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EXISTENCE AND UNIQUENESS OF SOLUTION OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING k-RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. The present study deals with the existence and uniqueness of solution of nonlinear fractional differential equations involving k-Riemann-Liouville fractional derivative with boundary conditions. Green's function and Banach contraction principle approach is used to prove solution of nonlinear fractional differential equations involving k-Riemann-Liouville fractional derivative with boundary conditions . Fractional differential equation with boundary conditions is reduced to the problem of Volterra integral equations. The equivalence of solution of fractional differential equations involving k-Riemann-Liouville fractional derivative with boundary conditions and Volterra integral equations is also proved. The properties of k-gamma functions , k-beta functions and $k-$ Riemann Liouville fractional deirvatives are considered. The Green's function is obtained to prove the existence and uniqueness of solution of the nonlinear boundary value problem involving $k-$ Riemann Liouville fractional deirvatives. Some properties of the Green's theorem for the existence and uniqueness of solution of nonlinear fractional differential equations involving k-Riemann-Liouville derivative with boundary conditions are considered.

1. Introduction

During last four decades many researchers attracted to the area of fractional differential equations due to wide range of applications in applied sciences, economics, engineering and technology etc. [7, 11, 15]. The existence, uniqueness and stability results were studied in [6, 16] using Green function, fixed point theorems, monotone iterative techniques and Lie group symmetry etc. Boundary value problems of various types of fractional value problems have been studied by many authors [1]-[5] and [12]-[19].

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Zou et al.[2017][19] studied uniqueness result for the solution of the following nonlinear boundary value problem

$$\begin{cases} D^{\alpha}v(x) + f(x, v(x)) = 0, \quad 2 < \alpha \leq 3, \quad x \in (0,1), \\ v(0) = v'(0) = v(1) = 0. \end{cases}$$

In 2021, Bachar et al. [2] obtained existence and uniqueness results for a class of fractional nonlinear boundary value problems under mild assumptions. The existence and uniqueness of solutions of fractional differential equations with different conditions were studied by researchers [8]. Very few studies on the existence and uniqueness results for nonlinear fractional differential equations with k-Riemann-Liouville fractional derivative is rare in the literature. Therefore, investigating the existence and uniqueness of solutions is interesting research topic makes our results novel and worthy.

Motivated by above mentioned work, we generalize the results obtained in [2] for the solution of following nonlinear fractional boundary value problem :

$$\begin{cases} {}^{kRL}D^{\alpha}v(s) + f(s, v(s)) = 0, 2 < \alpha \leq 3, s \in (0,1) \\ v(0) = v'(0) = v(1) = 0, \end{cases} \tag{1}$$

where ${}^{kRL}D^{\alpha}$ is the k-Riemann-Liouville fractional derivative of order α and $f \in \big((0,1) \times \mathbb{R}, \mathbb{R}\big)$ satisfies the following assumptions:
(A_1) $\int_0^1 (1-s)^{\frac{\alpha}{k}-2} |f(s,0)| ds < \infty$
(A_2) There exists $q \in C((0,1), [0,\infty))$ such that

$$|f(s,v) - f(s,w)| \leq q(s)|v - w|, \quad \forall\ s \in (0,1), v, w \in \mathbb{R},$$

and $0 < M_{q,\frac{\alpha}{k}} < \infty$,where $0 < \int_0^1 q(s) ds < \infty$ and

$$0 < M_{q,\frac{\alpha}{k}} = \frac{1}{\Gamma(\frac{\alpha}{k} - 1)} \int_0^1 s^{\frac{\alpha}{k}-1} (1-s)^{\frac{\alpha}{k}-1} q(s) ds.$$

Consider the following :

- $h(s) = s^{\frac{\alpha}{k}-1}(1-s), s \in [0,1], \quad \alpha \in [2,3]$.
- $G_{\alpha}(s,t)$ be the Green's function of the operator $v \to -{}^{kRL}D^{\alpha}v$ with boundary conditions $v(0) = v'(0) = v(1)$.
- $E = \{a > 0 : \int_0^1 G_{\alpha}(s,t)h(t)dt \leq ah(s), s \in [0,1]\}$ be nonempty and
$$M = infE \tag{2}$$
- For $a \in \mathbb{R}$, $a^+ = max(a, 0)$.
- $C([0,1]) = \{v \in C([0,1])$: there is $\sigma > 0$ such that $|v(s)| \leq \sigma h(s), s \in [0,1]\}$.

