE(α,ζ,δ) family, as defined in (2.1) with parent r.v. having CDF and PDF given by (2.2) and (2.3), respectively. Then the density function of X can be expressed as a **countable linear combination** of parent densities defined by the following relationship:*

$$f(x;\alpha,\zeta,\delta) = \sum_{j=0}^{\infty} w_j g(x;(j+1)\alpha,\zeta), \tag{2.4}$$

where

$$w_j = \frac{1}{\delta}(1-\frac{1}{\delta})^j. \tag{2.5}$$

Proof. From (2.1), the PDF of the MOEGIE family is given by

$$f(x;\alpha,\zeta,\delta)=\frac{\delta\alpha h^*(x;\zeta)\exp[-\alpha H(x;\zeta)]}{\{1-\bar{\delta}(1-\exp[-\alpha H(x;\zeta)])\}^2};x>0,\alpha,\zeta,\delta>0,\bar{\delta}=1-\delta, \tag{2.6}$$

Now, we shall discuss the validity of (2.4) for all $\delta>0$.
(i)For $0<\delta<1$, it follows that $|\bar{\delta}(1-\exp[-\alpha H(x;\zeta)])|<1$. Applying the generalized binomial expansion:

$$(1-z)^{-a}=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k, \tag{2.7}$$

where $(a)_k=a(a+1)...(a+k-1)$, $a>0$, and $|z|<1$.
Thus, $f(x;\alpha,\zeta,\delta)$ in (2.6) can be rewritten as:

$$f(x;\alpha,\zeta,\delta)=\sum_{k=0}^{\infty}\delta\bar{\delta}^k(k+1)\alpha h^*(x;\zeta)\exp[-\alpha H(x;\zeta)](1-\exp[-\alpha H(x;\zeta)])^k,$$

Using the binomial expansion, we obtain:

$$\begin{aligned}f(x;\alpha,\zeta,\delta)&=\sum_{k=0}^{\infty}\delta\bar{\delta}^k(k+1)\alpha h^*(x;\zeta)\exp[-\alpha H(x;\zeta)]\sum_{j=0}^{k}\binom{k}{j}(-1)^j(\exp[-j\alpha H(x;\zeta)])\\&=\sum_{j=0}^{\infty}(-1)^j\delta\sum_{k=j}^{\infty}\bar{\delta}^k\binom{k+1}{j+1}(j+1)\alpha h^*(x;\zeta)\exp(-(j+1)\alpha H(x;\zeta)),\end{aligned}$$

Using (2.3),we have:

$$f(x;\alpha,\zeta,\delta)=\sum_{j=0}^{\infty}(-1)^j\delta\sum_{k=j}^{\infty}\binom{k+1}{j+1}\bar{\delta}^k g(x;(j+1)\alpha,\zeta). \tag{2.8}$$

It is easy to show that

$$\sum_{k=j}^{\infty}\binom{k+1}{j+1}\bar{\delta}^k=\bar{\delta}^j(1-\bar{\delta})^{-j-2}.$$

Substituting with the above result in (2.8), we get:

$$f(x;\alpha,\zeta,\delta)=\sum_{j=0}^{\infty}w_j g(x;(j+1)\alpha,\zeta),\ \text{for}\ 0<\delta<1. \tag{2.9}$$

(ii) For $\delta=1$, $f(x;\alpha,\zeta,\delta)$ reduces to the parent distribution. Thus, $f(x;\alpha,\zeta,\delta)$ can be expressed as:

$$f(x;\alpha,\zeta,\delta)=\sum_{j=0}^{\infty}w_j g(x;(j+1)\alpha,\zeta), \tag{2.10}$$

where

$$w_j=\begin{cases}1, & \text{for } j=0,\\ 0, & \text{for all } j=1,2,\ldots\end{cases}$$

(iii) For $\delta > 1$, $f(x;\alpha,\zeta,\delta)$ could be written as:

$$f(x;\alpha,\zeta,\delta) = \frac{\alpha h^*(x;\zeta)\exp[-\alpha H(x;\zeta)]}{\delta\{1-(1-\frac{1}{\delta})\exp[-\alpha H(x;\zeta)]\}^2}, \tag{2.11}$$

where $|(1-\frac{1}{\delta})\exp(-\alpha H(x;\zeta))| < 1$. Using the power series (2.7), the PDF (2.11) becomes:

$$f(x;\alpha,\zeta,\delta) = \sum_{j=0}^{\infty}\frac{(1-\frac{1}{\delta})^j}{\delta}(j+1)\alpha h^*(x;\zeta)\exp[-(j+1)\alpha H(x;\zeta)], \tag{2.12}$$

From (2.3) and (2.12), we obtain:

$$f(x;\alpha,\zeta,\delta) = \sum_{j=0}^{\infty} w_j g(x;(j+1)\alpha,\zeta), \text{ for } \delta > 1. \tag{2.13}$$

Combining (2.9), (2.10) and (2.13), we obtain (2.4). The proof is completed. □

Corollary 2.2. *If $\delta > 1$, then the density function $f(x;\alpha,\zeta,\delta)$ in (2.4) represents a* ***countable mixture*** *of the parent distributions, where the weights $\{w_j\}_{j=0}^{\infty}$ form a valid probability distribution, i.e., $0 < w_j < 1$ and $\sum_{j=0}^{\infty} w_j = 1$.*

Proof. For $\delta > 1$, we verify that $\{w_j\}_{j=0}^{\infty}$ forms a valid probability distribution.

First, observe that $0 < \frac{1}{\delta} < 1$ implies $0 < 1-\frac{1}{\delta} < 1$, and consequently each weight satisfies:

$$w_j = \frac{1}{\delta}\left(1-\frac{1}{\delta}\right)^j \in (0,1) \text{ for all } j \geq 0.$$

Second, the series of weights converges absolutely to 1 since it is a geometric series:

$$\sum_{j=0}^{\infty} w_j = \frac{1}{\delta}\sum_{j=0}^{\infty}\left(1-\frac{1}{\delta}\right)^j = \frac{1/\delta}{1-(1-1/\delta)} = 1.$$

Therefore, $f(x;\alpha,\zeta,\delta)$ represents a proper countable mixture of the parent densities. □

2.2.1. Moments, mean residual and mode of MOEGIE

Formula (2.4) allows us to calculate the mathematical quantities related to the MOEGIE distribution, such as moments and others, by leveraging the known quantities of the parent distribution.
Using (2.4), we could calculate the following quantities.

1. The *moment generating function* (MGF) of a MOEGIE family member, X, as

$$M_X(t) = \sum_{j=0}^{\infty} w_j M_{Y((j+1)\alpha,\zeta)}(t), \tag{2.14}$$

where $M_{Y((j+1)\alpha,\zeta)}(t)$ is the MGF (if it exists) of the r.v. $Y((j+1)\alpha,\zeta)$ with PDF (2.3).

