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HOMOTOPY ANALYSIS INTEGRAL TRANSFORM METHOD FOR THE SOLUTIONS OF FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The main thrust of this research is to propose a reliable method for the solution of a class of fractional order integro-differential equations with difference kernel. The integro-differential equations considered are both linear and nonlinear type with the fractional order derivative interpreted in Caputo sense. The proposed method combined Shehu transform with the Homotopy Analysis Method. The essence of the HAM is to overcome any nonlinearity that may be encountered in the problem with the aid of Homotopy derivative, while Shehu transform is chosen as result of the unique advantage that it handles bothe constant and variable coefficients problems, unlike the Laplace transform. The Homotopy Analysis Integral Transform Method (HAITM) developed is applied to some problems in the literature and the results are either the exact solution (when such exits) or at the minimum in truncated series which in all cases agree with those in the literature. The results are presented in tabular form, as well as in 2D graphs. The computations are implemented in Mathematica 13.3.

1. Introduction

Mathematical modeling is essential for understanding and predicting real-world phenomena across various disciplines, including physics, astronomy, chemistry, biology, economics, and engineering. These models help researchers analyze complex systems, explore theoretical concepts, and develop practical applications. Traditionally, integer-order ordinary and partial differential equations have been the foundation for modeling dynamic processes in these fields, providing well-established methodologies for problem-solving and analysis [1]. However, many natural and

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engineered systems exhibit complexities that integer-order models cannot fully capture. These complexities often arise from memory effects, non-local interactions, and anomalous diffusion, which require more advanced mathematical tools for accurate characterization. To address these limitations, researchers have increasingly adopted fractional differential equations, which extend classical models by incorporating derivatives of non-integer order. Unlike integer-order equations, fractional differential equations account for hereditary properties and long-range dependencies, making them particularly useful for modeling complex systems and control processes [2].

Fractional calculus has recently gained significant attention for its effectiveness in modeling anomalous diffusion, viscoelastic behavior, control systems, and diverse biological and economic phenomena, surpassing the capabilities of traditional methods [3, 4, 5, 6]. The development of fractional calculus is credited to several key mathematicians, including Gottfried Wilhelm Leibniz, Joseph Liouville, and Bernhard Riemann. Initially viewed as a theoretical extension of classical calculus [7, 8], its practical applications have only recently been explored extensively. Studies have demonstrated the growing significance of fractional calculus in modeling complex physical phenomena. For example, Meng [9] used fractional calculus to study the nonlinear viscoelastic and dielectric properties of ferroelectric polymer composites, demonstrating its ability to capture detailed material characteristics. Similarly, Chauhan, Bansal, and Sircar [10] applied a fractional framework to analyze the stability of viscoelastic subdiffusive channel flows, providing deeper insights into flow behavior. Additionally, Di Paola, Reddy, and Ruocco [11] explored its role in formulating the viscoelastic Reddy beam. These studies showcase the adaptability and effectiveness of fractional calculus in solving problems across various scientific and engineering fields. Despite its broad applications, researchers such as Oloniju *et al.*[12], Mohammed [13], and Maitama and Zhao [14] have noted that solving fractional differential equations analytically remains challenging due to their non-local properties. Consequently, various analytical and semi-analytical techniques have been developed to overcome the said challenges.

Integral transforms, such as Laplace, Fourier, Elzaki, Sumudu, Shehu transforms, etc, have also been widely applied to simplify and solve linear integral equations. Recently, the Homotopy Analysis Integral Transform Method (HAITM) has emerged as an effective approach, combining the strengths of homotopy analysis and integral transforms to provide accurate and efficient solutions to nonlinear integral equations. This study explores the HAITM for solving fractional-order integro-differential equations. Unlike perturbation-based methods, HAITM does not require small parameters, making it a more flexible and robust framework for handling nonlinear problems.

This paper is organized in such a way that Section 2 discusses the literature review, brief discussion on Shehu transform and homotopy analysis method. In Section 3, we present the statement of problem for the linear Volterra integro-differential equations (VIDEs), the methodology and numerical examples for the class of problem considered there. The statement of problem for the nonlinear VIDEs, methodology and numerical examples are presented in Section 4. The results that are presented in both tabular and graphical forms, discussion of results, conclusion and future research are in Sections 5.

2. Literature Review

Researchers have proposed various approaches to solving problems in fractional calculus. Recent studies have highlighted the effectiveness of integral transform techniques, such as Fourier and Laplace transforms, in solving fractional order differential equations (FDEs) in some domains. For example, Rahimkhani and Ordokhani [16] combined the Hahn wavelets collocation method with the Laplace transform to address fractional integro-differential equations, demonstrating the practicality of this approach. Similarly, Uchenna (2024) applied the Fourier transform method to complex variables in non-homogeneous FDEs, providing explicit solutions and showcasing the method's effectiveness. Additionally, Boiti and Franceschi (2024) explored generalizations of the Fourier transform to extend its applicability to fractional models, offering a broader framework for solving such equations.

These methods offer efficient alternatives to exact analytical solutions, producing rapidly convergent series solutions with reduced computational complexity, making them valuable tools in fractional calculus. Similarly, Yang and Wang [20] developed an improved version of HPM, which was shown through various test examples to be a powerful approach for solving local fractional differential equations while avoiding cumbersome computations. Ishag *et al.* [21] applied HPM to nonlinear fractional reaction-diffusion systems, demonstrating its rapid convergence to exact solutions and its ability to effectively handle multi-dimensional problems.