The paper is organized as follows :

In Section 2, we recall some basic definitions of fractional calculus and give some useful preliminary results. In section 3, we prove the existence and uniqueness results for solution of nonlinear k-Riemann-Liouville fractional differential equations with boundary conditions (1).

2. Definitions and Basic Results

In this section, we recall some definitions of k-gamma function, k-beta function, k-Riemann-Liouville fractional integral, k-Riemann-Liouville fractional derivative and results which are useful in our study.

Definition 2.1. [9] *For $k > 0$, the k-gamma function Γ_k is given by*

$$\Gamma_k(\alpha) = \lim_{x\to\infty} \frac{n!k^n(nk)^{\frac{\alpha}{k}-1}}{\alpha_{n,k}}$$

For $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$ and $k > 0 (k \in \mathbb{R})$ the k-gamma function is defined by

$$\Gamma_k(\alpha) = \int_0^\infty s^{\alpha-1} e^{-\frac{s^k}{k}} ds$$

Theorem 2.1. [9] *The k-gamma function $\Gamma_k(.)$ satisfies the following properties :*
(i) $\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})$
(ii) $\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$
(iii) $\Gamma_k(k) = 1$ **Proof.(i)** Let $s = \frac{t^k}{k}$, then $ds = t^{k-1}dt$
and

$$\begin{aligned}\Gamma_k(\alpha) &= \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt\\ &= \int_0^\infty t^{\alpha-k} e^{\frac{-t^k}{k}} t^{k-1} dt\\ &= \int_0^\infty t^{k(\frac{\alpha}{k}-1)} e^{\frac{-t^k}{k}} t^{k-1} dt\\ &= \int_0^\infty (sk)^{(\frac{\alpha}{k}-1)} e^{-s} ds\\ &= k^{\frac{\alpha}{k}-1} \int_0^\infty s^{(\frac{\alpha}{k}-1)} e^{-s} ds\\ &= k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})\end{aligned}$$

(ii) From (i), we have

$$\begin{aligned}\Gamma_k(\alpha + k) &= k^{\frac{\alpha+k}{k}-1}\Gamma(\frac{\alpha + k}{k})\\ &= k^{\frac{\alpha}{k}} \frac{\alpha}{k}\Gamma(\frac{\alpha}{k})\\ &= \alpha\Gamma_k(\alpha).\end{aligned}$$

(iii) By definition of the k-gamma function, we have

$$\begin{aligned}\Gamma_k(k) &= \lim_{x\to\infty} \frac{n!k^n(nk)^{\frac{k}{k}-1}}{(k)_{n,k}}\\ &= \lim_{x\to\infty} \frac{n!k^n}{(k)_{n,k}}\\ &= 1\end{aligned}$$

Definition 2.2. [9]*Let $\alpha, \beta \in \mathbb{C}$ with $Re(\alpha) > 0$ and $Re(\beta) > 0$. Then, the k-beta function $B_k(\alpha, \beta)$ is defined by*

$$B_k(\alpha, \beta) = \frac{1}{k}\int_0^1 s^{\frac{\alpha}{k}-1}(1 - s)^{\frac{\beta}{k}-1} ds$$

Note that beta function and k-beta function have the following relation

$$B_k(\alpha,\beta)=\frac{1}{k}B(\frac{\alpha}{k},\frac{\beta}{k})$$

Definition 2.3. [13] *Let $f(t)$ be an integrable function defined on $[a,b]$ and $k>0$. The k-Riemann-Liouville fractional integral of order $\alpha>0(\alpha\in R)$ of the function $f(s)$ is given by*

$$^{kRL}I^{\alpha}_{a^+}f(s)=\frac{1}{k\Gamma_k(\alpha)}\int_a^s(s-t)^{\frac{\alpha}{k}-1}f(t)dt.$$