2. The k^{th} ordinary moment of X as

$$E(X^k) = \sum_{j=0}^{\infty} w_j E(Y^k((j+1)\alpha, \zeta)). \tag{2.15}$$

3. The k^{th} incomplete moment of X as

$$I_k(z) = \sum_{j=0}^{\infty} w_j \int_0^z x^k g(x; (j+1)\alpha, \zeta) dx. \tag{2.16}$$

4. The mean residual of X as

$$m(t; \alpha, \zeta, \delta) = \frac{1}{\bar{F}(t; \alpha, \zeta, \delta)} \sum_{j=0}^{\infty} w_j \int_0^{\infty} y g(y + t; (j+1)\alpha, \zeta) dy. \tag{2.17}$$

5. The mode of X is determined by solving the following equation

$$\sum_{j=0}^{\infty} w_j (j+1)\alpha [\frac{\partial h^*(x;\zeta)}{\partial x} \exp(-(j+1)\alpha H(x;\zeta)) + (h^*(x;\zeta))^2 (j+1)\alpha \exp(-(j+1)\alpha H(x;\zeta)] = 0. \tag{2.18}$$

2.3. The hazard rate function (HRF) of MOEGIE family

The HRF considers the risk of instantaneous failure at some time t. The HRF of a MOEGIE family member, X, is

$$\lambda(x; \alpha, \zeta, \delta) = \frac{\lambda_G(x)}{1 - \bar{\delta}\bar{G}(x)} = \frac{\alpha h^*(x;\zeta) \exp[-\alpha H(x;\zeta)]}{(1 - \exp[-\alpha H(x;\zeta)])(1 - \bar{\delta}(1 - \exp[-\alpha H(x;\zeta)]))}; x > 0, \alpha, \delta > 0. \tag{2.19}$$

where $\lambda_G(x)$ is HRF of the parent distribution. From (2.19), we note that $\frac{\lambda(x;\delta)}{\lambda_G(x)}$ is increasing in x for $\delta > 1$, decreasing in x for $0 < \delta < 1$; and is equal to 1 for $\delta = 1$. It is obvious that the chosen parent distribution influences the formula of the HRF of the MOEGIE family.

2.4. The quantile function of the MOEGIE family

Since $F(x; \alpha, \zeta, \delta)$ in (2.1) is continuous and strictly increasing, the quantile function $X_u = F^{-1}(u)$: $0 < u < 1$ can be obtained by inverting (2.1), and expressed in terms of $H^{-1}(\cdot)$. Thus, we have

$$X_u = H^{-1}(\frac{1}{\alpha} \log \frac{1 - \bar{\delta}u}{\delta u}, \zeta). \tag{2.20}$$

Clearly, $\log \frac{1-\bar{\delta}u}{\delta u} > 0$ since $\frac{1-\bar{\delta}u}{\delta u} > 1$, for $\delta > 0$, and $0 < u < 1$.
Setting $u = 0.5$ in (2.20) gives the median.
Furthermore, Bowley's skewness [6] and Moors' kurtosis [13] can be measured using (2.20) through the following equations

$$SK = \frac{X_{\frac{3}{4}} + X_{\frac{1}{4}} - 2X_{\frac{1}{2}}}{X_{\frac{3}{4}} - X_{\frac{1}{4}}}, \tag{2.21}$$

and

$$K = \frac{X_{\frac{7}{8}} - X_{\frac{5}{8}} + X_{\frac{3}{8}} - X_{\frac{1}{8}}}{X_{\frac{6}{8}} - X_{\frac{2}{8}}}, \tag{2.22}$$

where SK and K are the skewness and the kurtosis respectively.
These measures of skewness and kurtosis are less susceptible to outliers than moment-based measures and can be computed even for distributions without moments.

3. MOEMIR Distribution

The MIR r.v. represents the maximum of two independent r.v.s: an inverse exponential variable with parameter α and an IR variable with parameter $\alpha\zeta$. The CDF of MIR is given by (1.2), and its PDF is given by

$$g(x;\alpha,\zeta) = \frac{\alpha}{x^2}(1 + \frac{2\zeta}{x})\exp(-\alpha(\frac{1}{x} + \zeta\frac{1}{x^2})), x > 0, \alpha, \zeta > 0. \tag{3.1}$$

Clearly, from (1.2), (2.2), (2.3), and (3.1), we see that MIR distribution has an inverse exponential form with

$$H(X;\zeta) = \frac{1}{x} + \zeta\frac{1}{x^2}, \tag{3.2}$$

and

$$h^*(X;\zeta) = \frac{1}{x^3}(x + 2\zeta). \tag{3.3}$$

Consequently, the Marshall-Olkin extended distribution of the MIR (MOEMIR) given in (1.3) is a member of the MOEGIE family (2.1) with $H(X;\zeta)$ in (3.2). Substituting (3.2) and (3.3) into (2.6), the PDF of the MOEMIR is

$$f(x;\alpha,\zeta,\delta) = \frac{\alpha\delta(x + 2\zeta)\exp(-\alpha(\frac{1}{x} + \frac{\zeta}{x^2}))}{x^3\{\delta + \bar{\delta}\exp[-\alpha(\frac{1}{x} + \frac{\zeta}{x^2})]\}^2} : x > 0,\ \alpha, \zeta, \delta > 0. \tag{3.4}$$

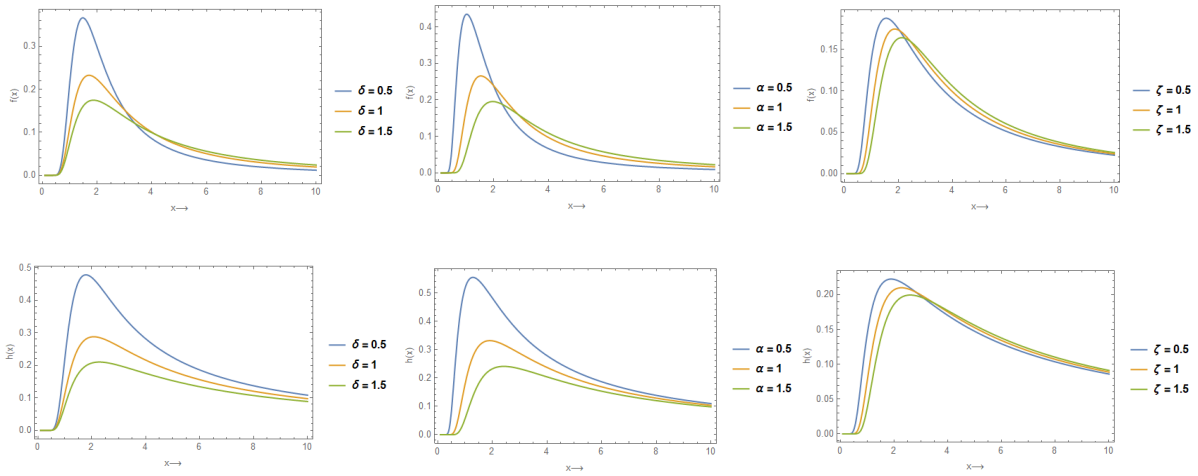
If $\delta = 1$, the special case of the MIR distribution is obtained. In addition, if $\zeta = 0$, the Marshall-Olkin extended inverse exponential distribution is obtained.
The survival function and HRF of the MOEMIR distribution are given respectively by