In this present work, the advantage in the applications of integral transform, especially the Shehu transform which has a unique property of solving both constant and variable coefficients problems together with the fact that it generalizes the earlier transforms such as Laplace and Sumudu, is incorporated into Homotopy Analysis method to handle Volterra Integro-differential Equations (VIDEs). The method proposed in this work is Homotopy Analysis Integral Transform Method (HAITM), and it has been successfully applied to the nonlinear as well as linear VIDEs seamlessly with solutions that compared favourably with results in the existing literature.

2.1. **The Shehu Transform. Definition** [27, 33]

The Shehu transform of a function $g(x)$ which is of an exponential order is defined as

$$\begin{aligned}\mathbb{S}\left[g(t)\right] &= \int_0^\infty exp\left(\frac{-s}{u}t\right) g(t)dt = G(s,u).\\ \mathbb{S}\left[g(t)\right] &= \lim_{\eta\to\infty}\int_0^\eta exp\left(\frac{-s}{u}t\right) g(t)dt; \quad s>0, u>0.\end{aligned} \tag{1}$$

If the limit of the integral in (1) exists, it converges, otherwise it diverges.

The inverse of Shehu transform is given by

$$\mathbb{S}^{-1}\left[G(s,u)\right] = g(t), \ \text{ for } t \geq 0. \tag{2}$$

Which can as well be stated as

$$v(x) = \mathbb{S}^{-1}\left[G(s,u)\right] = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{u}exp\left(\frac{st}{u}\right)G(s,u)ds, \tag{3}$$

where s and u are the Shehu transform variables, and η is a real constant and the integral in (1) is taken along $s=\eta$ in the complex plane $s=x+iy$.

The present study explored the advantage of Shehu transform over some other transforms such as Laplse, due to the fact that it does not have restriction in application to constant coefficients problems. For examples on the applications of Shehu transform to variable coefficient problems see [33]. Another superiority of the transform adopted in this work is that it generalizes both Laplace and Sumudu transforms [14, 33].

2.2. **Homotopy Analysis Method.** Homotopy Analysis Method (HAM) introduced by Liao [26] is an essential tool for solving both linear and nonlinear, ordinary and partial differential differential equations. Since the introduction of the method, its scope of application has been expanded to cover linear and nonlinear fractional order problems, including systems of differential equations [34]. HAM has two deformation equations referred to as zeroth and nth order deformation equations. The advantage in the use of latter is the reduction in volume of computation. HAM has a parameter, h which is referred to as the convergence control parameter. This parameter takes the value of -1 or 1 depending on which gives quick convergence. For details on the algorithm of HAM and its implementation, interested reader should consult [5, 26, 34]. The present work leveraged on the use of homotopy derivatives in overcoming the nonlinear terms encountered in the problems considered.

3. Linear Fractional Order Integro-differential Equations

In this section, the statement of problem for the class of linear problem, the methodology and numerical examples presented. Discussions on the nonlinear Volterra integro-differential equations are reserved for Section 4 of the work.

3.1. **Statement of the Problem.** Consider the general linear fractional Volterra integro-differential equation

$$D^{\xi}u(t) = q(t) + \lambda \int_0^t G(t-\tau)\, u(\tau)\, d\tau, \tag{4}$$

where, D^{ξ} is the non-integer order operator, $q(t)$ represents inhomogeneous parameter, λ is the langrange multiplier, $G(t-\tau)$ is the nucleus of the equation, which is a smooth function of two variables, and $u(t)$ is the unknown function. Here, the Kernel being considered is a difference kernel.

3.2. **The Methodology.** The method proposed in the present work is explained in the sequel. The Shehu transform of all the terms in (4) are taken as follows

$$\mathbb{S}\left\{D^{\xi}u(t)\right\} = \mathbb{S}\left\{q(t)\right\} + \mathbb{S}\left\{\int_0^t G(t-\tau)\, u(\tau)\, dt\right\}. \tag{5}$$

$$\left(\frac{s}{u}\right)^{\xi} U(s,u) - \sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} u^{(i)}(0) = Q(s,u) + \mathbb{S}\left\{G(t-\tau)\right\} * \mathbb{S}\left\{u(t)\right\}. \tag{6}$$

Dividing (3) through by $\left(\frac{s}{u}\right)^{\xi}$ we have

$$U(s,u) - \left(\frac{u}{s}\right)^{\xi} \sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} u^{(i)}(0) = \left(\frac{u}{s}\right)^{\xi} Q(s,u) + \left(\frac{u}{s}\right)^{\xi} \mathbb{S}\left\{G(t-\tau) * \mathbb{S}\left\{u(t)\right\}\right\}. \tag{7}$$

$$U(s,u)-\left(\frac{u}{s}\right)^{\xi}\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)}u^{(i)}(0)-\left(\frac{u}{s}\right)^{\xi}Q(s,u)-\left(\frac{u}{s}\right)^{\xi}\mathbb{S}\left\{G(t-\tau)\right\}*\mathbb{S}\left\{u(t)\right\}=0. \tag{8}$$

The result above shall be implemented in the k^{th} order deformation equation

$$L\left[U_k\left((s,u);q\right)-\zeta_k U_{k-1}\left((s,u);q\right)\right]=hD_{k-1}\left[N\left(U\left((s,u)\right);q\right)\right]. \tag{9}$$

The auxiliary linear term is

$$L\left[U_k\left((s,u);q\right)\right]=U_k(s,u). \tag{10}$$

Likewise, the general nonlinear term $N\left(U\left((s,u)\right);q\right)$ is obtained from (5) as

$$N\left(U\left((s,u)\right);q\right)=U(s,u)-\left(\frac{u}{s}\right)^{\xi}\left(\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)}u^{(i)}(0)-Q(s,u)-\mathbb{S}\left\{G(t-\tau)\right\}*\mathbb{S}\left\{u(t)\right\}\right) \tag{11}$$