Definition 2.4. [10] *Let $k,\alpha\in R_+=(0,\infty)$ and $n\in N$ such that $n=[\frac{\alpha}{k}]$ and $f(t)$ be an integrable function defined on $[a,b]$. Then, the k-Riemann-Liouville fractional derivative of order $\alpha>0$ of the function $f(t)$ is*

$$^{kRL}D^{\alpha}_{a_+}f(s)=(k\frac{d}{dx})^n\ ^{kRL}I^{nk-\alpha}_{a^+}f(t)dt$$

Lemma 2.1. *Let $\alpha>0$, and $v(x)\in\mathbb{C}(0,1)\cap L(0,1)$, then the fractional differential equation $^{kRL}D^{\alpha}v(x)=0$ has unique solution*

$$v(x)=c_1x^{\frac{\alpha}{k}-1}+c_2x^{\frac{\alpha}{k}-2}+...+c_Nx^{\frac{\alpha}{k}-N},\quad c_i\in\mathbb{R},\quad i=1,2,....,N$$

.

Note: As $^{kRL}D^{\alpha}\ ^{kRL}I^{\alpha}v=v$, for all $v\in\mathbb{C}(0,1)\cap L(0,1)$. From Lemma 2.1 we deduce the following law of composition:

Lemma 2.2. *Assume that $v\in\mathbb{C}(0,1)\cap L(0,1)$, with a fractional derivative of order $\alpha>0$ that belongs to $\mathbb{C}(0,1)\cap L(0,1)$ then*

$$^{k}I^{\alpha k}D^{\alpha}v(x)=v(x)+c_1x^{\frac{\alpha}{k}-1}+c_2x^{\frac{\alpha}{k}-2}+....+c_Nx^{\frac{\alpha}{k}-N}$$

for some $c_i\in\mathbb{R},\quad i=1,2,...N$.

Lemma 2.3. *If $f(t)\in\mathbb{C}[0,1]$ and $2<\alpha\leq 3$, the unique solution of problem (1) is*

$$v(x)=\int_0^1 G(s,t)f(t)dt,$$

where

$$G_\alpha(s,t)=\frac{1}{k\Gamma_k(\alpha)}\begin{cases}[s(1-t)]^{\frac{\alpha}{k}-1}-(s-t)^{\frac{\alpha}{k}-1}, & 0\leq t\leq s\leq 1\\ [s(1-t)]^{\frac{\alpha}{k}-1}, & 0\leq s\leq t\leq 1.\end{cases}\tag{3}$$

Proof. We apply Lemma 2.2 to reduce problem (1) to an equivalent integral equation

$$v(s)=\ ^{kRL}I^{\alpha}_{0^+}f(s)+c_1s^{\frac{\alpha}{k}-1}+c_2s^{\frac{\alpha}{k}-2}+c_3s^{\frac{\alpha}{k}-3}\text{ for some }c_1,c_2,c_3\in\mathbb{R}.$$

Consequently, the general solution of equation (1) is

$$v(s)=-\int_0^s\frac{(s-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(t)dt+c_1s^{\frac{\alpha}{k}-1}+c_2s^{\frac{\alpha}{k}-2}+c_3s^{\frac{\alpha}{k}-3}$$

By Lemma 2.2, we have $c_2=c_3=0$ therefore

$$c_1=\int_0^1\frac{(1-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(t)dt.$$

Therefore, the unique solution of problem (1) is

$$\begin{aligned} v(s) &= -\int_0^s \frac{(s-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt + \int_0^1 \frac{\left[s(1-t)\right]^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt \\ &= \int_0^s \frac{\left[s(1-t)\right]^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt - \int_0^s \frac{(s-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt + \int_s^1 \frac{\left[s(1-t)\right]^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt \\ &= \int_0^s \frac{\left[s(1-t)\right]^{\frac{\alpha}{k}-1} - (s-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} f(t)dt \\ &= \int_0^1 G_\alpha(s,t) f(t)dt. \end{aligned}$$