$$S(t) = \frac{\delta(1 - \exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})])}{\delta + \bar{\delta}\exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})]} : t \geq 0,\ \alpha, \zeta, \delta > 0, \tag{3.5}$$

and

$$\lambda(t) = \frac{\alpha(t + 2\zeta)\exp(-\alpha(\frac{1}{t} + \frac{\zeta}{t^2}))}{t^3(\delta + \bar{\delta}\exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})])(1 - \exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})])} : t \geq 0,\ \alpha, \zeta, \delta > 0. \tag{3.6}$$

Figure 1 displays various patterns of the PDF and HRF for the MOEMIR distribution, where cases 1, 2, and 3 correspond to $(\alpha = 2, \beta = 1)$, $(\beta = 2, \delta = 1.5)$, and $(\alpha = 2, \delta = 1.5)$, respectively. As depicted, the MOEMIR is positively skewed and unimodal, while its HRF follows an upside-down bathtub curve. Additionally, the impact of ζ on the shape of the distribution is relatively minor compared to that of α and δ. The graphs clearly depict a heavy-tailed distribution with long tails extending far from the peak, indicating a higher probability of extreme values. Consequently, the MOEMIR distribution is



(a) Case 1 **(b)** Case 2 **(c)** Case 3

Figure 1. Plots of the MOEMIR PDF (top row) and HRF (bottom row) for different parameter values.

appropriate for modeling a variety of practical fields. In reliability engineering, for example, it can represent the lifetimes of systems subject to both wear and sudden shocks. While in survival analysis, it can represent patient survival times and remission data for clinical use.

3.1. Statistical properties of MOEMIR distribution

3.1.1. Mean

If $Y((j+1)\alpha, \zeta)$ is an MIR distributed random variable with PDF given by (3.1) with parameters $(j+1)\alpha$ and ζ, it is easy to show that

$$E(Y((j+1)\alpha,\zeta)) = \sum_{m=0}^{\infty} \frac{(-1)^m((j+1)\alpha)^{1-m}\zeta^m}{m!}[\Gamma(2m) + \frac{2\zeta}{(j+1)\alpha}\Gamma(2m+1)]. \tag{3.7}$$

Setting $k = 1$, in Equation (2.15) and substituting with Equation (3.7), the mean of the MOEMIR distribution can be expressed as:

$$E(X) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_j \frac{(-1)^m((j+1)\alpha)^{1-m}\zeta^m}{m!}[\Gamma(2m) + \frac{2\zeta}{(j+1)\alpha}\Gamma(2m+1)], \tag{3.8}$$

where w_j is defined in (2.5) and $\Gamma(.)$ is the gamma function. It should be noted that the MOEMIR distribution does not possess finite moments of order greater than 1, emphasizing its heavy-tailed nature.

3.1.2. Mean residual life

Using (2.17), the mean residual life is given by

$$m(t;\alpha,\zeta,\delta) = \frac{\delta + \bar{\delta}\exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})]}{\delta^2(1-\exp[-\alpha(\frac{1}{t} + \frac{\zeta}{t^2})])}\sum_{j=0}^{\infty}(1-\frac{1}{\delta})^j A_j, \tag{3.9}$$

where

$$A_j = \sum_{m=0}^{\infty} \frac{(-1)^m((j+1)\alpha)^{-m}\zeta^m}{m!}[(j+1)\alpha\Gamma(2m) + (2\zeta - t)\Gamma(2m+1) - \frac{2\zeta t}{(j+1)\alpha}\Gamma(2m+2)].$$

In practical computations, the infinite sums in Equations (3.8) and (3.9) are truncated using a convergence criterion. Specifically, the summation is terminated when the absolute difference between consecutive partial sums falls below a predefined tolerance level of 10^{-6}. Numerical validation demonstrates that this criterion achieves relative errors under 0.1% with summation limits of $j = 0$ to 50 for the outer series and $m = 0$ to 20 for the inner series, ensuring robust accuracy while optimizing computational efficiency.

3.1.3. The quantile function

Using (2.20), the quantile function of the MOEMIR distribution can be obtained as follows:

$$X_u = \frac{2\zeta}{\sqrt{1 + \frac{4\zeta}{\alpha}\log(\frac{1-\bar{\delta}u}{\delta u})} - 1}, 0 < u < 1. \tag{3.10}$$

Clearly, the argument of the square root in the expression above is always positive since $\frac{1-\bar{\delta}u}{\delta u} > 1$, and hence $\log \frac{1-\bar{\delta}u}{\delta u} > 0$.
Substituting $u = 1/2$ in (3.10), the median of MOEMIR distribution is obtained as follows:

$$median = \frac{2\zeta}{\sqrt{1 + \frac{4\zeta}{\alpha}\log(1 + \frac{1}{\delta})} - 1}. \tag{3.11}$$

Using (3.10) in (2.21) and (2.22), the skewness and kurtosis are obtained respectively.

Table 1. Some statistical measures of MOEMIR distribution for different values of δ ($\alpha = 0.01, \zeta = 0.5$)

δ	mode	skewness	kurtosis	mean	median
0.01	0.032	0.1765	1.4588	0.0735	0.034
0.2	0.045	0.2963	1.6358	0.7985	0.0557
0.7	0.0557	0.3333	1.6725	2.651	0.0809
1.0	0.06	0.3372	1.679	3.9355	0.0925
1.2	0.0625	0.3409	1.6817	4.7053	0.0994
2.7	0.0776	0.3486	1.7171	10.4587	0.1428
5.0	0.0956	0.4446	2.0541	18.3546	0.1953

Using (2.18), (2.21), (2.22), (3.8) and (3.11), we numerically calculated the mode, skewness, kurtosis, mean and median, respectively, for the parameters $\alpha = 0.01$, $\zeta = 0.5$ and different values of δ, as shown in Table 1. From Table 1, it is evident that the MOEMIR distribution is positively skewed and leptokurtic. Additionally, as δ increases, the mode, skewness, kurtosis, mean and median also show an increasing trend.