Using (7) and (8) in (6), we have

$$U_k(s,u)-\zeta_k U_{k-1}(s,u)=hD_{k-1}\left[U(s,u)-\left(\frac{u}{s}\right)^{\xi}\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)}u^{(i)}(0)-\left(\frac{u}{s}\right)^{\xi}F(s,u)\right.$$
$$\left.-\left(\frac{u}{s}\right)^{\xi}\mathbb{S}\left\{u(x)\right\}*\mathbb{S}\left\{u(t)\right\}\right] \tag{12}$$

With the application of Homotopy Derivative D_{k-1} on the right hand side of (9), we have

$$U_k(s,u)=\zeta_k U_{k-1}(s,u)+h\left[U_{k-1}(s,u)-\left(1-\zeta_{k-1}\right)\left(\frac{u}{s}\right)^{\xi}\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)}u^{(i)}(0)\right.$$
$$\left.+D_{k-1}\left[\left(\frac{s}{u}\right)^{\xi}F(s,u)-\left(\frac{u}{s}\right)^{\xi}\mathbb{S}\left\{u(x)\right\}*\mathbb{S}\left\{u(t)\right\}\right]\right] \tag{13}$$

where

$$\zeta_k=\begin{cases}0, & k\leq 1\\ 1 & k>1\end{cases}\quad,\qquad \zeta_{k-1}=\begin{cases}0, & k-1<1\\ 1 & k-1\geq 1\end{cases} \tag{14}$$

The expected solution of (4) is finally obtained as

$$u(x)=u_0+u_1+u_2+... \tag{15}$$

3.3. **Numerical Examples.** The algorithm presented in Section 3.1 is applied to the following problems, all of which are taken from the existing literature.

Problem 3.1 [24]

Solve the system of linear integro-differential equations

$$D^{\xi}u(x)-\frac{3x^2\xi\Gamma\left(3\xi\right)}{\Gamma\left(1+2\xi\right)}-\int_0^x\left(x-t\right)u\left(t\right)dt-\int_0^x\left(x-t\right)v\left(t\right)dt=0 \tag{17a}$$

$$D^{\xi}v(x)+\frac{2x^{2+3\xi}\xi\Gamma\left(3\xi\right)}{2+9\xi+9\xi^2}+\frac{3x^2\xi\xi\Gamma\left(3\xi\right)}{\Gamma\left(1+2\xi\right)}-\int_0^x\left(x-t\right)u\left(t\right)dt-\int_0^x\left(x-t\right)v\left(t\right)dt=0 \tag{17b}$$

Subject to conditions $u(0)=v(0)=0$.

Solution. The Shehu transform is applied to equation $(17a)$ as follows:

$$\mathbb{S}\left\{D^{\xi}u(x)\right\}-\mathbb{S}\left\{\frac{3x^{2}\xi\Gamma\left(3\xi\right)}{\Gamma\left(1+2\xi\right)}\right\}-\mathbb{S}\left\{\int_{0}^{x}\left(x-t\right)u\left(t\right)dt\right\}-\mathbb{S}\left\{\int_{0}^{x}\left(x-t\right)v\left(t\right)dt\right\}=\mathbb{S}\left\{0\right\} \tag{18}$$

When the Shehu transform of derivative is implemented on the first term of (18) it gives

$$\mathbb{S}\left\{D^{\xi}u(x)\right\}=\left(\frac{s}{u}\right)^{\xi}U(s,u)-\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)}u^{(i)}(0)=0 \tag{19}$$

Using the initial condition, $u(0)=0$ (19) reduced to

$$\mathbb{S}\left\{D^{\xi}u(x)\right\}=\left(\frac{s}{u}\right)^{\xi}U(s,u) \tag{20}$$

Using (20) in $(17a)$, gives

$$\left(\frac{s}{u}\right)^{\xi}U(s,u)-\frac{3\xi\Gamma\left(3\xi\right)}{\Gamma(1+2\xi)}\mathbb{S}\left\{x^{2\xi}\right\}-\mathbb{S}\left\{\int_{0}^{x}\left(x-t\right)u\left(t\right)dt\right\}-\mathbb{S}\left\{\int_{0}^{x}\left(x-t\right)v\left(t\right)dt\right\}=\mathbb{S}\left\{0\right\} \tag{21}$$

$$\left(\frac{s}{u}\right)^{\xi}U(s,u)-\frac{3\xi\Gamma\left(3\xi\right)\Gamma\left(1+2\xi\right)}{\Gamma(1+2\xi)}\left(\frac{u}{s}\right)^{1+2\xi}-\mathbb{S}(x)*\mathbb{S}\left\{u(x)\right\}-\mathbb{S}\left\{x\right\}*\mathbb{S}\left\{v(x)\right\}=0 \tag{22}$$

$$\left(\frac{s}{u}\right)^{\xi}U(s,u)-\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{2\xi+1}-\left(\frac{u}{s}\right)^{2}U(s,u)-\left(\frac{u}{s}\right)^{2}V(s,u)=0 \tag{21}$$

Dividing (21) through by $\left(\frac{s}{u}\right)^{\xi}$ yields

$$U(s,u)-\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}-\left(\frac{u}{s}\right)^{2+\xi}U(s,u)-\left(\frac{u}{s}\right)^{2+\xi}V(s,u)=0 \tag{22}$$