Lemma 2.4. *The Green function $G_\alpha(s,t)$ has the following properties :*
(i) $G_\alpha(s,t)$ is a non-negative continuous function on $[0,1]\times[0,1]$
(ii) For all $s,t\in[0,1]$ we have

$$H_\alpha(s,t) \le G_\alpha(s,t) \le \Big(\frac{\alpha}{k}-1\Big) H_\alpha(s,t) \tag{4}$$

where, $H_\alpha(s,t) = \frac{1}{k\Gamma_k(\alpha)} s^{\frac{\alpha}{k}-2}(1-t)^{\frac{\alpha}{k}-2} min(s,t)\big[1-max(s,t)\big]$

Proof. It is obvious that property (i) holds. Now we prove property (ii), from equation (3), for all $s,t\in(0,1)$, we have

$$\begin{aligned} k\Gamma_k(\alpha)G_\alpha(s,t) &= s^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1} - (s-t)^{\frac{\alpha}{k}-1} \\ &= s^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}\left(1-\left(\frac{s-t}{s(1-t)}\right)^{\frac{\alpha}{k}-1}\right) \end{aligned} \tag{5}$$

since for $\lambda>0$ and $t\in[0,1]$, $min(1,\lambda)(1-t)\le 1-t^\lambda \le max(1,\lambda)(1-t)$.
We deduce that,

$$1-\frac{s-t}{s(1-t)} \le 1-\Big(\frac{s-t}{s(1-t)}\Big)^{\frac{\alpha}{k}-1} \le \Big(\frac{\alpha}{k}-1\Big)\Big(1-\frac{s-t}{s(1-t)}\Big)$$

$$\frac{s(1-t)-(s-t)}{s(1-t)} \le \frac{k\Gamma_k(\alpha)G_\alpha(s,t)}{s^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}} \le \frac{\Big(\frac{\alpha}{k}-1\Big)\Big(s(1-t)-(s-t)\Big)}{s(1-t)}.$$

Using this and equation (5), we obtain

$$s(1-t)-(s-t) \le \frac{k\Gamma_k(\alpha)G_\alpha(s,t)}{s^{\frac{\alpha}{k}-2}(1-t)^{\frac{\alpha}{k}-2}} \le \Big(\frac{\alpha}{k}-1\Big)\Big(s(1-t)-(s-t)\Big)$$

Hence

$$H_\alpha(s,t) \le G_\alpha(s,t) \le \Big(\frac{\alpha}{k}-1\Big) H_\alpha(s,t)$$

where $s(1-t)-(s-t) = min(s,t)\big[1-max(s,t)\big]$

Lemma 2.5. *Let $q\in C((0,1),[0,\infty))$ and assume that $0<M_{q,\frac{\alpha}{k}}$ then*

$M_{q,\frac{\alpha}{k}+1} \le M \le M_{q,\frac{\alpha}{k}}$.

Proof. Let

$$E = \int_0^1 G_\alpha(s,t)h(t)q(t)dt \le ah(s), \quad s\in[0,1]: a>0,$$

where $h(s) = s^{\frac{\alpha}{k}-1}(1-s), \quad s \in [0,1]$.
By equation (3), we obtain

$$\begin{aligned}\int_0^1 G_\alpha(s,t)h(t)q(t)dt &\leq \int_0^1 (\frac{\alpha}{k}-1)H_\alpha(s,t)h(t)q(t)dt \\ &\leq (\frac{\alpha}{k}-1)\int_0^1 \frac{1}{\Gamma_k(\alpha)}s^{\frac{\alpha}{k}-2}(1-t)^{\frac{\alpha}{k}-2}P(t)t^{\frac{\alpha}{k}-1}(1-t)q(t)dt \\ &\leq \frac{s^{\frac{\alpha}{k}-2}}{\Gamma_k(\frac{\alpha}{k}-1)}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}P(t)q(t)dt \\ &\leq M_{q,\frac{\alpha}{k}}h(s),\end{aligned}$$