3.2. Parameters estimation using an adaptive Type-II progressive censoring scheme

The adaptive Type-II progressive scheme, proposed by Ng et al. [14], is a mixture of Type-I censoring and progressive Type-II censoring. This approach enhances flexibility in life-testing experiments by adjusting the termination criteria based on the experimental timeline and observed failures. In a traditional Type-I censoring setup, the experiment ends at a fixed time, irrespective of the number of failures. In contrast, a progressive Type-II censoring scheme begins with n units under test and an effective sample size $m < n$ fixed in advance. When the first failure occurs at time $X_{1:m:n}$, R_1 units are randomly withdrawn from the remaining $n-1$ units. The process is repeated at the second failure at time $X_{2:m:n}$, where R_2 units are removed from the remaining $n-2-R_1$ units. This continues until the m^{th} failure at time $X_{m:m:n}$ occurs, after which all remaining $n-m-R_1-\cdots-R_{m-1}$ units are withdrawn, concluding the experiment. The values of R_i, $i = 1, 2, \ldots, m-1$ may be predetermined or randomly assigned. For a detailed discussion on progressive censoring methods, see Balakrishnan [3] and Balakrishnan with Aggarwala [4], which provide comprehensive reviews.

The adaptive Type-II progressive censoring scheme extends these two concepts by introducing flexibility in the termination criteria. The experimenter specifies an ideal total test duration T. If the m^{th} progressively censored failure occurs before T, i.e., if $X_{m:m:n} < T$, the experiment concludes at time $X_{m:m:n}$. However, if the test reaches time T before observing m failures, the remaining removals are adjusted to end the experiment promptly. Specifically, if j failures occur before T, i.e., if $X_{j:m:n} < T < X_{j+1:m:n}$, where $j = 0, 1, ..., m$, then $R_{j+1} = ... = R_{m-1} = 0$, and $R_m = n - m - \sum_{i=1}^{j} R_i$ are set to terminate the experiment as quickly as possible. More paper used this scheme as [2, 5].

In extreme cases, if $T \to \infty$, time becomes irrelevant, reducing the scheme to a standard progressive Type-II censoring. Conversely, if $T = 0$, the experiment must conclude as quickly as possible, resulting in conventional Type-II censoring.

Assume the failure times of the n test items follow a continuous population with PDF $f(x;\boldsymbol{\theta})$ and CDF $F(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of parameters and $x > 0$. For $J = j$, the likelihood function of the observed failure times $x_{1:m:n}, ..., x_{m:m:n}$ will be

$$l(\boldsymbol{\theta}|J=j) = d_j \prod_{i=1}^{m} f(x_{i:m:n};\boldsymbol{\theta}) \prod_{i=1}^{j} [1-F(x_{i:m:n};\boldsymbol{\theta})]^{R_i} [1-F(x_{m:m:n};\boldsymbol{\theta})]^{n-m-\sum_{i=1}^{j} R_i}, \tag{3.12}$$

where

$$d_j = \prod_{i=1}^{m} [n-i+1-\sum_{k=1}^{\min\{i-1,j\}} R_k], x_{1:m:n} < x_{2:m:n} < ... < x_{m:m:n}. \tag{3.13}$$

It is to be noticed that there is a misprint in Ng et al. [14] in the equation of likelihood function where the max there should be a min as given here (the letter of communication between us and Prof. Ng is attached as per request).

3.2.1. Maximum likelihood estimation

Let $\underline{X}$=$(X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ represent an adaptive Type-II progressive censored sample from a MOEMIR population with discrete uniformly distributed random removals $R_i, i = 1, ..., m-1$, and $R_m = n - m - \sum_{i=1}^{m-1} R_i$. Thus, we assume that

$$P(R_1 = r_1) = \frac{1}{n-m+1}; 0 \le r_1 \le n-m, \tag{3.14}$$

$$P(R_i = r_i|R_{i-1} = r_{i-1}, .., R_1 = r_1) = \frac{1}{n - m - \sum_{j=1}^{i-1} r_j + 1}; 0 \leq r_i \leq n - m - \sum_{j=1}^{i-1} r_j, i = 2, ..., m - 1. \quad (3.15)$$

Using (3.12), the likelihood function is given by

$$l(\boldsymbol{\theta}|J = j) = l_1(\underline{x};\boldsymbol{\theta})P(R_1 = r_1, R_2 = r_2, .., R_m = r_m) = l_1(\underline{x};\boldsymbol{\theta})(\frac{1}{n - m + 1}\prod_{i=1}^{m-1}\frac{1}{n - m - \sum_{j=1}^{i-1} r_j + 1}), \quad (3.16)$$

where

$$l_1(\underline{x};\boldsymbol{\theta}) = \frac{d_j\alpha^m\delta^n\prod_{i=1}^{m}\frac{x_i+2\zeta}{x_i^3}\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2}))\prod_{i=1}^{j}(1 - \exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2})))^{r_i}(1 - \exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j} r_i}}{\prod_{i=1}^{m}(\delta + \bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^2\prod_{i=1}^{j}(\delta + \bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^{r_i}(\delta + \bar{\delta}\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j} r_i}}. \quad (3.17)$$

and for simplicity x_i is used instead of $x_{i:m:n}$.
Clearly, the joint PDF of R_i's, $i = 1, ..., m$ is free of unknown parameters. Thus, the log-likelihood function for the vector of parameters $\boldsymbol{\theta}$=$(\alpha, \zeta, \delta)^T$ is given by

$$\begin{aligned} L_1(\underline{x};\boldsymbol{\theta}) = const + n\log(\delta) + m\log(\alpha) + \sum_{i=1}^{m}\log(\frac{x_i + 2\zeta}{x_i^3}) - 2\sum_{i=1}^{m}\log(\delta + \bar{\delta}\exp(-\alpha\frac{x_i + \zeta}{x_i^2})) \\ - \alpha\sum_{i=1}^{m}\log(\frac{x_i + \zeta}{x_i^2}) + \sum_{i=1}^{j} r_i\log(1 - \exp(-\alpha\frac{x_i + \zeta}{x_i^2})) - \sum_{i=1}^{j} r_i\log(\delta + \bar{\delta}\exp(-\alpha\frac{x_i + \zeta}{x_i^2})) \\ + (n - m - \sum_{i=1}^{j} r_i)\log(1 - \exp(-\alpha\frac{x_m + \zeta}{x_m^2})) + (n - m - \sum_{i=1}^{j} r_i)\log(\delta + \bar{\delta}\exp(-\alpha\frac{x_m + \zeta}{x_m^2})). \end{aligned} \quad (3.18)$$

The MLEs of α, ζ, and δ are derived by taking partial derivatives of Equation (3.18) with respect to each parameter, equating them to zero, and solving the resulting nonlinear system of equations numerically, by applying the Newton-Raphson or the bisection method.