Thus, we write the k^{th} order deformation as

$$L\left[U_{k}\left((s,u);q\right)-\zeta_{k}U_{k-1}\left((s,u);q\right)\right]=hD_{k-1}\left[N\left(U(s,u);q\right)\right] \tag{23}$$

$$U_{k}(s,u)-\zeta_{k}U_{k-1}(s,u)=hD_{k-1}\left[U(s,u)-\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}-\left(\frac{u}{s}\right)^{2+\xi}U(s,u)-\left(\frac{u}{s}\right)^{2+\xi}V(s,u)\right] \tag{24}$$

$$U_{k}(s,u)=\zeta_{k}U_{k-1}(s,u)+hD_{k-1}\left[U(s,u)-\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}-\left(\frac{u}{s}\right)^{2+\xi}U(s,u)-\left(\frac{u}{s}\right)^{2+\xi}V(s,u)\right] \tag{25}$$

Simplifying terms in the bracket, and taking the convergence control parameter $h=-1$, gives

$$U_{k}(s,u)=\zeta_{k}U_{k-1}(s,u)-U_{k-1}(s,u)+(1-\zeta_{k-1})\left[\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}\right]+\left(\frac{u}{s}\right)^{2\xi}U_{k-1}(s,u)+\left(\frac{u}{s}\right)^{2\xi}V_{k-1}(s,u) \tag{26}$$

Recall that $u(0)=v(0)=0$, when $k=1:$

$$U_{1}(s,u)=\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi} \tag{27}$$

The inverse Shehu transform of (27) as

$$S^{-1}\left\{U_1(s,u)\right\} = \Gamma\left(1+3\xi\right) S^{-1}\left\{\left(\frac{u}{s}\right)^{1+3\xi}\right\} \tag{28}$$

gives

$$u_1(x) = x^{3\xi}. \tag{29}$$

We shall repeat the procedure followed in (18) to (29) for $v(x)$ in ($17b$) as follows:

$$\mathbb{S}\left\{D^{\xi}v(x)\right\} + \frac{2}{\left(3\xi+1\right)\left(2+3\xi\right)}\mathbb{S}\left\{x^{3\xi+2}\right\} + \frac{3\xi\Gamma\left(3\xi\right)}{\Gamma\left(2\xi+1\right)}\mathbb{S}\left\{x^{2}\right\} - \mathbb{S}\left\{x\right\} * \mathbb{S}\left\{u(x)\right\} - \mathbb{S}\left\{x\right\} * \mathbb{S}\left\{v(x)\right\} = \mathbb{S}\left\{0\right\} \tag{30}$$

which gives

$$\left(\frac{s}{u}\right)^{\xi} V(s,u) + \frac{2\Gamma\left(3\xi+3\right)}{\left(3\xi+1\right)\left(3\xi+2\right)}\left(\frac{u}{s}\right)^{3\xi+3} + \frac{3\xi\Gamma\left(3\xi\right)\Gamma\left(2\xi+1\right)}{\Gamma\left(2\xi+1\right)}\left(\frac{u}{s}\right)^{2\xi+1} - \left(\frac{u}{s}\right)^{2} U(s,u) - \left(\frac{u}{s}\right)^{2} V(s,u) \tag{31}$$

$$V(s,u) + \frac{2\Gamma\left(3\xi+2\right)}{\left(3\xi+1\right)}\left(\frac{u}{s}\right)^{4\xi+3} + \Gamma\left(3\xi+1\right)\left(\frac{u}{s}\right)^{3\xi+1} - \left(\frac{u}{s}\right)^{2+\xi} U(s,u) - \left(\frac{u}{s}\right)^{2+\xi} V(s,u) \tag{32}$$

Then, the k^{th} order deformation is written as

$$L\left[V_k\left((s,u);q\right) - \zeta_k V_{k-1}\left((s,u);q\right)\right] = hD_{k-1}\left[N\left(V(s,u);q\right)\right]$$

$$V_k(s,u) - \zeta_k V_{k-1}(s,u) = hD_{k-1}\left[V(s,u) + 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{3+4\xi} + \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi} - \left(\frac{u}{s}\right)^{2+\xi} U(s,u) + \left(\frac{u}{s}\right)^{2+\xi} V(s,u)\right] \tag{33}$$

Taking $h=-1$,

$$V_k(s,u) = \zeta_k V_{k-1}(s,u) - D_{k-1}\left[V(s,u) + 2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3+4\xi} + \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi} - \left(\frac{u}{s}\right)^{2+\xi} U(s,u) + \left(\frac{u}{s}\right)^{2+\xi} V(s,u)\right] \tag{34}$$

$$V_k(s,u) = \zeta_k V_{k-1}(s,u) - V_{k-1}(s,u) - \left(1-\zeta_{k-1}\right)\left[2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3+4\xi} + \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}\right] - \left(\frac{u}{s}\right)^{2+\xi} U_{k-1} + \left(\frac{u}{s}\right)^{2+\xi} V_{k-1} \tag{35}$$

Then,

$$V_k(s,u) = -\left(1-\zeta_k\right)V_{k-1} - \left(1-\zeta_{k-1}\right)\left[2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3+4\xi} + \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi}\right] - \left(\frac{u}{s}\right)^{2+\xi} U_{k-1} + \left(\frac{u}{s}\right)^{2+\xi} V_{k-1} \tag{36}$$

