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Berwald and Chern Connections under Anisotropic Conformal Transformations on Conic Pseudo-Finsler Surfaces

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Abstract

This paper extends our previous research on anisotropic conformal changes $F(x,y) \mapsto \overline{F}(x,y) = e^{\phi(x,y)}F(x,y)$. The study focuses on the behavior of the Berwald connection, which measures the deviation of a Finsler structure from a Riemannian one, and the Chern-Rund connection on conic pseudo-Finsler surfaces under this anisotropic conformal transformation, along with the dynamical covariant derivative. In particular, we express the Landsberg tensor of the transformed Finsler metric $\overline{F}$ in terms of the difference between the horizontal coefficients of the Berwald and Chern-Rund connections. Consequently, we find the necessary and sufficient conditions under which the Landsbergian property is preserved under the anisotropic conformal transformation. Our findings shed light on the relationship between anisotropic conformal transformations and the intrinsic geometry of Finsler surfaces. Additionally, we provide explicit formulas for the anisotropic conformal transformation of the dynamical covariant derivatives in the context of conic pseudo-Finsler surfaces.

Keywords: anisotropic conformal change; conic pseudo-Finsler surface; modified Berwald frame; dynamical covariant derivative; Berwald connection.

Introduction

Anisotropic conformal transformations generalize classical (isotropic) conformal changes by allowing the conformal factor to depend not only on the position but also on the direction. In the Finsler geometry, this transformation takes the form $F(x,y) \mapsto \overline{F}(x,y) = e^{\phi(x,y)}F(x,y)$, where the function $\phi(x,y)$ encodes the directional anisotropy. This anisotropy is important in various physical scenarios such as enables richer models of describing the universe, new solutions to gravitational field equations, light propagation in anisotropic materials and potential explanations for phenomena not addressed by isotropic theories [Friedl-Sz´asz et al., 2025; Heefer et al., 2023; Hohmann et al., 2020; Pfeifer and Wohlfarth, 2012; Savvopoulos and Stavrinos, 2023; Voicu et al., 2023; Youssef et al., 2024; Youssef et al., 2025b].

The anisotropic conformal transformation of a conic pseudo-Finsler surface has been introduced and thoroughly investigated in [Youssef et al., 2024; Youssef et al., 2025a; Youssef et al., 2025b]. Unlike isotropic conformal changes, anisotropic conformal changes do not necessarily yield a pseudo-Finsler metric. Consequently, we have determined the necessary and sufficient conditions for $(M,\overline{F})$ to remain a conic pseudo-Finsler surface under such transformations. Notably, there exist non-homothetic conformal factors $\phi(x,y)$ that preserve the geodesic spray. Moreover, it is possible to transform a pseudo-Finsler metric into a pseudo-Riemannian one, and vice versa. These results highlight the greater geometric flexibility and significance of anisotropic conformal transformations. We have further investigated the relationships between key geometric objects of F and their counterparts for $\overline{F}$, including the Berwald, Landsberg, and Douglas tensors, as well as the T-tensor. In particular, we have determined the conditions under which the geodesic spray of a two-dimensional pseudo-Berwald metric $\overline{F}$ is metrizable by a two-dimensional pseudo-Riemannian metric F. Furthermore, we study the Cartan connection and derive several identities satisfied under the anisotropic conformal transformation.

In Finsler geometry, the Berwald and Chern-Rund connections are canonical linear connections defined on the pullback bundle or the tangent bundle of a Finsler manifold [Miron and Anastasiei 2012; YOUSSEF, 2008; Youssef et al., 2009] The Berwald connection generalizes the Levi-Civita connection of Riemannian geometry and measures how far a Finsler structure departs from being Riemannian. A Finsler space is called a Berwald space if its Berwald curvature vanishes [Antonelli et al., 2013; Bao et al., 2012; Bidabad and Tayebi, 2011; Youssef et al., 2010; Shen and Shen, 2016; Bucataru and Miron, 2007]. Both Berwald and Chern connections are torsion-free and only slightly fail to be fully metric-compatible, an expected feature in the Finsler setting. These connections coincide when the underlying Finsler structure is Landsbergian and when the structure is of Berwald type, they reduce to a linear connection on the manifold M that acts directly on the tangent bundle TM.

In this paper, we explore specific properties of the Berwald connection on conic pseudo-Finsler surfaces (M,F). We express the connection coefficients in terms of the modified Berwald frame. The transformation of the horizontal coefficients under anisotropic conformal changes has already been studied in our previous work [Youssef et al., 2024]. Here, we further examine the dynamical covariant derivative and provide an alternative proof that the dynamical covariant derivative of the modified Berwald frame vanishes. Additionally, we analyze how the dynamical covariant derivative transforms under anisotropic conformal changes. Furthermore, we study the Chern-Rund connection, express its horizontal coefficients using the modified Berwald frame and derive their transformation under anisotropic conformal changes.

It is known that the difference between the Berwald and Chern-Rund connections is the Landsberg tensor. Therefore, we drive the anisotropic conformal transformation of the Landsberg tensor. As a result, we determine the necessary and sufficient conditions under which the Landsbergian property is preserved under the anisotropic conformal transformation Proposition 4.7.

Notation and preliminaries

Let M be a smooth manifold of dimension n. The tangent bundle of M is denoted by (TM,π_M,M), where TM is the disjoint union of all tangent spaces at each point of M, and $\pi_M: TM \to M$ is the the canonical projection onto the base manifold. The slit tangent bundle is defined as $\mathcal{T}M = TM\backslash\{0\}$, which is the tangent bundle with the zero section removed. A local coordinate system on M is denoted by (x^i), which induces local coordinates (x^i, y^i) on TM, where x^i are coordinates on the base and y^i are the components of tangent vectors in each fiber.

A smooth function $f \in C^\infty(\mathcal{T}M)$ is considered to be positively homogeneous of degree r in the fiber coordinates y, denoted $f \in h(r)$, if it satisfies

$$f(x,\lambda y)=\lambda^r f(x,y),\quad \forall\,\lambda>0.$$

A conic sub-bundle of TM is a non-empty open subset $\mathcal{A}\subseteq \mathcal{T}M$ such that $\pi_M(\mathcal{A})=M$ and $\mathcal{A}$ is invariant under positive scaling of the fiber coordinates; that is, for any $(x,y)\in\mathcal{A}$ and any $\lambda>0$, one has $(x,\lambda y)\in\mathcal{A}$.

Definition 2.1 *A conic pseudo-Finsler metric on M is a smooth function $F:\mathcal{A}\to\mathbb{R}$, with $F\in h(1)$, defined on a conic sub-bundle $\mathcal{A}\subseteq\mathcal{T}M$. For each point $(x,y)\in\mathcal{A}$, the Hessian matrix*

$$g_{ij}(x,y):=\frac{1}{2}\dot{\partial}_i\dot{\partial}_jF^2(x,y),$$

$$\text{where}\quad \dot{\partial}_i:=\frac{\partial}{\partial y^i},$$

must be nondegenerate. The pair (M,F) is called a conic pseudo-Finsler manifold.