3.2.2. Bayesian estimation

The unknown parameters α, ζ, and δ are estimated via a Bayesian approach, utilizing an adaptive Type-II progressive censored sample with discrete uniformly distributed random removals. The estimation is performed under both the SEL and LINEX functions, incorporating informative priors.
The SEL function, a widely used symmetric loss measure, is defined for an estimator $\hat{\rho}$ of parameter ρ as:

$$L(\rho;\hat{\rho}) = (\hat{\rho} - \rho)^2. \quad (3.19)$$

Under SEL, the Bayes estimator of any function $g(\boldsymbol{\theta})$ of the vector parameter $\boldsymbol{\theta} = (\alpha, \zeta, \delta)^T$ is given by

$$\hat{g}_{BS}(\boldsymbol{\theta}|data) = E_{\boldsymbol{\theta}|data}(g(\boldsymbol{\theta})). \quad (3.20)$$

where the expectation is taken over the posterior distribution of $\boldsymbol{\theta}$.
In contrast, the LINEX loss function $L(\triangle)$ for a parameter ρ is defined as:

$$L(\triangle) = e^{c\triangle} - c\triangle - 1, c \neq 0 \; and \; \triangle = \hat{\rho} - \rho. \tag{3.21}$$

The corresponding Bayes estimator under the LINEX loss function of any function $g(\boldsymbol{\theta})$ of the vector parameter $\boldsymbol{\theta}$ is

$$\hat{g}_{BL}(\boldsymbol{\theta}|data) = -\frac{1}{c}\log[E_{\boldsymbol{\theta}|data}(\exp(-cg(\boldsymbol{\theta})))], c \neq 0. \tag{3.22}$$

The parameters α, ζ, and δ are assigned independent gamma prior distributions. The gamma distribution is chosen as the prior distribution due to its natural suitability for positive-valued parameters, its flexibility in encoding prior beliefs through shape and rate parameters, and its computational tractability for MCMC sampling. Formally, the priors are specified as:

$$\pi_1(\alpha) \propto (\alpha)^{a_1-1}\exp(-b_1\alpha),\ \pi_2(\zeta) \propto (\zeta)^{a_2-1}\exp(-b_2\zeta) \text{ and } \pi_3(\delta) \propto (\delta)^{a_3-1}\exp(-b_3\delta);\ \alpha,\zeta,\delta > 0. \tag{3.23}$$

Combining the likelihood function with the priors yields the joint posterior distribution of the vector parameter $\boldsymbol{\theta}$ denoted by $\pi^*(\boldsymbol{\theta}|\text{data})$ as

$$\pi^*(\boldsymbol{\theta}|\text{data}) \propto$$

$$\frac{\pi_1(\alpha)\pi_2(\zeta)\pi_3(\delta)\alpha^m\delta^n\prod_{i=1}^{m}\frac{x_i+2\zeta}{x_i^3}\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2}))\prod_{i=1}^{j}(1-\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2})))^{R_i}(1-\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}R_i}}{\prod_{i=1}^{m}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^2\prod_{i=1}^{j}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^{R_i}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}R_i}}. \tag{3.24}$$

The posterior conditionals of α, ζ and δ are, respectively,

$$\pi_1^*(\alpha \mid \zeta,\delta,data) \propto \frac{\pi_1(\alpha)\alpha^m\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2}))\prod_{i=1}^{j}(1-\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2})))^{r_i}(1-\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}r_i}}{\prod_{i=1}^{m}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^2\prod_{i=1}^{j}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^{r_i}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}r_i}}, \tag{3.25}$$

$$\pi_2^*(\zeta \mid \alpha,\delta,data) \propto \frac{\pi_2(\zeta)\prod_{i=1}^{m}\frac{x_i+2\zeta}{x_i^3}\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2}))\prod_{i=1}^{j}(1-\exp(-\alpha\sum_{i=1}^{m}(\frac{x_i+\zeta}{x_i^2})))^{r_i}(1-\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}r_i}}{\prod_{i=1}^{m}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^2\prod_{i=1}^{j}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^{r_i}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}r_i}}, \tag{3.26}$$

and

$$\pi_3^*(\delta \mid \alpha,\zeta,data) \propto \frac{\pi_3(\delta)\delta^n}{\prod_{i=1}^{m}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^2\prod_{i=1}^{j}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_i+\zeta}{x_i^2})))^{r_i}(\delta+\bar{\delta}\exp(-\alpha(\frac{x_m+\zeta}{x_m^2})))^{n-m-\sum_{i=1}^{j}r_i}}. \tag{3.27}$$

As we see, the conditional posterior distributions of α, ζ, and δ are not standard distributions. As a result, the M-H Algorithm is applied for MCMC implementation. The following steps show how the M-H algorithm is applied to obtain posterior samples and subsequently get the Bayes estimates, as suggested by Tierney [16].

1. Start with initial guess $\varphi^{(0)} = \alpha^{(0)}, \zeta^{(0)}, \delta^{(0)}$.
2. Put $j = 1$.
3. For each parameter, generate proposal value φ^* (where φ can be α, ζ or δ) from a proposal density

$$q_i(\varphi^{(j-1)}, \varphi^{(j)}) = p(\varphi^{(j-1)} \to \varphi^{(j)}),$$

   where $p(\varphi^{(j-1)} \to \varphi^{(j)})$ is the probability of obtaining a value $\varphi^{(j)}$ from a previous value $\varphi^{(j-1)}$. Here, $i = 1$ for α, $i = 2$ for ζ and $i = 3$ for δ.
4. Evaluate the acceptance probabilities for each parameter as follows:

$$\nu_\alpha = \min[1, \frac{\pi_1^*(\alpha^*|\zeta^{(j-1)}, \delta^{(j-1)}, data)q_1(\alpha^*, \alpha^{(j-1)})}{\pi_1^*(\alpha^{(j-1)}|\zeta^{(j-1)}, \delta^{(j-1)}, data)q_1(\alpha^{(j-1)}, \alpha^*)}], \tag{3.28}$$

$$\nu_\zeta = \min[1, \frac{\pi_2^*(\zeta^*|\alpha^{(j)}, \delta^{(j-1)}, data)q_2(\zeta^*, \zeta^{(j-1)})}{\pi_2^*(\zeta^{(j-1)}|\alpha^{(j)}, \delta^{(j-1)}, data)q_2(\zeta^{(j-1)}, \zeta^*)}], \tag{3.29}$$

   and

$$\nu_\delta = \min[1, \frac{\pi_3^*(\delta^*|\alpha^{(j)}, \zeta^{(j)}, data)q_3(\delta^*, \delta^{(j-1)})}{\pi_3^*(\delta^{(j-1)}|\alpha^{(j)}, \zeta^{(j)}, data)q_3(\delta^{(j-1)}, \delta^*)}]. \tag{3.30}$$