where $P(t) = min(s,t)\big[1-max(s,t)\big]$. It follows that $E \neq \phi$ and $M \leq M_{q,\frac{\alpha}{k}}$, where $M = infE$.
On the other hand, using equation (3) and
$min(s,t)(1-max(s,t)) \geq st(1-s)(1-t)$ for $s,t \in [0,1]$, we deduce that for any $a \in E$,

$$\begin{aligned}ah(s) &\geq \frac{1}{\Gamma_k(\alpha)}s^{\frac{\alpha}{k}-2}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}min(s,t)\big[1-max(s,t)\big]q(t)dt, \\ &\geq \frac{1}{\Gamma_k(\alpha)}s^{\frac{\alpha}{k}-2}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}st(1-s)(1-t)q(t)dt, \\ &= h(s)M_{q,\frac{\alpha}{k}+1}.\end{aligned}$$

Hence for each $a \in E$, $a \geq M_{q,\frac{\alpha}{k}+1}$.
Therefore $M \geq M_{q,\frac{\alpha}{k}+1}$, that is, $M \in [M_{q,\frac{\alpha}{k}+1}, M_{q,\frac{\alpha}{k}}]$.

Lemma 2.6. *If $\alpha \in (2,3)$ and let ϕ be a function such that*

$$s \to (1-s)^{\frac{\alpha}{k}-1}\phi(s) \in C((0,1)) \cap L^1((0,1)).$$

Then the unique continuous solution of the problem

$$\begin{cases} {}^{kRL}D^\alpha v(s) = -\phi(s), \quad s \in (0,1) \\ v(0) = v'(0) = v(1) = 0 \end{cases} \tag{6}$$

is

$$V\phi(s) = \int_0^1 G_\alpha(s,t)\phi(t)dt.$$

Proof. Let ϕ be a function such that

$$s \to (1-s)^{\frac{\alpha}{k}-1}\phi(s) \in C((0,1)) \cap L^1((0,1))$$

. By Lemma 2.2, $G_\alpha(s,t)$ belongs to $C([0,1]\times[0,1])$ with

$$0 \leq G_\alpha(s,t) \leq \frac{1}{k\Gamma(\frac{\alpha}{k}-1)}(1-t)^{\frac{\alpha}{k}-1}.$$

We deduce by the dominated convergence theorem that $V\phi \in C([0,1])$ and
$V\phi(0) = V\phi(1) = 0$. Therefore ${}^{kRL}I^{3-\alpha}(V|\phi|)$ is bounded on $[0,1]$.

By Fubini's theorem, we obtain

$$\begin{aligned}
{}^{kRL}I^{3-\alpha}(V\phi)(s) &= \frac{1}{k\Gamma_k(3-\alpha)}\int_0^s (s-t)^{\frac{3-\alpha}{k}-1}V\phi(t)dt \\
&= \frac{1}{k\Gamma_k(3-\alpha)}\int_0^s (s-t)^{\frac{3-\alpha-k}{k}}V\phi(t)dt \\
&= \frac{1}{k\Gamma_k(3-\alpha)}\int_0^s \int_0^1 (s-t)^{\frac{3-\alpha-k}{k}}G_\alpha(t,r)\phi(r)drdt \\
&= \int_0^1 \Big[\frac{1}{k\Gamma_k(3-\alpha)}\int_0^s (s-t)^{\frac{3-\alpha-k}{k}}G_\alpha(t,r)dt\Big]\phi(r)dr \\
&= \int_0^1 K(s,r)\phi(r)dr,
\end{aligned}$$

where

$$\begin{aligned}
K(s,r) &= \frac{1}{k\Gamma_k(3-\alpha)}\int_0^s (s-t)^{\frac{3-\alpha-k}{k}}G_\alpha(t,r)dt \\
&= \frac{1}{k\Gamma_k(3-\alpha)}\int_0^s (s-t)^{\frac{3-\alpha-k}{k}}\left[t^{\frac{\alpha}{k}-1}(1-r)^{\frac{\alpha}{k}-1}-(t-r)^{\frac{\alpha}{k}-1}\right]dt \\
&= \frac{[3-\alpha-(n+1)!k]}{k\Gamma_k(3-\alpha)(\alpha+n!k)}\int_0^s (s-t)^{\frac{3-\alpha-(n+2)!k}{k}}T(t)dt,
\end{aligned}$$