In terms of the Finsler metric F, there exists a unique nonlinear Cartan (Ehresmann) connection in the conic sub-bundle $\mathcal{A}\subset TM$ with coefficients determined by

$$G_i^j=\tfrac{1}{4}\dot{\partial}_i\big[g^{jk}\big(y^m\partial_m\dot{\partial}_kF^2-\partial_kF^2\big)\big].$$

This nonlinear connection defines the horizontal derivatives $\delta_i:=\partial_i-G_i^j\dot{\partial}_j$. The coefficients of the geodesic spray coefficients of F can be expressed as

$$G^i=\tfrac{1}{4}g^{ik}\big(y^m\partial_m\dot{\partial}_kF^2-\partial_kF^2\big).$$

It is evident that G^i are smooth and positively homogeneous of degree 2 in $\mathcal{A}$; furthermore, the geodesic spray can be defined globally on $\mathcal{T}M$ as $S=y^i\partial_i-2G^i\dot{\partial}_i$.

We are concerned with a two-dimensional Finsler space (M,F) with coordinates $x=(x^i)$, $y=(y^i)$, where $i=1,2$. Then we have

$$\ell^i=\frac{1}{F}y^i,\quad \ell_i=\dot{\partial}_iF,\qquad h_{ij}=F\dot{\partial}_i\dot{\partial}_jF,$$

$$g_{ij}=\ell_i\ell_j+h_{ij}.\tag{2.1}$$

The angular metric tensor h_{ij} of an n-dimensional Finsler space has the matrix (h_{ij}) of rank $n-1$. In a two-dimensional space, the angular metric has a matrix of rank one and we have

$$\det(h_{ij})=h_{11}h_{22}-(h_{12})^2=0.\tag{2.2}$$

If $h_{11}=h_{22}=0$, then (2.2) implies $h_{12}=0$, leading to a contradiction $h_{ij}=0$. Therefore, we assume $h_{ij}\neq 0$ and choose the sign $\varepsilon=\pm1$ for h_{11}. We find that $\varepsilon h_{11}=(m_1)^2$ uniquely determines a non-zero m_1 up to the sign of h_{11}. Subsequently, $\varepsilon h_{12}=m_1m_2$ determines m_2, and (2.2) gives $\varepsilon h_{22}=(m_2)^2$. Consequently, we have (m_1,m_2) and the sign ε, satisfying

$$h_{ij}=\varepsilon m_im_j.\tag{2.3}$$

Henceforward, we work in a two-dimensional conic pseudo-Finsler space equipped with a modified Berwald frame (ℓ_i,m_i). The components g_{ij} of the metric tensor are given by

$$g_{ij}=\ell_i\ell_j+\varepsilon m_im_j\tag{2.4}$$

and its inverse as

$$g^{ij}=\ell^i\ell^j+\varepsilon m^im^j.\tag{2.5}$$

Consequently, the Kronecker delta takes the form

$$\delta_j^i=\ell^i\ell_j+\varepsilon m^im_j,\tag{2.6}$$

where ε is called the signature of F. The two vector fields $\ell=(\ell^1,\ell^2)$ and $m=(m^1,m^2)$ have been chosen in such a way that they satisfy

$$g(\ell,\ell)=1,\quad g(\ell,m)=0,\quad g(m,m)=\varepsilon.$$

The main scalar $\mathcal{I}(x,y)$ is a $h(0)$-smooth function that is derived from the Cartan tensor [Antonelli et al., 2013] and defined by

$$FC_{ijk}=\mathcal{I}m_im_jm_k.\tag{2.7}$$

For a smooth function $f\in C^\infty(\mathcal{T}M)$, the vertical and horizontal scalar derivatives $(f_{;1},f_{;2})$ and $(f_{,1},f_{,2})$ in a Finsler surface (M,F) are given by

$$\left.\begin{aligned}F\dot{\partial}_if&=f_{;1}\ell_i+f_{;2}m_i,\\ \delta_if&=f_{,1}\ell_i+f_{,2}m_i\end{aligned}\right\}\tag{2.8}$$

where

$$f_{;1}=y^i\dot{\partial}_if,\ f_{;2}=\varepsilon F\big(\dot{\partial}_if\big)m^i,$$

$$f_{,1}=(\delta_if)\ell^i,\ f_{,2}=\varepsilon(\delta_if)m^i.$$

In particular, if f is $h(r)$, then $f_{;1}=rf$.

Lemma 2.2 *[Matsumoto, 2003] Let (M,F) be a conic pseudo-Finsler surface equipped with modified Berwlad frames. Then, we have the following:*

(i) $\ell^im_i=\ell_im^i=0,\ m^im_i=\varepsilon,$
$\ell^i\ell_i=1,$

(ii) $F\dot{\partial}_j\ell_i=\varepsilon m_im_j=h_{ij},$
$F\dot{\partial}_j\ell^i=\varepsilon m^im_j,$

(iii) $F\dot{\partial}_jm_i=-(\ell_i-\varepsilon\mathcal{I}m_i)m_j,$
$F\dot{\partial}_jm^i=-(\ell^i+\varepsilon\mathcal{I}m^i)m_j.$

Definition 2.3 [Youssef et al., 2024] *An anisotropic conformal change of a conic pseudo-Finsler metric F is defined by*

$$F\longmapsto\overline{F}(x,y)=e^{\phi(x,y)}F(x,y),\tag{2.9}$$

where the conformal factor $\phi(x,y)$ is a smooth $h(0)$-function on $\mathcal{A}$. Under this transformation, the following condition holds:

$$F^2\big(\dot{\partial}_i\dot{\partial}_j\phi+(\dot{\partial}_i\phi)(\dot{\partial}_j\phi)\big)m^im^j+\varepsilon=\sigma-(\phi_{;2})^2+\varepsilon\neq 0,$$

with $\sigma=\phi_{;2;2}+\varepsilon\mathcal{I}\phi_{;2}+2(\phi_{;2})^2$.

In this case, we say that F is anisotropically conformally changed to $\overline{F}$. The transformation (2.9) is called proper if the conformal factor $\phi(x,y)$ is neither isotropic nor homothetic, i.e., $\phi_{;2} \neq 0$.

Now, we define the v-scalar derivatives $(f_{;a}, f_{;b})$ and h-scalar derivatives $(f_{,a}, f_{,b})$ in $(M,\overline{F})$ for f by:

$$\overline{F}\dot{\partial}_i f = f_{;a}\overline{\ell}_i + f_{;b}\overline{m}_i,$$
$$\overline{\delta}_i f = f_{,a}\overline{\ell}_i + f_{,b}\overline{m}_i,$$

where

$$f_{;a} = y^i\dot{\partial}_i f, \;\; f_{;b} = \varepsilon\overline{F}(\dot{\partial}_i f)\overline{m}^i,$$
$$f_{,a} = (\overline{\delta}_i f)\overline{\ell}^i, \;\; f_{,b} = \varepsilon(\overline{\delta}_i f)\overline{m}^i.$$