   Note that when the proposal density $q_i(\varphi^*, \varphi^{(j-1)})$ is symmetric, the acceptance probability simplifies to:

$$\nu_\varphi = \min[1, \frac{\pi_i^*(\varphi^*)}{\pi_i^*(\varphi^{(j-1)})}]. \tag{3.31}$$

5. For each parameter $\varphi = \alpha, \zeta, \delta$, draw $u_\varphi \sim$ Uniform$(0, 1)$ and perform the acceptance/rejection step:

$$\phi^{(j)} = \begin{cases} \varphi^* & \text{if } u_\varphi < \nu_\varphi, \\ \varphi^{(j-1)} & \text{otherwise,} \end{cases}$$

   where ν_φ is the acceptance probability for φ.
6. Increment $j = j + 1$ and repeat Steps (3)-(5) for N iterations to obtain $\alpha^{(i)}, \zeta^{(i)}, \delta^{(i)}$, $i = 1, 2, ...N$.
7. Compute the approximate Bayes estimates under SEL and LINEX loss functions, respectively, as:

$$\hat{\varphi}_{\mathbf{BS}} = \frac{1}{N - M} \sum_{j=M+1}^{N} \varphi^{(j)}, \tag{3.32}$$

   and

$$\hat{\varphi}_{\mathbf{BL}} = \frac{-1}{c} \log \left(\frac{1}{N - M} \sum_{j=M+1}^{N} \exp(-c\varphi^{(j)}) \right). \tag{3.33}$$

   where M denotes burn-in period, and φ represents either α, ζ or δ.

3.3. Simulation study

A simulation study is conducted to evaluate the performance of the estimators derived earlier. 1,000 adaptive Type-II progressive censoring samples of varying sizes $n = 40, 80$ are generated from the MOEMIR distribution, with random discrete uniformly distributed removals. The true parameters are set as $\alpha = 0.05$, $\zeta = 0.3$, $\delta = 0.5$, and $T = 5$. In this study, failure information percentages ($\frac{m}{n} \times 100\%$) are considered at 50% and 75%. For the Bayesian estimation, independent gamma priors are assigned to the parameters α, ζ, and δ, as specified in (3.23). In most practical situations, the available information is usually very limited, so the hyperparameters are selected to reflect weak prior information, ensuring the posterior distribution would be dominated by the data. Specifically, we chose $a_1 = 0.1, b_1 = 0.1$ for α; $a_2 = 0.2, b_2 = 0.2$ for ζ; and $a_3 = 0.3, b_3 = 0.3$ for δ. This specification ensures that the posterior distribution is data-dominated, aligning with the principle of weakly informative priors, which promotes robustness while minimizing undue influence on the estimates, see Gelman et al. [7].

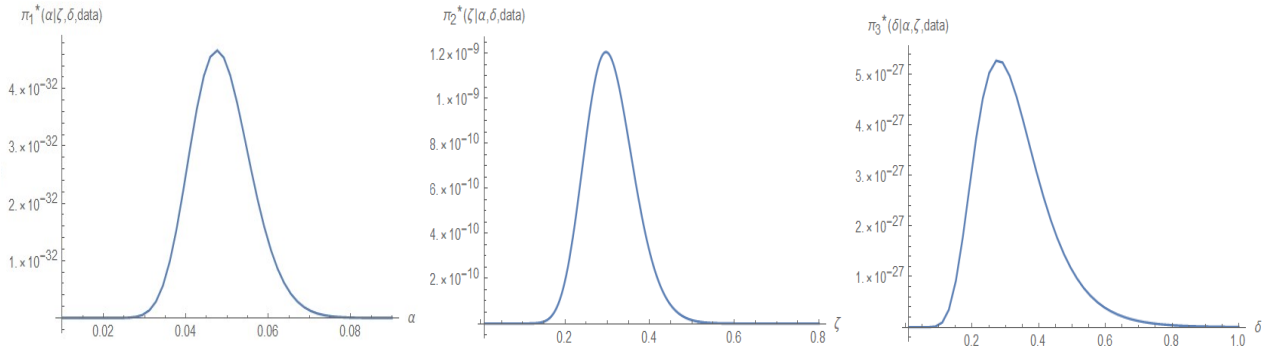


Figure 2. Samples of graphs of the conditional posterior density function of α, ζ, and δ

As illustrated in Figure 2, the conditional posterior distributions of α, ζ, and δ are approximately bell-shaped and symmetric, indicating that the proposal value φ^* can be appropriately sampled from a normal distribution: $\varphi^* \sim N(\varphi^{(j-1)}, \sigma^2_\varphi)$, where σ^2_φ represents proposal variances derived from the inverse observed information matrix.

The study primarily compares the performance of MLEs and Bayesian estimators based on the SEL and LINEX loss functions, focusing on bias and *mean squared error* (MSE). Table 2 reveals that the performance of Bayesian estimates is clearly influenced by the choice of loss function. Specifically, LINEX with ($c = 5$) outperforms SEL with respect to both bias and MSE, whereas LINEX with ($c = -5$) may sometimes underperform relative to SEL. In many cases, Bayesian estimates using the LINEX ($c = 5$) often exhibit lower biases than the MLEs, indicating greater accuracy in estimating the true parameter values. Furthermore, as both n and m increase, all estimates tend to exhibit reduced bias, suggesting that larger sample sizes improve the accuracy of all methods, in line with statistical estimation principles. Overall, Bayesian estimates tend to outperform MLEs for smaller sample sizes, providing better accuracy and precision. However, as sample sizes grow, MLEs become more competitive, with their performance converging to that of Bayesian methods.