Using the values of ζ_k and ζ_{k-1} as given in (11) and with ($k=1$), (36) reduces to

$$V_1(s,u) = -V_0(s,u) - 2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{4\xi+3} - \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3\xi+1} + \left(\frac{u}{s}\right)^{2+\xi} U_{k-1} - \left(\frac{u}{s}\right)^{2+\xi} V_{k-1} \tag{37}$$

But $V_0(s,u)=0$, thus we have

$$V_1(s,u) = -2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{4\xi+3} - \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3\xi+1} \tag{38}$$

Taking the Inverse Shehu transform

$$S^{-1}\left\{V_1(s,u)\right\} = -2\Gamma\left(1+3\xi\right) S^{-1}\left\{\left(\frac{u}{s}\right)^{3+4\xi}\right\} - \Gamma\left(1+3\xi\right) S^{-1}\left\{\left(\frac{u}{s}\right)^{1+3\xi}\right\} \tag{39}$$

$$v_1(x) = -\frac{2\Gamma\left(1+3\xi\right) x^{2+4\xi}}{\Gamma\left(3+4\xi\right)} - \frac{\Gamma\left(1+3\xi\right) x^{3\xi}}{\Gamma\left(1+3\xi\right)}. \tag{40}$$

When $k = 2$:

$$U_2(s,u) = \left(\frac{u}{s}\right)^{2+\xi} U_1(s,u) + \left(\frac{u}{s}\right)^{2+\xi} V_1(s,u) \tag{41}$$

$$U_2(s,u) = \left(\frac{u}{s}\right)^{2+\xi} \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi} + \left(\frac{u}{s}\right)^{2+\xi}\left[-2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{4\xi+3} - \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3\xi+1}\right] \tag{42}$$

$$U_2(s,u) = \left(\frac{u}{s}\right)^{3+4\xi} \Gamma\left(1+3\xi\right) - 2\left(\frac{u}{s}\right)^{5+5\xi} \Gamma\left(1+3\xi\right) - \left(\frac{u}{s}\right)^{3+4\xi} \Gamma\left(1+3\xi\right) \tag{43}$$

$$U_2(s,u) = -2\left(\frac{u}{s}\right)^{5+5\xi} \Gamma\left(1+3\xi\right). \tag{44}$$

Taking the inverse Shehu transform of (44),

$$u_2(x) = -\frac{2\Gamma\left(1+3\xi\right) x^{4+5\xi}}{\Gamma\left(5+5\xi\right)}. \tag{45}$$

Also,

$$V_2(s,u) = \left(\frac{u}{s}\right)^{2+\xi} U_1(s,u) - \left(\frac{u}{s}\right)^{2+\xi} V_1(s,u) \tag{46}$$

$$V_2(s,u) = \left(\frac{u}{s}\right)^{2+\xi} \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{1+3\xi} - \left(\frac{u}{s}\right)^{2+\xi}\left[-2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{4\xi+3} - \Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{3\xi+1}\right] \tag{47}$$

$$V_2(s,u) = \left(\frac{u}{s}\right)^{3+4\xi} \Gamma\left(1+3\xi\right) + 2\left(\frac{u}{s}\right)^{5+5\xi} \Gamma\left(1+3\xi\right) + \left(\frac{u}{s}\right)^{3+4\xi} \Gamma\left(1+3\xi\right) \tag{48}$$

$$V_2(s,u) = 2\left(\frac{u}{s}\right)^{5+5\xi} \Gamma\left(1+3\xi\right) + 2\left(\frac{u}{s}\right)^{3+4\xi} \Gamma\left(1+3\xi\right). \tag{49}$$

Taking the inverse Shehu transform yields

$$S^{-1}\left\{V_2(s,u)\right\} = 2\Gamma\left(1+3\xi\right) S^{-1}\left\{\left(\frac{u}{s}\right)^{5+5\xi}\right\} + 2\Gamma\left(1+3\xi\right) S^{-1}\left\{\left(\frac{u}{s}\right)^{3+4\xi}\right\}. \tag{50}$$

$$v_2(x) = \frac{2\Gamma\left(1+3\xi\right) x^{2+4\xi}}{\Gamma\left(3+4\xi\right)} + \frac{2\Gamma\left(1+3\xi\right) x^{4+5\xi}}{\Gamma\left(5+5\xi\right)}. \tag{51}$$

When $k = 3$:

$$U_3(s,u) = \left(\frac{u}{s}\right)^{2+\xi} U_2(s,u) + \left(\frac{u}{s}\right)^{2+\xi} V_2(s,u). \tag{52}$$

$$U_3(s,u) = \left(\frac{u}{s}\right)^{2+\xi}\left[2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{5+5\xi}\right] + \left(\frac{u}{s}\right)^{2+\xi}\left[2\Gamma\left(3+4\xi\right)\left(\frac{u}{s}\right)^{3+4\xi} + 2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{5+5\xi}\right] \tag{53}$$

$$u_3(s,u) = -2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{6+7\xi} + 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{5+5\xi} + 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{6+7\xi} \tag{54}$$

Simplifying, gives

$$U_3(s,u) = 2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{5+5\xi} \tag{16}$$

$$u_3(x) = \frac{2\Gamma\left(1+3\xi\right) x^{4+5\xi}}{\Gamma\left(5+5\xi\right)} \tag{55}$$

Also,

$$V_3(s,u) = \left(\frac{u}{s}\right)^{2+\xi} U_2(s,u) - \left(\frac{u}{s}\right)^{2+\xi} V_2(s,u).$$