In [Youssef et al., 2024], we studied the anisotropic conformal change of a conic pseudo-Finsler surface (M,F) equipped with a modified Berwald frame and determined how this change affects the components of the Berwald frame (ℓ, m) of F, that is,

$$\overline{\ell}_i = e^{\phi}\,[\ell_i + \phi_{;2}\, m_i], \qquad \overline{\ell}^i = e^{-\phi}\,\ell^i, \quad (2.10)$$

$$\left.\begin{aligned} \overline{m}_i &= e^{\phi}\sqrt{\tfrac{\varepsilon}{\rho}}\, m_i, \\ \overline{m}^i &= e^{-\phi}\sqrt{\varepsilon\rho}\,[m^i - \varepsilon\phi_{;2}\ell^i]. \end{aligned}\right\} \quad (2.11)$$

$$\rho = \frac{1}{\sigma+\varepsilon-(\phi_{;2})^2} \quad (2.12)$$

Furthermore, the anisotropic conformal change of the geodesic spray coefficients, geodesic spray, Barthel connection coefficients and Berwald connection coefficients are given, respectively, by

$$\overline{G}^i = G^i + Q\,m^i + P\ell^i, \quad (2.13)$$
$$\overline{S} = S - 2(Q\,m^i + P\ell^i)\dot{\partial}_i, \quad (2.14)$$
$$\overline{G}^i_j = G^i_j + \frac{1}{F}\{2P\ell^i\ell_j + (P_{;2} - Q)\ell^i m_j + 2Q\ell_j m^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i m_j\}, \quad (2.15)$$
$$\overline{G}^i_{jk} = G^i_{jk} + \frac{1}{F^2}[(2P\ell^i + 2Qm^i)\ell_j\ell_k + \{(P_{;2} - Q)\ell^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i\}(\ell_j m_k + \ell_k m_j) + \{(\varepsilon P + P_{;2;2} - 2Q_{;2} + \varepsilon\mathcal{I}P_{;2})\ell^i + (2\varepsilon P_{;2} + \varepsilon Q + Q_{;2;2} - \varepsilon\mathcal{I}_{;2}Q - \varepsilon\mathcal{I}Q_{;2})m^i\}m_j m_k], \quad (2.16)$$

where

$$2Q = \varepsilon\rho F^2(\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2}), \quad (2.17)$$
$$2P = -\rho F^2\phi_{;2}(\phi_{;2}\phi_{,1} + \phi_{,1;2} - 2\phi_{,2}) + F^2\phi_{,1}, \quad (2.18)$$
$$2\varepsilon\phi_{;2}Q + 2P = F^2\phi_{,1}. \quad (2.19)$$

Also, we have the following identities [Youssef et al., 2025a]:

$$\frac{\phi_{;2}}{2\rho}(F^2\rho_{,1} - 2\varepsilon\rho_{;2}Q) = P_{;2} + 2\varepsilon\phi_{;2;2}Q - F^2\phi_{;2,1} + Q, \quad (2.20)$$
$$\frac{\phi_{;2}}{2\rho}(F^2\rho_{,1} - 2\varepsilon\rho_{;2}Q) = -P_{;2} - 2\varepsilon\phi_{;2}Q_{;2} + F^2\phi_{,2} + Q. \quad (2.21)$$
$$\frac{1}{2\rho}(F^2\rho_{,1} - 2\varepsilon\rho_{;2}Q) = P - \varepsilon Q_{;2} - \mathcal{I}Q. \quad (2.22)$$
$$\phi_{;2}P + P_{;2} + \varepsilon\phi_{;2}Q_{;2} - \mathcal{I}\phi_{;2}Q - F^2\phi_{,2} - Q = 0. \quad (2.23)$$

Berwald connection and dynamical covariant derivative

Definition 3.1 [Antonelli et al., 2013] *For a conic pseudo-Finsler surface (M,F) there exists a unique linear connection $\mathcal{B}\Gamma = (G^i_{jk}, G^i_j, 0)$, that satisfies the following:*

- **B1:** $y^i_{|j} = 0$ i.e. $G^i_j = G^i_{jk}y^k$,
- **B2:** $\mathcal{B}\Gamma$ is h-metric: $F_{|i} = 0$,
- **B3:** $\mathcal{B}\Gamma$ is h-symmetric: $T^i_{jk} = 0$, i.e. $G^i_{jk} = G^i_{kj}$,
- **B4:** the $(v)hv$-torsion tensor: $P^i_{jk} = 0$, i.e. $\dot{\partial}_k G^i_j = G^i_{jk}$,
- **B5:** (h) hv-torsion vanishes, i.e. $C^i_{jk} = 0$.

This connection $\mathcal{B}\Gamma = (G^i_{jk}, G^i_j, 0)$, is called Berwald connection. The horizontal and vertical covariant derivatives of X^i_j with respect to Berwald connection are given respectively by:

$$\nabla^B_{\delta_k} X^i_j = X^i_{j|k} := \delta_k X^i_j + X^r_j G^i_{rk} - X^i_r G^r_{jk},$$
$$\nabla^B_{\dot{\partial}_k} X^i_j = X^i_j|_k := \dot{\partial}_k X^i_j.$$

Proposition 3.2 *Let (M,F) be a pseudo-Finsler surface. Then, for the Berwald connection $\mathcal{B}\Gamma = (G^i_{jk}, G^i_j, 0)$, the following relations hold:*

- **(i)** $\ell^i_{|j} = 0$, $\ell_{i|j} = 0$, $m^i_{|j} = \varepsilon\mathcal{I}_{,1}m^i m_j$, $m_{i|j} = -\varepsilon\mathcal{I}_{,1}m_i m_j$,
- **(ii)** $Fm^i|_j = -\ell^i m_j - \varepsilon\mathcal{I}m^i m_j$, $Fm_i|_j = -\ell_i m_j + \varepsilon\mathcal{I}m_i m_j$,
- **(iii)** $F\ell^i|_j = \varepsilon m^i m_j$, $F\ell_i|_j = \varepsilon m_i m_j$,
- **(iv)** $(\ell_i m_j + \ell_j m_i)_{|k} = -\varepsilon\mathcal{I}_{,1}(\ell_i m_j + \ell_j m_i)m_k$.
- **(v)** $G^i_j = -F\delta_j\ell^i$.

Proof.

(i) Since $y^i = F\ell^i$, take the horizontal covariant derivative of both sides and using **B1** and **B2**, we obtain $\ell^i_{|j} = 0$. The Berwald connection is not h-metrical, then

$$g_{ij|k} = -2\mathcal{I}_{,1} m_i m_j m_k \quad (3.1)$$

Moreover, since $\ell_i = g_{ij}\ell^j$, then from (3.1) and Lemma 2.2 **(i)** along with $\ell^i_{|j} = 0$, we have

$$\ell_{i|r} = g_{ij|r}\ell^j + g_{ij}\ell^j_{|r}$$
$$= -2\mathcal{I}_{,1} m_i m_j m_r \ell^j = 0.$$

Next, differentiating $\ell_i m^i = 0$ and $g_{ij}m^i m^j = \varepsilon$ horizontally, leads to $\ell_i m^i_{|j} = 0$ and $m_i m^i_{|j} = \mathcal{I}_{,1} m_j$. Thus we get

$$m^i_{|j} = \varepsilon \mathcal{I}_{,1} m^i m_j.$$

From $m_i = g_{ir}m^r$ and (3.1), we get

$$m_{i|j} = g_{ir|j}m^r + g_{ir}m^r_{|j}$$
$$= -2\mathcal{I}_{,1} m_i m_r m_j m^r + \varepsilon \mathcal{I}_{,1} g_{ir} m^r m_j$$
$$= -\varepsilon \mathcal{I}_{,1} m_i m_j.$$