where $T(t)=\left[(1-r)^{\frac{\alpha}{k}-1}t^{\frac{\alpha+n!k}{k}}-(t-r)^{\frac{\alpha+n!k}{k}}\right]$. Hence, for $s\in(0,1)$, we have

$$ {}^{kRL}I^{3-\alpha}(V\phi)(s) = \frac{[3-\alpha-(n+1)!k]}{k\Gamma_k(3-\alpha)(\alpha+n!k)}\int_0^s (s-t)^{\frac{3-\alpha-(n+2)!k}{k}}T(t)dt\phi(r)dr. $$

This implies that

$$ {}^{kRL}D^{3-\alpha}\Big({}^{kRL}I^{3-\alpha}(V\phi)\Big)(s) = -\phi(s) $$

. Now, since for each $t\in(0,1)$,

$$ \lim_{s\to 0}\frac{G_\alpha(s,t)}{s}=0 \quad \textit{and} \quad 0\le \frac{G_\alpha(s,t)}{s}\le \frac{1}{k\Gamma(\frac{\alpha}{k}-1)}(1-t)^{\frac{\alpha}{k}-1} $$

by the Dominated convergent theorem we obtain $(V\phi)'(0)=0$.

To prove the uniqueness, let $v,w\in C([0,1])$ be two solutions of problem (6) and set $\theta(s)=v(s)-w(s)$. Then $\theta(s)\in C([0,1])$, and we have

$$ \begin{cases} {}^{kRL}D^\alpha\theta(s)=0, & s\in(0,1) \\ \theta(0)=\theta'(0)=\theta(1)=0. \end{cases} $$

By Corollary 2.1[11], there exist $c_1,c_2,c_3\in\mathbb{R}$ such that

$$ \theta(s)=c_1s^{\alpha-1}+c_2s^{\alpha-2}+c_3s^{\alpha-3}. $$

Applying the boundary conditions, we obtain $c_3=c_2=c_1=0$. Thus $v(s)=w(s)$.

3. Main Results

In this section we obtain the existence and uniqueness results of the solutions of nonlinear fractional boundary value problems involving k-Riemann-Liouville fractional derivative.

Theorem 3.2. *Assume that (A_1) and (A_2) hold. If $M < 1$, then problem* (1) *has a unique solution v in $C_h([0,1])$. In addition, for any $v_0 \in C_h([0,1])$, the iterative sequence $v_i(s) = \int_0^1 G_\alpha(s,t)f(t,v_{i-1}(t))dt$ converges to v with respect to the h-norm, and we have*

$$\|v_i(s) - v(s)\|_h \leq \frac{M^i}{1-M}\|v_1(s) - v_0(s)\|_h. \tag{7}$$

Proof To this end, define the operator T by

$$Tv(s) = \int_0^1 G_\alpha(s,t)f\Big(t, v(t)\Big)dt, \quad s \in [0,1], \quad v \in C\Big([0,1]\Big). \tag{8}$$

We claim that T is a contraction operator from $\Big(C([0,1]), \|.\|\Big)$ into itself.
Let $v \in C\Big([0,1]\Big)$, and let $\sigma > 0$ be such that $|v(s)| \leq \sigma h(s)$ for all $s \in [0,1]$.
Since by Lemma 2.2 (ii), $0 \leq G_\alpha(s,t) \leq \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}(1-t)^{\frac{\alpha}{k}-2}$, it follows from ($A_2$) that

$$\begin{aligned}
\big|G_\alpha(s,t)f\big(t,v(t)\big)\big| &\leq \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}(1-t)^{\frac{\alpha}{k}-2}\big(|f\big(t,v(t)\big) - f(t,0)| + |f(t,0)|\big) \\
&\leq \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}(1-t)^{\frac{\alpha}{k}-2}\Big(q(t)|v(t)| + |f(t,0)|\Big) \\
&\leq \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}\Big(\sigma t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\alpha}{k}-1}q(t) + (1-t)^{\frac{\alpha}{k}-2}|f(t,0)|\Big)
\end{aligned}$$

Since $G_\alpha(s,t)$ is continuous on $[0,1]\times[0,1]$, by (A_1)-(A_2) and the dominated convergence theorem we deduce that $Tv \in C([0,1])$.