Substituting values of $U_2(s,u)$ and $V_2(s,u)$ gives:

$$V_3(s,u) = \left(\frac{u}{s}\right)^{2+\xi}\left[-2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{5+5\xi}\right] - \left(\frac{u}{s}\right)^{2+\xi}\left[-2\Gamma\left(3+4\xi\right)\left(\frac{u}{s}\right)^{3+4\xi} + 2\Gamma\left(1+3\xi\right)\left(\frac{u}{s}\right)^{5+5\xi}\right]. \tag{56}$$

$$V_3(s,u) = -2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{6+7\xi} - 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{5+5\xi} - 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{6+7\xi}. \tag{57}$$

$$V_3(s,u) = -4\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{6+7\xi} - 2\Gamma(1+3\xi)\left(\frac{u}{s}\right)^{5+5\xi}. \tag{58}$$

Taking the inverse Shehu transform to obtain

$$v_3(x) = \frac{-4\Gamma(1+3\xi)\,x^{6+6\xi}}{\Gamma(5+5\xi)} - \frac{2\Gamma(1+3\xi)\,x^{4+5\xi}}{\Gamma(4+5\xi)}. \tag{59}$$

Conclusively,

$$u(x) = \sum_{i=0}^{\infty} u_i(x) \tag{60}$$

$$u(x) = x^{3\xi} - \frac{2\Gamma(1+3\xi)\,x^{4+5\xi}}{\Gamma(5+5\xi)} - \frac{2\Gamma(1+3\xi)\,x^{8+7\xi}}{\Gamma(9+7\xi)} + \frac{2\Gamma(1+3\xi)\,x^{4+5\xi}}{\Gamma(5+5\xi)} + \ldots \tag{61}$$

$$u(x) = x^{3\xi}. \tag{62}$$

Also $v(x)$ becomes

$$v(x) = -\frac{2\Gamma(1+3\xi)x^{2+4\xi}}{\Gamma(3+4\xi)} - \frac{\Gamma(1+3\xi)\,x^{3\xi}}{\Gamma(1+3\xi)} + \frac{2\Gamma(1+3\xi)\,x^{4+5\xi}}{\Gamma(4+5\xi)} + \frac{2\Gamma(1+3\xi)\,x^{2+4\xi}}{\Gamma(3+4\xi)} - \frac{4\Gamma(1+3\xi)x^{6+6\xi}}{\Gamma(5+5\xi)} - \frac{2\Gamma(1+3\xi)x^{4+5\xi}}{\Gamma(4+5\xi)} \tag{63}$$

$$v(x) = -x^{3\xi} - \frac{4\Gamma(1+3\xi)x^{6+6\xi}}{\Gamma(5+5\xi)}\ldots \tag{64}$$

The results for both $u(x)$ and $v(x)$ coincide with the exact solution, and this validates the algorithm.

Problem 3.2 [30]

Consider the following system of linear fractional order integro-differential equations

$$D^{\xi}y_1(t) = 1 + t - \frac{t^3}{3} + \int_0^t [(t-s)y_1(s) + (t-s)y_2(s)]ds \tag{65a}$$

$$D^{\xi}y_2(t) = 1 - t - \frac{t^4}{12} + \int_0^t [(t-s)y_1(s) - (t-s)y_2(s)]ds \tag{65b}$$

Subject to the initial conditions $y_1(0) = 0$ and $y_2(0) = 0$. The exact solutions are $y_1(t) = t + \frac{t^2}{2}$ and $y_2(t) = t - \frac{t^2}{2}$.

When the procedure described in problem (3.1) is followed, the following results are arrived at

$$y_1(t) = \frac{t^{\xi}}{\Gamma(1+\xi)} + \frac{t^{1+\xi}}{\Gamma(2+\xi)} - \frac{2t^{3+\xi}}{\Gamma(4+\xi)} + \frac{2t^{2+2\xi}}{\Gamma(3+2\xi)} - \frac{2t^{5+2\xi}}{\Gamma(6+2\xi)} - \frac{2t^{6+2\xi}}{\Gamma(7+2\xi)}. \tag{66}$$

$$y_2(t) = \frac{t^{\xi}}{\Gamma(1+\xi)} - \frac{t^{1+\xi}}{\Gamma(2+\xi)} - \frac{2t^{4+\xi}}{\Gamma(5+\xi)} - \frac{2t^{2+2\xi}}{\Gamma(3+2\xi)} + \frac{2t^{5+2\xi}}{\Gamma(6+2\xi)} + \frac{2t^{6+2\xi}}{\Gamma(7+2\xi)}. \tag{67}$$

Problem 3.3 [30]

Consider the following nonlinear Volterra integro-differential equation.

$$D^{\xi}y_1(t) - 2t^2 - \int_0^t [(t-s)y_1 + (t-s)y_2]\,ds = 0. \tag{68}$$

$$D^{\xi}y_2(t) + 3t^2 + \frac{t^5}{5} - \int_0^t [(t-s)y_1 - (t-s)y_2]\,ds = 0. \tag{69}$$

Subject to the initial conditions: $y_1(0) = y_2(0) = 1$. The exact solutions of are $y_1(t) = 1 + t^3$ and $y_2(t) = 1 - t^3$.

When the algorithm in Section 3.1 is implemented, the results obtained are:

$$y_1(t) = 1 + \frac{6t^{\xi+2}}{\Gamma(\xi+3)} - \frac{24t^{2\xi+7}}{\Gamma(2\xi+8)} - \frac{48t^{3\xi+9}}{\Gamma(3\xi+10)}\ldots \tag{70}$$

$$y_2(t) = 1 - \frac{6t^{\xi+2}}{\Gamma(\xi+3)} - \frac{24t^{\xi+5}}{\Gamma(\xi+6)} - \frac{24t^{2\xi+7}}{\Gamma(2\xi+8)}\ldots \tag{71}$$

4. Nonlinear Volterra Integro-differential Equations

For the sake of completeness, this section of our work is dedicated to the solution of nonlinear Volterra integro-differential equations (NVIDEs).