Consequently, we obtain

$$\ell^i_{|j} = 0, \quad \ell_{i|j} = 0,$$
$$m^i_{|j} = \varepsilon \mathcal{I}_{,1} m^i m_j, \quad m_{i|j} = -\varepsilon \mathcal{I}_{,1} m_i m_j.$$

(ii) From Lemma 2.2 **(iii)** and the vertical covariant derivative of $\mathcal{B}\Gamma$, we obtain

$$Fm^i|_j = F\dot{\partial}_j m^i = -\ell^i m_j - \varepsilon \mathcal{I} m^i m_j.$$
$$Fm_i|_j = F\dot{\partial}_j m_i = -\ell_i m_j + \varepsilon \mathcal{I} m_i m_j.$$

(iii) Similarly to **(ii)**, from Lemma 2.2 **(i)** and the vertical covariant derivative of $\mathcal{B}\Gamma$, we have

$$F\ell^i|_j = F\dot{\partial}_j \ell^i = \varepsilon m^i m_j.$$
$$F\ell_i|_j = F\dot{\partial}_j \ell_i = \varepsilon m_i m_j.$$

(iv) Follows directly from **(i)**.

(v) From **(i)**, we have , then

$$\ell^i_{|j} = 0, \quad \text{so} \quad \delta_j \ell^i + \ell^k G^i_{jk} = 0.$$

By **B1**,

$$0 = F\delta_j \ell^i + y^k G^i_{jk} = F\delta_j \ell^i + G^i_j.$$

Therefore,

$$G^i_j = -F\delta_j \ell^i.$$

Definition 3.3 *A nonlinear connection induces a dynamical covariant derivative ∇_D that acts on functions $f \in C^\infty(\mathcal{A})$ and Finsler vector fields $X \in \Gamma(\pi^*TM)$ as follows:*

$$\nabla_D f = y^i \delta_i f = S(f), \quad (3.2)$$
$$\nabla_D X = (\nabla X^i)\partial_i, \quad \nabla X^i = y^j \delta_j X^i + G^i_j X^j, \quad (3.3)$$

where $X = X^i \partial_i$ in local coordinates. This definition allows ∇_D to map functions and Finsler vector fields to their corresponding counterparts.

Let $T^{r_1 r_2 \dots r_p}_{s_1 s_2 \dots s_q}(x, y)$ be the components of an (p, q)-d-tensor field T. Then the dynamical covariant derivative ∇_D maps T to a tensor field with components:

$$\nabla_D T^{r_1 r_2 \dots r_p}_{s_1 s_2 \dots s_q} = y^k \delta_k T^{r_1 r_2 \dots r_p}_{s_1 s_2 \dots s_q} + G^{r_1}_m T^{m r_2 \dots r_p}_{s_1 s_2 \dots s_q} |$$
$$+ \dots + G^{r_p}_m T^{r_1 r_2 \dots m}_{s_1 s_2 \dots s_q}$$
$$-G^m_{s_1} T^{r_1 r_2 \dots r_p}_{m s_2 \dots s_q} - \dots - G^m_{s_q} T^{r_1 r_2 \dots r_p}_{s_1 s_2 \dots m}. \quad (3.4)$$

A scalar function with vanishing dynamical covariant derivative is constant along geodesics, and therefore is a first integral of the geodesic spray S.

Lemma 3.4 *In Finsler surfaces, a scalar function $f \in C^\infty(\mathcal{A})$ is constant along the geodesic, that is, it is a first integral of the geodesic spray if and only if $f_{,1} = 0$.*

Proof. A function f is constant along geodesics if its total derivative along the spray vector field vanishes:

$$\frac{df}{dt} = \nabla_D f = S(f) = 0.$$

From (2.8), we have

$$0 = y^i \delta_i f = y^i (f_{,1}\ell_i + f_{,2} m_i).$$

The conclusion follows from Lemma 2.2 **(i)**, which implies that this holds if and only if $f_{,1} = 0$.

Lemma 3.5 *The dynamical covariant derivative ∇_D, the Berwald connection ∇^B are related by*

$$y^k \nabla^B_{\delta_k} = \nabla_D.$$

We present an alternative proof of the well-known result that the dynamical covariant derivative of the Berwald frame of a conic pseudo-Finsler surface vanishes [Bucataru and Miron, 2007].

Lemma 3.6 *Let (M, f) be a conic pseudo-Finsler surface equipped with a Berwald frame (ℓ, m). Then the dynamical covariant derivatives of (ℓ, m) vanish.*

Proof. From Proposition 3.2 **(i)**, we have

$$\ell^i_{|j} = 0, \quad \ell_{i|j} = 0,$$
$$m^i_{|j} = \varepsilon \mathcal{I}_{,1} m^i m_j, \quad m_{i|j} = -\varepsilon \mathcal{I}_{,1} m_i m_j$$

Along with Lemma 3.5, we get

$$\nabla_D \ell^i = y^j \ell^i_{|j} = 0,$$
$$\nabla_D \ell_i = y^j \ell_{i|j} = 0,$$
$$\nabla_D m^i = y^j m^i_{|j} = \varepsilon F \mathcal{I}_{,1} m^i m_j \ell^j = 0,$$
$$\nabla_D m_i = y^j m_{i|j} = -\varepsilon F \mathcal{I}_{,1} m_i m_j \ell^j = 0.$$

Hence, the dynamical covariant derivative of the Berwald frame vanishes.

Under the anisotropic conformal transformation $F \longrightarrow \overline{F} = e^{\phi}F$, we define the dynamical covariant derivatives in $(M, \overline{F})$ as

$$\overline{\nabla}_D f = y^i \, \overline{\delta}_i f = \overline{S}(f), \quad (3.5)$$

$$\overline{\nabla}_D X^i_j = y^k\,\overline{\delta}_k X^i_j + \overline{G}^i_k X^k_j - \overline{G}^k_j X^i_k, \tag{3.6}$$

where $\overline{S}$ and $\overline{G}^i_j$ are given by equations (2.14) and (2.15), respectively, and the horizontal derivative $\overline{\delta}_j$ is defined by

$$F\overline{\delta}_j = F\delta_j - (2P\ell^i\ell_j + (P_{;2} - Q)\ell^i m_j + 2Q\ell_j m^i + (\varepsilon P + Q_{;2} - \varepsilon \mathcal{I} Q)m^i m_j)\dot{\partial}_i. \tag{3.7}$$

Proposition 3.7 *Let the conic pseudo-Finsler metric F be anisotropically conformal to $\overline{F} = e^{\Phi}F$. Then we have:*

$$\overline{\nabla}_D f = \nabla_D f - \frac{2}{F}(f_{;1}P + \varepsilon f_{;2}Q). \tag{3.8}$$

$$\overline{\nabla}_D X^i = \nabla_D X^i + \frac{1}{F}[(-2P\,A_{;1} - 2\varepsilon Q\,A_{;2} + \varepsilon QB + 2P\,A + \varepsilon P_{;2}B)\ell^i + (-2P\,B_{;2} - 2\varepsilon Q\,B_{;2} + PB + \varepsilon Q_{;2}B + Q\mathcal{I}B)m^i]. \tag{3.9}$$

Proof. Firstly, from (3.5), (2.14) and (2.8) for a scalar function $f \in C^\infty(M)$, we have

$$\overline{\nabla}_D f = \big(S - 2(P\,\ell^i + Q\,m^i)\dot{\partial}_i\big)(f) = S(f) - 2(P\,\ell^i + Q\,m^i)(f_{;1}\ell_i + f_{;2}m_i)$$

By using Lemma 2.2 **(i)**, we obtain

$$\overline{\nabla}_D f = \nabla_D f - \frac{2}{F}(f_{;1}P + \varepsilon f_{;2}Q)$$

Secondly, on a Finsler surface, any tangent vector can be expressed in the form

$X^i = A(x,y)\ell^i + B(x,y)m^i$, with $A, B \in C^\infty(TM)$.