Table 2. Average estimates of parameters (bias and MSE in parentheses): $T = 5$

n	m		MLE	SEL	LINEX	
					$c = -5$	$c = 5$
40	40	$\hat{\alpha}$	0.0789	0.0797	0.0799	0.0795
			(0.0289, 0.005)	(0.0297, 0.0053)	(0.0299, 0.0053)	(0.0295, 0.0052)
		$\hat{\zeta}$	0.3805	0.3936	0.4414	0.3523
			(0.0805, 0.0936)	(0.0936, 0.158)	(0.1414, 0.3097)	(0.0523, 0.0963)
		$\hat{\delta}$	0.4799	0.4847	0.5	0.4699
			(-0.0201, 0.1187)	(-0.0153, 0.1414)	(0.0, 0.19)	(-0.0301, 0.118)
	30	$\hat{\alpha}$	0.0767	0.0774	0.0776	0.0772
			(0.0267, 0.0044)	(0.0274, 0.0047)	(0.0276, 0.0048)	(0.0272, 0.0047)
		$\hat{\zeta}$	0.39	0.4591	0.5408	0.3771
			(0.09, 0.1032)	(0.1591, 0.4171)	(0.2408, 0.8299)	(0.0771, 0.1312)
		$\hat{\delta}$	0.4905	0.5127	0.5366	0.4884
			(-0.0095, 0.1251)	(0.0127, 0.2264)	(0.0366, 0.3386)	(-0.0116, 0.1442)
	20	$\hat{\alpha}$	0.0698	0.0715	0.0725	0.0708
			(0.0198, 0.0036)	(0.0215, 0.0045)	(0.0225, 0.0056)	(0.0208, 0.004)
		$\hat{\zeta}$	0.4133	0.5527	0.7203	0.395
			(0.1133, 0.1055)	(0.2527, 3.1334)	(0.4203, 9.1702)	(0.095, 0.1485)
		$\hat{\delta}$	0.5233	0.5347	0.5844	0.4909
			(0.0233, 0.172)	(0.0347, 0.2948)	(0.0844, 0.5969)	(-0.0091, 0.1714)
80	80	$\hat{\alpha}$	0.0755	0.0759	0.0759	0.0758
			(0.0255, 0.0039)	(0.0259, 0.004)	(0.0259, 0.0041)	(0.0258, 0.004)
		$\hat{\zeta}$	0.3473	0.3554	0.376	0.3355
			(0.0473, 0.0754)	(0.0554, 0.1112)	(0.076, 0.1612)	(0.0355, 0.0758)
		$\hat{\delta}$	0.4715	0.4722	0.4759	0.4685
			(-0.0285, 0.0769)	(-0.0278, 0.0785)	(-0.0241, 0.0823)	(-0.0315, 0.076)
	60	$\hat{\alpha}$	0.0778	0.0779	0.0779	0.0778
			(0.0278, 0.0043)	(0.0279, 0.0044)	(0.0279, 0.0044)	(0.0278, 0.0044)
		$\hat{\zeta}$	0.3551	0.3786	0.4162	0.342
			(0.0551, 0.0831)	(0.0786, 0.1794)	(0.1162, 0.3089)	(0.042, 0.0947)
		$\hat{\delta}$	0.4583	0.4651	0.4721	0.4592
			(-0.0417, 0.0764)	(-0.0349, 0.0866)	(-0.0279, 0.1016)	(-0.0408, 0.0801)
	40	$\hat{\alpha}$	0.0767	0.0782	0.079	0.0774
			(0.0267, 0.0048)	(0.0282, 0.0061)	(0.029, 0.0081)	(0.0274, 0.0051)
		$\hat{\zeta}$	0.4082	0.4634	0.5247	0.4062
			(0.1082, 0.1225)	(0.1634, 0.3606)	(0.2247, 0.6392)	(0.1062, 0.1934)
		$\hat{\delta}$	0.485	0.5154	0.5525	0.4799
			(-0.015, 0.1064)	(0.0154, 0.8157)	(0.0525, 2.9488)	(-0.0201, 0.1099)

3.4. Real data illustration

We analyze a dataset of remission times (in months) for 128 bladder cancer patients from Lee and Wang [10]. The dataset spans disease stages I-IV, with remission times ranging from 0.08 to 79.05 months.

Table 3. Summary statistics of the dataset

Statistic	Mean	Median	Standard deviation	Skewness	Kurtosis
Value (months)	9.37	6.93	10.50	2.84	11.92

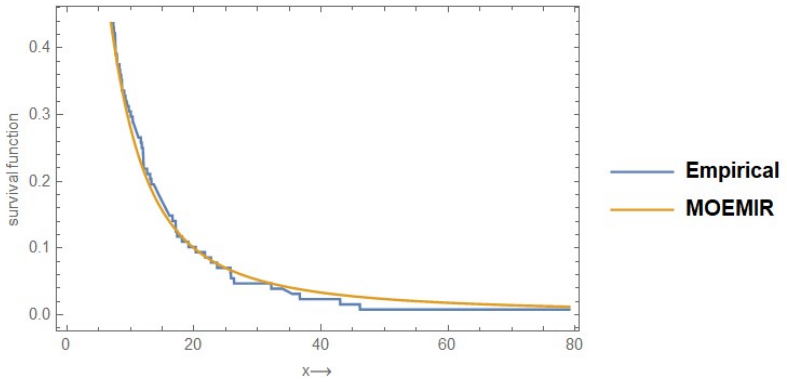


Figure 3. Bladder cancer data fitted to MOEMIR

Table 3 presents a descriptive statistical summary of the data. The results indicate that the data reveal right skewness. So, the MOEMIR distribution is suitable for modeling this data. Figure 3 plots the fitted survival functions, visually confirming close alignment of the MOEMIR distribution with the empirical distribution. To demonstrate the fitness of the MOEMIR distribution, we compare it with five alternative distributions: *Marshall-Olkin Extended inverse Rayleigh* (MOEIR), *Marshall-Olkin Extended Exponential* (MOEE), *Marshall-Olkin Generalized Weibull* (MOGW), MIR, and IR, all of which are flexible for survival analysis. Table 4 presents the *Kolmogorov-Smirnov* (K-S) statistic, P-value, *Akaike Information Criterion* (AIC), *Consistent AIC* (CAIC), *Hannan-Quinn Information Criterion* (HQIC), and *Bayesian Information Criterion* (BIC) for the six competing distributions. From Table 4 we observe that MOEMIR distribution provides an appropriate fit to the data with the highest p-value (0.9429) for K-S (0.0467). It also yields the lowest AIC, CAIC, HQIC, and BIC among the MIR, IR, MOEE and MOGW distributions, with no significant difference compared to MOEIR.
Additionally, the unknown parameters of the MOEMIR distribution as well as its survival function and HRF at month 1 are estimated via maximum likelihood estimation for complete data and adaptive Type-II progressive censored samples with random discrete uniformly distributed removals. For $m = 96$ (25% elimination), the adaptive progressive sample is:{0.08, 0.2, 0.4, 0.5, 1.19, 1.26, 1.35, 1.4, 1.46, 2.02, 2.02, 2.07, 2.09, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.75, 2.83, 2.87, 3.25, 3.31, 3.36, 3.36, 3.52, 3.57, 3.64, 3.82, 4.18, 4.23, 4.26, 4.33, 4.34, 4.4, 4.5, 4.51, 5.09, 5.17, 5.32, 5.34, 5.49, 5.62, 5.71, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.62, 7.63, 7.66, 7.93, 8.26, 8.53, 8.65, 9.02, 9.22, 10.34, 10.66, 10.75, 11.64, 11.79, 11.98, 12.03, 12.07, 12.63, 13.11, 13.29, 13.8, 14.24, 14.76, 14.77, 15.96, 17.12, 17.14, 17.36, 18.1, 19.13, 22.69, 23.63, 25.74, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05} and the corresponding random removals are $\{29, 2, 1, 0^{93}\}$. Similarly, for