4.1. **Statement of the Problem.** Consider the nonlinear Volterra integro-differential equation

$$D^{\xi}y(x) = f(x) + \lambda \int_0^x K(x-t)\, N(y(t))\, dt, \tag{72}$$

where $N(y(t))$ represents the nonlinear term, $f(x)$ denotes the inhomogeneous source term and $K(x-t)$ is the kernel (nucleus) of integral equation. The parameter λ is taken as 1 throughout this work (Yisa & Adelabu, 2018).

4.2. **Method of Solution.** We shall take the Shehu transform of both sides of (72) to have

$$\mathbb{S}\left\{D^{\xi}y(x)\right\} = \mathbb{S}\{f(x)\} + \mathbb{S}\left\{\int_0^x K(x-t)\, N(y(t))\, dt\right\} \tag{73}$$

$$\left(\frac{s}{u}\right)^{\xi} Y(s,u) - \sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} y^{(i)}(0) = F(s,u) + \mathbb{S}\{K(x-t)\} * \mathbb{S}\{N(y(t))\} \tag{74}$$

Dividing through by $\left(\frac{s}{u}\right)^{\xi}$ gives

$$Y(s,u) - \left(\frac{s}{u}\right)^{\xi}\left(\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} y^{(i)}(0)\right) - \left(\frac{s}{u}\right)^{\xi} F(s,u) - \left(\frac{s}{u}\right)^{\xi}\mathbb{S}\{K(x-t)\} * \mathbb{S}\{N(y(t))\} = 0 \tag{75}$$

In order to effectively handle the nonlinear term in (72) above, that is $N(y(t))$, the homotopy derivative shall be applied. The k^{th} order deformation equation is,

$$L\left[Y_k((s,u);q) - \zeta_k Y_{k-1}((s,u);q)\right] = hD_{k-1}\left[N((s,u);q)\right], \tag{76}$$

where $q \in [0,1]$ is an embedding parameter. But

$$L\left[Y_k((s,u);q)\right] = Y_i(s,u) \tag{77}$$

Thus, (4.4) becomes

$$Y_k(s,u) = \zeta_k Y_{k-1}(s,u) + hD_k - 1\left[Y(s,u) - \left(\frac{s}{u}\right)^{\xi}\left(\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} y^{(i)}(0)\right) - \left(\frac{s}{u}\right)^{\xi} F(s,u) - \left(\frac{s}{u}\right)^{\xi}\mathbb{S}\{K(x-t)\} * \mathbb{S}\{N(y(t))\}\right] \tag{78}$$

Making use of Homotopy Derivative D_{k-1}, and taking $h = -1$, the result gives

$$Y_k(s,u) = \zeta_k Y_{k-1}(s,u) - Y_{k-1}(s,u) - (1-\zeta_{k-1})\left(\frac{s}{u}\right)^{\xi}\left(\sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} y^{(i)}(0)\right) + D_{k-1}\left[\left(\frac{s}{u}\right)^{\xi}\mathbb{S}\{K(x-t)\} * \mathbb{S}\{N(y(t))\}\right] \tag{79}$$

Then the series solution of (72) is

$$y(x) = y_0 + y_1 + y_2 + ... \tag{80}$$

4.3. **Numerical Examples on Nonlinear Volterra Integro-differential Equations.** The algorithm presented in Section 4.2 is applied in the solution of following nonlinear Volterra integro-differential equation.

Problem 4.1 [31]

Solve the following non-linear NVIDE

$$D^{\xi}y(x) - \int_0^x \exp^{-t}[y(t)]^2 dt = 1, \qquad 0 \le x \le 1, \qquad 3 < \xi \le 4 \tag{81}$$

Subject to the initial conditions: $y(0) = y'(0) = y''(0) = y'''(0) = 1$.
Solution
Taking the Shehu transform of (81) to have

$$\mathbb{S}\{D^{\xi}\ y(x)\} - \mathbb{S}\left\{\int_0^x \exp^{-t}[y(t)]^2 dt\right\} = S\{1\} \tag{82}$$

But

$$\mathbb{S}\{D^{\xi} y(x)\} = \left(\frac{s}{u}\right)^{\xi} Y(s,u) - \sum_{i=0}^{\xi-1}\left(\frac{s}{u}\right)^{\xi-(i+1)} y^{(i)}(0) \tag{83}$$

Implementing the given conditions, we have

$$\mathbb{S}\{D^{\xi} y(x)\} = \left(\frac{s}{u}\right)^{\xi} Y(s,u) - \left(\frac{s}{u}\right)^{\xi-1} y(0) - \left(\frac{s}{u}\right)^{\xi-2} y'(0) - \left(\frac{s}{u}\right)^{\xi-3} y''(0) - \left(\frac{s}{u}\right)^{\xi-4} y'''(0) \tag{84}$$

$$\mathbb{S}\{D^{\xi} y(x)\} = \left(\frac{s}{u}\right)^{\xi} Y(s,u) - \left(\frac{s}{u}\right)^{\xi-1} - \left(\frac{s}{u}\right)^{\xi-2} - \left(\frac{s}{u}\right)^{\xi-3} - \left(\frac{s}{u}\right)^{\xi-4} \tag{85}$$