Consequently, From (2.8) and Lemma 2.2 **(ii)** and **(iii)**, we get

$$F\dot{\partial}_j X^i = F\dot{\partial}_j\big(A\,\ell^i + B\,m^i\big) = A_{;1}\ell^i\ell_j + \big(A_{;2} - B\big)\ell^i m_j + B_{;1}m^i\ell_j + (B_{;2} + \varepsilon A - \varepsilon\mathcal{I}B)m^i m_j. \tag{3.10}$$

According to (3.6), we get

$$\overline{\nabla}_D X^i = y^k\,\overline{\delta}_k X^i + \overline{G}^i_k X^k = \overline{S}(X^i) + \overline{G}^i_k X^k = S(X^i) - 2(P\,\ell^j + Q\,m^j)\,\dot{\partial}_j(X^i + G^i_k X^k + \frac{1}{F}(2P\ell^i\ell_k + (P_{;2} - Q)\ell^i m_k + 2Q\ell_k m^i + (\varepsilon P + Q_{;2} - \varepsilon\mathcal{I}Q)m^i m_k)(A\,\ell^k + B\,m^k)$$

From (3.10) and Lemma 2.2 **(i)**, we obtain

$$\overline{\nabla}_D X^i = \nabla_D X^i + \frac{1}{F}[(-2P\,A_{;1} - 2\varepsilon Q\,A_{;2} + \varepsilon QB + 2P\,A + \varepsilon P_{;2}B)\ell^i + (-2P\,B_{;2} - 2\varepsilon Q\,B_{;2} + PB + \varepsilon Q_{;2}B + Q\mathcal{I}B)m^i].$$

Chern-Rund connection and anisotropic change

Definition 4.1 [Antonelli et al., 2013] *For a conic pseudo-Finsler surface (M, F) there exists a unique linear connection $\mathcal{R}\Gamma = (\overset{\star}{\Gamma}{}^i_{jk}, G^i_j, 0)$, that satisfies the following:*

R1: $y^i_{\overset{\star}{|} j} = 0$, i.e. $G^i_j = \overset{\star}{\Gamma}{}^i_{jk}y^k$,,

R2: $\mathcal{R}\Gamma$ is h-metric: $g_{ij\overset{\star}{|}k} = 0$,

R3: $\mathcal{R}\Gamma$ is h-symmetric: $T^i_{jk} = 0$, i.e. $\overset{\star}{\Gamma}{}^i_{jk} = \overset{\star}{\Gamma}{}^i_{kj}$,

R4: (h) hv-torsion vanishes, i.e. $C^i_{jk} = 0$,

This linear connection is called Chern (Rund) connection. The geometric objects associated with Chern connection will be marked by a star.

The horizontal and vertical covariant derivatives of X^i_j w.r.t. Chern connection are given respectively by:

$$X^i_{j\overset{\star}{|}k} := \delta_k X^i_j + X^r_j\overset{\star}{\Gamma}{}^i_{rk} - X^i_r\overset{\star}{\Gamma}{}^r_{jk}, \quad X^i_j\overset{\star}{|}_k := \dot{\partial}_k X^i_j.$$

Lemma 4.2 *Let (M, F) be a conic pseudo-Finsler surface. Then the modified Berwald frame satisfies*

$$\ell^i\,(\delta_k\ell_j) + \varepsilon\,m^i(\delta_k m_j) = -\ell_j(\delta_k\ell^i) - \varepsilon\,m_j(\delta_k m^i)$$

Proof. Since we have

$$\delta^i_j = \ell^i\ell_j + \varepsilon m^i m_j.$$

taking the horizontal derivative yields

$$0 = \delta_k(\delta^i_j) = \delta_k(\ell^i\ell_j + \varepsilon m^i m_j) = \ell_j(\delta_k\ell^i) + \ell^i\,(\delta_k\ell_j) + \varepsilon\,m_j(\delta_k m^i) + \varepsilon\,m^i(\delta_k m_j)$$

Consequently, we have

$$\ell^i\,(\delta_k\ell_j) + \varepsilon\,m^i(\delta_k m_j) = -\ell_j(\delta_k\ell^i) - \varepsilon\,m_j(\delta_k m^i)$$

Proposition 4.3 *Let (M, F) be a pseudo-Finsler surface. Then, for the Chern-Rund connection $\mathcal{R}\Gamma = (\overset{\star}{\Gamma}{}^i_{jk}, G^i_j, 0)$, the following relations hold:*

- **(i)** $\ell^i_{\overset{\star}{|} j} = 0$, $\ell_{i\overset{\star}{|}j} = 0$, $m^i_{\overset{\star}{|} j} = 0$, $m_{i\overset{\star}{|}j} = 0$,
- **(ii)** $Fm^i\overset{\star}{|}_j = -\ell^i m_j - \varepsilon\mathcal{I}m^i m_j$,
  $Fm_i\overset{\star}{|}_j = -\ell_i m_j + \varepsilon\mathcal{I}m_i m_j$,
- **(iii)** $F\ell^i\overset{\star}{|}_j = \varepsilon m^i m_j$, $F\ell_i\overset{\star}{|}_j = \varepsilon m_i m_j$,
- **(iv)** $F\,(\ell_i m_j - \ell_j m_i)\overset{\star}{|}k = \varepsilon\,\mathcal{I}(\ell_i m_j - \ell_j m_i)m_k$.
- **(v)** $\overset{\star}{\Gamma}{}^i_{kj} = \ell^i\delta_k\ell_j + \varepsilon m^i\delta_k m_j$.

Proof.

(i) Since the Finsler metric satisfies $F^2 = g_{ij}y^i y^j$, it follows from **R1** and **R2** that

$$F_{\overset{\star}{|}i} = 0. \tag{4.1}$$

Since $y^i = F\,\ell^i$, differentiating horizontally gives

$$0 = y^i_{\overset{\star}{|}j} = F_{\overset{\star}{|}j}\ell^i + F\,\ell^i_{\overset{\star}{|}j}.$$

By (4.1), we have $\ell^i_{\overset{\star}{|}j} = 0$.

Next, taking the horizontal covariant derivative of the covector $\ell_i = g_{ij}\ell^j$, we obtain

$$\ell_{i\overset{\star}{|}r} = g_{ij\overset{\star}{|}r}\,\ell^j + g_{ij}\,\ell^j_{\overset{\star}{|}r}.$$

By **R2** and $\ell^j_{\overset{\star}{|}r} = 0$, we have $\ell_{i\overset{\star}{|}r} = 0$.