m=64 (50% elimination), the adaptive Type-II progressive censored sample is: {0.08, 0.4, 0.51, 0.81, 1.05, 1.19, 1.35, 1.4, 1.46, 2.07, 2.09, 2.23, 2.46, 2.75, 2.83, 3.31, 3.36, 3.36, 3.57, 3.64, 3.82, 4.18, 4.34, 4.4, 4.5, 4.51, 4.87, 4.98, 5.09, 5.32, 5.32, 5.34, 5.41, 5.49, 5.62, 5.85, 6.25, 6.54, 6.94, 6.97, 7.09, 7.26, 7.32, 7.59, 7.62, 7.87, 7.93, 8.37, 9.47, 10.75, 11.25, 12.02, 12.63, 13.29, 13.8, 14.24, 15.96, 17.12, 17.36, 18.1, 21.73, 25.74, 25.82, 26.31} with random removals $\{32, 2, 13, 13, 1, 3, 0^{58}\}$. From the results in Table 5, we may conclude that the probability of remission time being greater than one month is greater than 93%, since $S(\hat{1}) > 0.93$ remained robust across censoring schemes. This indicates that a large proportion of bladder cancer patients are expected to experience remission times longer than 1 month.

Table 4. Goodness-of-fit statistics for the competitive models

Distributions	MOEMIR	MOEIR	MOGW	MOEE	MIR	IR
k-s statistics	0.0467	0.0495	0.0783	0.1121	0.2316	0.7502
p-value	0.9429	0.9124	0.3510	0.1024	< 0.0001	< 0.0001
AIC	833.385	831.124	838.76	845.210	924.765	1550.68
CAIC	833.578	831.220	838.953	845.403	924.861	1550.71
HQIC	836.861	833.411	842.236	848.686	927.082	1551.84
BIC	841.941	836.828	848.322	852.774	930.469	1553.54

Table 5. MLEs of parameters, survival and hazard functions for bladder cancer data

m	$\hat{\alpha}$	$\hat{\zeta}$	$\hat{\delta}$	$S(\hat{1})$	$h(\hat{1})$
128	0.0149	3.3474	215.302	0.935	0.0012
96	0.0099	4.1218	311.513	0.9417	0.1078
64	0.0088	4.5457	267.574	0.9301	0.1302

4. Conclusion

This paper proposes the MOEGIE family of distributions and studies some of its important statistical properties. A new distribution, referred to as the MOEMIR distribution, which is a member of the proposed family and an extension of the MIR distribution, is introduced. Using the statistical properties of the MOEGIE family, the properties of the MOEMIR distribution are derived. Under an adaptive Type-II progressive censoring scheme with discrete uniformly distributed random removals, the MLEs and Bayesian estimators of the parameters of the MOEMIR distribution are determined. The Bayes estimates are obtained under two loss functions: symmetric (SEL) and asymmetric (LINEX). The performance of the various parameter estimates is compared using a Monte Carlo simulation. Results show that Bayesian estimators tend to outperform MLE in small sample sizes, providing better accuracy and precision. However, as sample sizes grow, MLE becomes more competitive, and its performance approaches that of Bayesian methods. Based on the theoretical developments, simulation results, quantile-based shape analysis, and model fitting statistics, we recommend adopting the MOEMIR distribution as a suitable model for positively skewed lifetime data, especially under censoring schemes.

Data Availability: The data that support the findings of this study are available upon request from the corresponding author.
Acknowledgments: The authors are grateful to the editor and the referees for their valuable comments, which resulted in an improved paper.
Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.
Conflicts of interest: The authors declare no conflicts of interest.

References

1. Ahmad, Z., Hamedani, G. G., & Butt, N. S. (2019). Recent developments in distribution theory: A brief survey and some new generalized classes of distributions. *Pakistan Journal of Statistics and Operation Research*, *15*(1), 87–110. https://doi.org/10.18187/pjsor.v15i1.2803
2. Alrumayh, A., Weera, W., Khogeer, H. A., & Almetwally, E. M. (2023). Optimal analysis of adaptive type-II progressive censored for new unit-lindley model. Journal of King Saud University-Science, 35(2), 102462.
3. Balakrishnan, N. (2007). Progressive censoring methodology: An appraisal. *Test*, *16*(2), 211–296. https://doi.org/10.1007/s11749-007-0061-y
4. Balakrishnan, N., & Aggarwala, R. (2000). *Progressive censoring: Theory, methods and applications*. Birkhäuser.
5. Haj Ahmad, H., Salah, M. M., Eliwa, M. S., Ali Alhussain, Z., Almetwally, E. M., & Ahmed, E. A. (2022). Bayesian and non-Bayesian inference under adaptive type-II progressive censored sample with exponentiated power Lindley distribution. Journal of Applied Statistics, 49(12), 2981-3001.
6. Galton, F. (1983). *Enquiries into human faculty and its development*. Macmillan.
7. Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B. (2013). *Bayesian data analysis* (3rd ed.). Chapman & Hall/CRC.
8. Jose, K. K. (2011). Marshall-Olkin family of distributions and their applications in reliability theory, time series modeling and stress-strength analysis. *Proceedings of the 58th World Statistics Congress of the International Statistical Institute*, 3918–3923.
9. Khan, M. S. (2014). Modified inverse Rayleigh distribution. *International Journal of Computer Applications*, *87*(13), 28–33. https://doi.org/10.5120/15271-3762
10. Lee, E. T., & Wang, J. W. (2003). *Statistical methods for survival data analysis* (3rd ed.). Wiley.
11. Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika*, *84*(3), 641–652. https://doi.org/10.1093/biomet/84.3.641
12. Mohie El-Din, M. M., Abdel-Aty, Y., & Shafay, A. R. (2011). Two sample Bayesian prediction intervals for order statistics based on the inverse exponential-type distributions using right censored sample. *Journal of the Egyptian Mathematical Society*, *19*, 102–105. https://doi.org/10.1016/j.joems.2011.12.002
13. Moors, J. J. A. (1988). A quantile alternative for kurtosis. *The Statistician*, *37*, 25–32. https://doi.org/10.2307/2348376

14. Ng, H. K. T., Kundu, D., & Chan, P. S. (2009). Statistical analysis of exponential lifetimes under an adaptive Type-II progressive censoring scheme. *Naval Research Logistics*, *56*(8), 687–698. https://doi.org/10.1002/nav.20371

15. Santos-Neto, M., Bourguignon, M., Zea, L. M., Nascimento, A. D., & Cordeiro, G. M. (2014). The Marshall-Olkin extended Weibull family of distributions. *Journal of Statistical Distributions and Applications*, *1*(1), 9. https://doi.org/10.1186/2195-5832-1-9

16. Tierney, L. (1994). Markov chains for exploring posterior distributions. *Annals of Statistics*, *22*(4), 1701–1728. https://www.jstor.org/stable/2242477