Furthermore, from Lemma 2.2 (ii), we have

$$0 \leq G_\alpha(s,t) \leq \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}h(s)(1-t)^{\frac{\alpha}{k}-2}, \tag{9}$$

Hence by using inequality (9) and similar arguments as before we obtain

$$\big|Tv(s)\big| \leq \Big[\sigma M_{q,\frac{\alpha}{k}} + \frac{1}{\Gamma_k(\frac{\alpha}{k}-1)}\int_0^1 (1-t)^{\frac{\alpha}{k}-2}\big|f(t,0)\big|\Big]h(t),$$

and thus $T(C([0,1])) \subset C([0,1])$.

Next, for any $v, w \in C([0,1])$, by using (A$_2$) we obtain that for $s \in [0,1]$,

$$\begin{aligned}\big|Tv(s) - Tw(s)\big| &\le \int_0^1 G_\alpha(s,t)\big|f(t,v(t)) - f(t,w(t))\big|dt \\ &\le \int_0^1 G_\alpha(s,t)q(t)\big|v(t) - w(t)\big|dt \\ &\le \|v-w\| \int_0^1 G_\alpha(s,t)q(t)h(t)dt \\ &\le M\|v-w\|h(s).\end{aligned}$$

Hence

$$\|Tv - Tw\| \le M\|v-w\|h(s)$$

Since $M < 1$, T becomes a contraction operator in $C([0,1])$. So there exists a unique $v \in C([0,1])$ satisfying

$$v(s) = \int_0^1 G_\alpha(s,t)f(t,v(t))dt, \quad s \in (0,1).$$

It remains to prove that $v(s)$ is a solution of problem (1). Indeed, it is clear that $s \to (1-s)^{\frac{\alpha}{k}-1}f(s,v(s)) \in C((0,1))$.
Next, by using (A$_2$) and $v \in C([0,1])$ we obtain

$$\begin{aligned}\big|(1-s)^{\frac{\alpha}{k}-1}f\big(s,v(s)\big)\big| &\le (1-s)^{\frac{\alpha}{k}-1}\Big|\bigg(f\big(s,v(s)\big) - f(s,0)\big| + \big|f(s,0)\big| \\ &\le (1-s)^{\frac{\alpha}{k}-1}\bigg(q(s)\big|v(s)\big| + \big|f(s,0)\big|\bigg) \\ &\le \sigma s^{\frac{\alpha}{k}-1}(1-s)^{\frac{\alpha}{k}-1}q(s) + (1-s)^{\frac{\alpha}{k}-2}\big|f(s,0)\big|.\end{aligned}$$

Therefore by (A$_1$) and (A$_2$) it follows that $s \to (1-s)^{\frac{\alpha}{k}-1}f\big(s,v(s)\big) \in L^1((0,1))$. Hence from Lemma 2.6, it follows that $v(s)$ is a solution of problem (1).

Finally, it is known that for any $v \in C([0,1])$, the iterative sequence $v_i(s) = \int_0^1 G_\alpha(s,t)f\big(t,v_{i-1}(t)\big)dt$ converges to $v(s)$, and we have

$$\|v_i(s) - v(s)\| \le \frac{M^i}{1-M}\|v_1(s) - v_0(s)\|.$$

This proves the theorem.

4. Conclusion

Green's function and Banach contraction principle is used to prove the existence and uniqueness of solution of nonlinear fractional differential equations with boundary condition involving k-Riemann-Liouville fractional derivative (1).

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