Moreover, differentiating $\ell_i m^i = 0$ and $g_{ij}m^i m^j = \varepsilon$ horizontally, we get $\ell_i m^i_{\overset{\star}{|}j} = 0$ and $m_i m^i_{\overset{\star}{|}j} = 0$. Thus we get $m^i_{\overset{\star}{|}j} = 0$.

Finally, from $m_i = g_{ir}m^r$ and **R2**, we get $m_{i\overset{\star}{|}j} = g_{ir\overset{\star}{|}j}m^r + g_{ir}m^r_{\overset{\star}{|}j} = 0$, where $m^i_{\overset{\star}{|}j} = 0$.

(ii) From Lemma 2.2 **(iii)** and the vertical covariant derivative of $\mathcal{R}\Gamma$, we obtain

$$F m^i\overset{\star}{|}_j = F\dot{\partial}_j m^i = -\ell^i m_j - \varepsilon\mathcal{J} m^i m_j.$$

$$F m_i\overset{\star}{|}_j = F\dot{\partial}_j m_i = -\ell_i m_j + \varepsilon\mathcal{J} m_i m_j.$$

(iii) Similarly to **(ii)**, from Lemma 2.2 **(i)** and the vertical covariant derivative of $\mathcal{R}\Gamma$, we have

$$F\ell^i\overset{\star}{|}_j = F\dot{\partial}_j\ell^i = \varepsilon m^i m_j.$$

$$F\ell_i\overset{\star}{|}_j = F\dot{\partial}_j\ell_i = \varepsilon m_i m_j.$$

(iv) Follows directly from **(ii)** and **(iii)**.

(v) From **(i)**, we have $\ell_{j\overset{\star}{|}k} = 0$ and $m_{j\overset{\star}{|}k} = 0$, then

$$\delta_k\ell_j - \ell_r\overset{\star}{\Gamma}{}^r_{kj} = 0. \tag{4.2}$$

$$\delta_k m_j - m_r\overset{\star}{\Gamma}{}^r_{kj} = 0. \tag{4.3}$$

Multiply (4.2) by ℓ^i and (4.3) by εm^i, then solve these equations to find $\overset{\star}{\Gamma}{}^i_{jk}$(taking into account $\delta^i_j = \ell^i\ell_j + \varepsilon m^i m_j$), we get

$$\overset{\star}{\Gamma}{}^i_{kj} = \ell^i\delta_k\ell_j + \varepsilon m^i\delta_k m_j$$

Remark 4.4

(a) From Proposition *4.3* ***(i)***, we have $\ell^i_{\overset{\star}{|}j} = 0$ and $m^i_{\overset{\star}{|}j} = 0$. Therefore, we obtain the following equations:

$$\delta_j\ell^i + \ell^k\overset{\star}{\Gamma}{}^i_{jk} = 0, \tag{4.4}$$

$$\delta_j m^i + m^k\overset{\star}{\Gamma}{}^i_{jk} = 0. \tag{4.5}$$

By multiplying (4.4) by ℓ_r and (4.5) by εm_r, and solving these equations to find $\overset{\star}{\Gamma}{}^i_{jk}$ (taking into account $\delta^i_j = \ell^i\ell_j + \varepsilon m^i m_j$), we get

$$\overset{\star}{\Gamma}{}^i_{jk} = -\ell_j\delta_k\ell^i - \varepsilon m_j\delta_k m^i.$$

(b) The symmetry of $\overset{\star}{\Gamma}{}^i_{jk}$ (i.e., $\overset{\star}{\Gamma}{}^i_{jk} = \overset{\star}{\Gamma}{}^i_{kj}$) is obtained from ***(a)***, Lemma *4.2*, and Proposition *4.3* ***(v)***.

Proposition 4.5 *Let (M, F) be conic pseudo-Finsler surface and (2.9) be the anisotropic conformal transformation.The transformation of the horizontal coefficients of Chern connection are given by*

$$\begin{aligned}
F^2\,\overline{\overset{\star}{\Gamma}}{}^i_{jk} &= F^2\overset{\star}{\Gamma}{}^i_{jk} + 2P\ell^i\ell_j\ell_k + 2Qm^i\ell_j\ell_k\\
&+(P_{;2} - Q)(\ell^i m_j\ell_k + \ell^i\ell_j m_k)\\
&+\Big(F^2\phi_{;2}\frac{\rho_{,2}}{2\rho} + F^2\phi_{;2,2} - (1 + \varepsilon\phi_{;2;2}\\
&+\varepsilon\phi_{;2}\frac{\rho_{;2}}{2\rho})(\varepsilon P + Q_{;2} - \varepsilon\mathcal{J}Q)\Big)\ell^i m_j m_k\\
&+(\varepsilon P + Q_{;2} - \varepsilon\mathcal{J}Q)(m^i\ell_j m_k + m^i m_j\ell_k\\
&+\Big(\varepsilon F^2\phi_{,2} - \varepsilon F^2\frac{\rho_{,2}}{2\rho} - \Big(\varepsilon\mathcal{J} + \phi_{;2} - \frac{\rho_{;2}}{2\rho}\Big)\\
&\qquad(\varepsilon P + Q_{;2} - \varepsilon\mathcal{J}Q)\Big)m^i m_j m_k.
\end{aligned}$$

Proof. The anisotropic transformation of Γ^i_{jk}, in view of Proposition 4.3 **(v)**, is given by

$$F^2\overline{\Gamma}{}^i_{jk} = -F^2\overline{\ell}_j\overline{\delta}_k\overline{\ell}{}^i - \varepsilon F^2\overline{m}_j\overline{\delta}_k\overline{m}{}^i \tag{4.6}$$

Firstly, using (2.10), (2.1) and (3.7), we can find the anisotropic transformation of the first term of (4.6).

$$\begin{aligned}
F^2\overline{\ell}{}^i\,\overline{\delta}_k\overline{\ell}_j &= e^{-\phi}\ell^i(F^2\delta_k - [2P\ell^r\ell_k + (P_{;2}\\
&\quad -Q)\ell^r m_k + 2Q\ell_k m^r + (\varepsilon P + Q_{;2}\\
&\quad -\varepsilon\mathcal{J}Q)m^r m_k]F\dot{\partial}_r)(e^{\phi}(\ell_j + \phi_{;2}m_j))
\end{aligned}$$

From (2.8), we get

$$\begin{aligned}
F^2\overline{\ell}{}^i\,\overline{\delta}_k\overline{\ell}_j &= e^{-\phi}\ell^i(e^{\phi}F^2[\delta_k\ell_j + (\phi_{,1}\ell_k\\
&\quad +\phi_{,2}m_k)(\ell_j + \phi_{;2}m_j) + \phi_{;2}\delta_k m_j\\
&\quad + (\phi_{;2,1}\ell_k + \phi_{;2,2}m_k)m_j]\\
&\quad - e^{\phi}[2P\ell^r\ell_k + (P_{;2} - Q)\ell^r m_k\\
&\quad + 2Q\ell_k m^r + (\varepsilon P + Q_{;2}\\
&\quad - \varepsilon\mathcal{J}Q)m^r m_k]\,[\phi_{;2}m_r(\ell_j + \phi_{;2}m_j)\\
&\quad + \varepsilon m_j m_r + \phi_{;2;2}m_j m_r + \phi_{;2}(-\ell_j m_r\\
&\quad + \varepsilon\mathcal{J}m_j m_r)])
\end{aligned}$$

By using Lemma 2.2 **(i)**, we obtain

$$\begin{aligned}
F^2\overline{\ell}{}^i\,\overline{\delta}_k\overline{\ell}_j &= F^2\ell^i\delta_k\ell_j + F^2\phi_{;2}\ell^i\delta_k m_j\\
&\quad +F^2\ell^i[(\phi_{,1}\ell_k + \phi_{,2}m_k)(\ell_j + \phi_{;2}m_j)\\
&\quad +(\phi_{;2,1}\ell_k + \phi_{;2,2}m_k)m_j] - \ell^i[2\varepsilon Q\ell_k
\end{aligned}$$

$$+(P+\varepsilon Q_{;2}-\mathcal{I}Q)m_k][\phi_{;2}(\ell_j+\phi_{;2}m_j)+\varepsilon m_j+\phi_{;2;2}m_j+\phi_{;2}(-\ell_j+\varepsilon\mathcal{I}m_j)]$$

From the formula of ρ (2.12), we get

$$F^2\overline{\ell}^i\,\overline{\delta}_k\overline{\ell}_j=F^2\ell^i\delta_k\ell_j+F^2\phi_{;2}\ell^i\delta_k m_j+F^2\ell^i[(\phi_{,1}\ell_k+\phi_{,2}m_k)(\ell_j+\phi_{;2}m_j)+(\phi_{;2,1}\ell_k+\phi_{;2,2}m_k)m_j]-\ell^i[2\varepsilon Q\ell_k+(P+\varepsilon Q_{;2}-\mathcal{I}Q)m_k]\frac{1}{\rho}\,m_j$$

$$=F^2\ell^i\delta_k\ell_j+F^2\phi_{;2}\ell^i\delta_k m_j+F^2(\phi_{,1}\ell_k+\phi_{,2}m_k)\ell^i\ell_j+[F^2\phi_{;2}\phi_{,1}+F^2\phi_{;2,1}-\frac{2\varepsilon}{\rho}Q]\ell^i m_j\ell_k+[F^2\phi_{;2}\phi_{,2}+F^2\phi_{;2,2}-\frac{1}{\rho}(P+\varepsilon Q_{;2}-\mathcal{I}Q)]\ell^i m_j m_k \tag{4.7}$$

Secondly, we find the second term of (4.6), by (2.10) and (2.11) along with (3.7), we have

$$\varepsilon F^2\,\overline{m}^i\,\overline{\delta}_k\,\overline{m}_j=\varepsilon(e^{-\phi}\sqrt{\varepsilon\rho}(m^i-\varepsilon\phi_{;2}\;\ell^i))(F^2\delta_k-[2P\ell^r\ell_k+(P_{;2}-Q)\ell^r m_k+2Q\ell_k m^r+(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q)m^r m_k]F\dot{\partial}_r)e^{\phi}\sqrt{\frac{\varepsilon}{\rho}}m_j.$$

From (2.8), we get

$$\varepsilon F^2\,\overline{m}^i\,\overline{\delta}_k\,\overline{m}_j=\varepsilon\sqrt{\varepsilon\rho}\big(m^i-\varepsilon\phi_{;2}\;\ell^i\big)\Bigg(F^2\sqrt{\frac{\varepsilon}{\rho}}m_j(\phi_{,1}\ell_k+\phi_{,2}m_k-\varepsilon F^2\sqrt{\varepsilon\rho}\,m_j\frac{(\rho_{,1}\ell_k+\rho_{,2}m_k)}{2\rho^2}+F^2\sqrt{\frac{\varepsilon}{\rho}}\delta_k m_j-[2P\ell^r\ell_k+(P_{;2}-Q)\ell^r m_k+2Q\ell_k m^r+(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q)m^r m_k]\Bigg[\sqrt{\frac{\varepsilon}{\rho}}\phi_{;2}m_j m_r-\varepsilon\sqrt{\varepsilon\rho}\,m_j\frac{\rho_{,2}m_r}{2\rho^2}+\sqrt{\frac{\varepsilon}{\rho}}(-\ell_j m_r+\varepsilon\mathcal{I}m_j m_r)\Bigg]\Bigg)$$

By applying Lemma 2.2 **(i)**, we can rewrite the expression as follows:

$$\varepsilon F^2\,\overline{m}^i\,\overline{\delta}_k\,\overline{m}_j=\varepsilon F^2 m^i\delta_k m_j-F^2\phi_{;2}\ell^i\delta_k m_j-2\varepsilon\phi_{;2}Q\ell^i\ell_j\ell_k-\phi_{;2}(P+\varepsilon Q_{;2}-\mathcal{I}Q)\ell^i\ell_j m_k+2Qm^i\ell_j\ell_k+(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q)m^i\ell_j m_k+(\varepsilon F^2\phi_{,2}-\varepsilon F^2\frac{\rho_{,2}}{2\rho}-(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q)(\phi_{;2}-\frac{\rho_{,2}}{2\rho}+\varepsilon\mathcal{I}))m^i m_j m_k+(\varepsilon F^2\phi_{,1}-\varepsilon F^2\frac{\rho_{,1}}{2\rho}-2Q(\phi_{;2}-\frac{\rho_{,2}}{2\rho}+\varepsilon\mathcal{I}))m^i m_j\ell_k+(-F^2\phi_{;2}\phi_{,1}+F^2\phi_{;2}\frac{\rho_{,1}}{2\rho}+2\varepsilon\phi_{;2}Q(\phi_{;2}-\frac{\rho_{,2}}{2\rho}+\varepsilon\mathcal{I}))\ell^i m_j\ell_k+(-F^2\phi_{;2}\phi_{,2}+F^2\phi_{;2}\frac{\rho_{,2}}{2\rho}+\phi_{;2}(P+\varepsilon Q_{;2}-\mathcal{I}Q)(\phi_{;2}-\frac{\rho_{,2}}{2\rho}+\varepsilon\mathcal{I}))\ell^i m_j m_k \tag{4.8}$$

From (4.7) and (4.8), we determine the anisotropic conformal transformation of the horizontal coefficients of Cartan connection

$$F^2\,\overline{\overset{\star}{\Gamma}}{}^i_{jk}=F^2\overset{\star}{\Gamma}{}^i_{jk}+\big(F^2\phi_{,1}-2\varepsilon Q\phi_{;2}\big)\ell^i\ell_j\ell_k+\Big(F^2\phi_{,2}-\varepsilon\phi_{;2}\big(\varepsilon P+Q_{;2}-\ \varepsilon\mathcal{I}Q\big)\Big)\ell^i\ell_j m_k+(F^2\phi_{;2}\frac{\rho_{,1}}{2\rho}-\varepsilon\phi_{;2}\frac{\rho_{;2}}{\rho}Q+F^2\phi_{;2,1}-2Q-2\varepsilon Q\phi_{;2;2})\ell^i m_j\ell_k+(F^2\phi_{;2}\frac{\rho_{,2}}{2\rho}+F^2\phi_{;2,2}-(1+\varepsilon\phi_{;2;2}+\varepsilon\phi_{;2}\frac{\rho_{;2}}{2\rho})(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q))\ell^i m_j m_k+2Qm^i\ell_j\ell_k+(\varepsilon F^2\phi_{,1}-\varepsilon F^2\frac{\rho_{,1}}{2\rho}+\frac{\rho_{;2}}{\rho}Q-2\varepsilon\mathcal{I}Q-2\phi_{;2}Q)m^i m_j\ell_k+(\varepsilon F^2\phi_{,2}-\varepsilon F^2\frac{\rho_{,2}}{2\rho}-(\varepsilon\mathcal{I}+\phi_{;2}-\frac{\rho_{;2}}{2\rho})(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q))m^i m_j m_k+(\varepsilon P+Q_{;2}-\varepsilon\mathcal{I}Q)m^i\ell_j m_k.$$

From (2.19) and (2.20)- (2.22) the formula of $\overline{\overset{\star}{\Gamma}}{}^i_{jk}$ can be obtained.

Proposition 4.6 *Let the conic pseudo-Finsler metric F be anisotropically conformal to $\overline{F}=e^{\phi}F$. Then the Landsberg tensor of $\overline{F}$ is given by*

$$\overline{L}{}^i_{jk}=L^i_{jk}+\frac{1}{F^2}[((\varepsilon P+P_{;2;2}-2Q_{;2}+\varepsilon\,\mathcal{I}P_{;2})\,\ell^i+(2\varepsilon P_{;2}+\varepsilon Q+Q_{;2;2}-\varepsilon\,\mathcal{I}_{;2}Q-\varepsilon\,\mathcal{I}Q_{;2})\,m^i)m_j m_k\ -(F^2\,\phi_{;2}\,\frac{\rho_{,2}}{2\rho}+F^2\,\phi_{;2,2}-(1+\varepsilon\,\phi_{;2;2}+\varepsilon\,\phi_{;2}\,\frac{\rho_{;2}}{2\rho})(\varepsilon P+Q_{;2}-\varepsilon\,\mathcal{I}Q))\ell^i m_j m_k\ -(\varepsilon F^2\,\phi_{,2}-\varepsilon F^2\,\frac{\rho_{,2}}{2\rho}-(\varepsilon\,\mathcal{I}+\phi_{;2}-\frac{\rho_{;2}}{2\rho})(\varepsilon P+Q_{;2}-\varepsilon\,\mathcal{I}Q))m^i m_j m_k].$$

Proof. Let (M,F) be a conic pseudo-Finsler surface equipped with the Chern connection

$\mathcal{R}\Gamma = (\overset{\star}{\Gamma}{}^i_{jk}, G^i_j, 0)$ and the Berwald connection $\mathcal{B}\Gamma = (G^i_{jk}, G^i_j, 0)$. The Landsberg tensor is defined as the difference between the horizontal coefficient of the two connections, consequently,

$$L^i_{jk} := G^i_{jk} - \overset{\star}{\Gamma}{}^i_{jk}.$$

Under the given anisotropic transformation, the transformed Landsberg tensor is given by

$$\overline{L}^i_{jk} = \overline{G}^i_{jk} - \overline{\overset{\star}{\Gamma}}{}^i_{jk}.$$

From (2.16) and Proposition 4.5 we get

$$\begin{aligned}\overline{L}^i_{jk} = L^i_{jk} + \frac{1}{F^2}[((\varepsilon P + P_{;2;2} - 2Q_{;2} \\ +\varepsilon\, \mathcal{I} P_{;2})\, \ell^i + (2\varepsilon P_{;2} + \varepsilon Q + Q_{;2;2} \\ -\varepsilon\, \mathcal{I}_{;2} Q - \varepsilon\, \mathcal{I} Q_{;2})\, m^i) m_j m_k \\ -(F^2\, \phi_{;2}\, \frac{\rho_{,2}}{2\rho} + F^2\, \phi_{;2,2} - (1 + \varepsilon\, \phi_{;2;2} \\ +\varepsilon\, \phi_{;2}\, \frac{\rho_{;2}}{2\rho})(\varepsilon P + Q_{;2} - \varepsilon\, \mathcal{I} Q)) \ell^i m_j m_k \\ -(\varepsilon F^2\, \phi_{,2} - \varepsilon F^2\, \frac{\rho_{,2}}{2\rho} - (\varepsilon\, \mathcal{I} + \phi_{;2} \\ -\frac{\rho_{;2}}{2\rho})(\varepsilon P + Q_{;2} - \varepsilon\, \mathcal{I} Q)) m^i m_j m_k].\end{aligned}$$

Proposition 4.7 *Let (M, F) be a conic pseudo-Finsler metric and (2.9) be the proper anisotropic conformal transformation, provided that the conformal factor is horizontally constant. Then the property of being Landsbergian is preserved if and only if $\phi_{;2,2} = 0$.*

Proof. As, if the conformal factor is horizontally constant ($\phi_{,1} = \phi_{,2} = 0$), that is $P = Q = 0$ [Youssef et al., 2024, Theorem 4.11]. From Proposition 4.6, we have

$$\begin{aligned}\overline{L}^i_{jk} = L^i_{jk} + \frac{1}{F^2}[-(F^2\, \phi_{;2}\, \frac{\rho_{,2}}{2\rho} \\ +F^2\, \phi_{;2,2}) \ell^i m_j m_k + \varepsilon F^2\, \frac{\rho_{,2}}{2\rho} m^i m_j m_k].\end{aligned}$$

Consequently the Landsbergian property is preserved under the anisotropic conformal transformation if and only if

$$\phi_{;2}\, \frac{\rho_{,2}}{2\rho} + \phi_{;2,2} = 0, \qquad \frac{\rho_{,2}}{2\rho} = 0.$$

Since (2.9) is proper i.e. $\phi_{;2} \neq 0$, then the Landsbergian property is preserved if and only if $\phi_{;2,2} = 0$.

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الملخص العربي

عنوان البحث: إتصالات بيرفلد وتشيرن تحت التحويلات التشاكلية الاتجاهية على الأسطح الفينسلرية

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تعتمد هذه الورقة على استكمال عملنا السابق حول التحويلات التشاكلية الاتجاهية

$$\boldsymbol{F(x,y) \to \bar{F}(x,y) = e^{\varphi(x,y)}F(x,y)}$$

الهدف الأساسي من هذه الدراسة هو دراسة سلوك اتصال بيرفلد، الذي يحدد مدى انحراف هندسة فينسلر عن الهندسة الريمانية، وكذلك تم دراسة اتصال تشيرن-روند على أسطح فينسلر. وتم دراسة تاثير التحويل التشاكلى الاتجاهي على هذين الاتصالين بوجه الخصوص، ومن ثم نستنتج ممتد لاندسبيرج لدالة فنسلر $\bar{F}$ المحولة من خلال التعبير عنه من حيث الفرق بين المعاملات الأفقية لاتصالات بيروالد وتشيرن–روند. وبالتالي، اوجدنا الشروط الضرورية والكافية التي يتم بموجبها الحفاظ على خاصية لاندسبيرجيان في ظل هذا التحول الاتجاهى حيث ان هذا الشرط يعتمد على عامل التحويل الاتجاهى. يوفر هذا النهج رؤية جديدة للتفاعل بين التحول الاتجاهى والهندسة الجوهرية لأسطح فينسلر.