Putting (85) in (81), to obtain

$$\left(\frac{s}{u}\right)^{\xi} Y(s,u) - \left(\frac{s}{u}\right)^{\xi-1} - \left(\frac{s}{u}\right)^{\xi-2} - \left(\frac{s}{u}\right)^{\xi-3} - \left(\frac{s}{u}\right)^{\xi-4} - \mathbb{S}\{1\} * \mathbb{S}\left\{[y(x)^2]\right\} = \left(\frac{u}{s}\right) \tag{86}$$

Rearranging and dividing through by $\left(\frac{s}{u}\right)^{\xi}$ gives:

$$Y(s,u) - \left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^3 - \left(\frac{u}{s}\right)^4 - (\frac{u}{s})^{1+\xi} S\{[y(x)^2]\} - \left(\frac{u}{s}\right)^{1+\xi} = 0 \tag{87}$$

The k^{th} order deformation equation is

$$L\left[Y_k\left((s,u);q\right) - \zeta_k Y_{k-1}\left((s,u);q\right)\right] = hD_{k-1}\left[N\left((s,u);q\right)\right] \tag{88}$$

$$Y_k(s,u) - \zeta_k Y_{k-1}(s,u) = hD_{k-1}\left[Y(s,u) - \left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^3 - \left(\frac{u}{s}\right)^4 - (\frac{u}{s})^{1+\xi}\left(S\{[y(x)^2]\} + 1\right)\right]. \tag{89}$$

Let $h = -1$ and implement the operator D_{k-1}

$$Y_k(s,u) = \zeta_k Y_{k-1}(s,u) - Y_{k-1}(s,u) + (1-\zeta_{k-1})\left[\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^{1+\xi}\right] + \left(\frac{u}{s}\right)^{1+\xi}\mathbb{S}\left\{\sum_{i=o}^{k-1} y_{k-1-i}y_i\right\}. \tag{90}$$

Upon simplification, we get

$$Y_k(s,u) = -(1-\zeta_k)Y_{k-1}(s,u) + (1-\zeta_{k-1})\left[\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^{1+\xi}\right] + \left(\frac{u}{s}\right)^{1+\xi} S\left\{\sum_{i=o}^{k-1} y_{k-1-i}\ y_i\right\}. \tag{91}$$

The initial approximation is obtained using the initial conditions as

$$y_0(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0). \tag{92}$$

$$y_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}. \tag{93}$$

Taking the inverse Shehu transform, we have

$$Y_0(s,u) = +\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4. \tag{94}$$

When $k = 1$:

$$Y_1(s,u) = -Y_0(s,u) + \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^{1+\xi} + \left(\frac{u}{s}\right)^{1+\xi} S\{\sum_{i=o}^{1-1} y_{1-1-i}\ y_i\} \tag{95}$$

$$Y_1(s,u) = -\left[\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4\right] + \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^{1+\xi} + \left(\frac{u}{s}\right)^{1+\xi}\mathbb{S}\{y_0 y_0\} \tag{96}$$

Which simplifies to give

$$Y_1(s,u) = \left(\frac{u}{s}\right)^{1+\xi} + \left(\frac{u}{s}\right)^{1+\xi} S\{y_0^2\}. \tag{97}$$

But

$$[y_0^2(x)] = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right),$$

That is

$$y_0^2(x) = 1 + 2x + 2x^2 + \frac{3x^3}{2!} + \frac{7x^4}{3!} + \frac{x^5}{3!} + \frac{x^6}{3!3!} \tag{99}$$

Thus,

$$Y_1(s,u) = \left(\frac{u}{s}\right)^{1+\xi} + \left(\frac{u}{s}\right)^{1+\xi} \mathbb{S}\left\{1 + 2x + 2x^2 + \frac{3x^3}{2!} + \frac{7x^4}{3!} + \frac{x^5}{3!} + \frac{x^6}{3!3!}\right\} \tag{100}$$

$$Y_1(s,u) = \left(\frac{u}{s}\right)^{1+\xi} + \left(\frac{u}{s}\right)^{2+\xi} + 2\left(\frac{u}{s}\right)^{3+\xi} + 4\left(\frac{u}{s}\right)^{4+\xi} + 9\left(\frac{u}{s}\right)^{5+\xi} + 28\left(\frac{u}{s}\right)^{6+\xi} + 20\left(\frac{u}{s}\right)^{7+\xi} + 20\left(\frac{u}{s}\right)^{8+\xi} \tag{101}$$

Taking the inverse Shehu transform, gives

$$y_1(x) = \frac{x^{\xi}}{\Gamma(1+\xi)} + \frac{x^{1+\xi}}{\Gamma(2+\xi)} + \frac{2x^{2+\xi}}{\Gamma(3+\xi)} + \frac{4x^{3+\xi}}{\Gamma(4+\xi)} + \frac{9x^{4+\xi}}{\Gamma(5+\xi)} + \frac{28x^{5+\xi}}{\Gamma(6+\xi)} + \frac{20x^{6+\xi}}{\Gamma(7+\xi)} + \frac{20x^{7+\xi}}{\Gamma(8+\xi)} \tag{102}$$

When $k = 2$:

$$Y_2(s,u) = \left(\frac{u}{s}\right)^{1+\xi} \mathbb{S}\{\sum_{i=o}^{2-1} y_{2-1-i}\ y_i\} \tag{103}$$

Simplifying the terms, gives

$$\begin{aligned}Y_2(s,u) &= 2\left(\frac{u}{s}\right)^{2+2\xi} + 2\left(\frac{u}{s}\right)^{3+2\xi} + 4\left(\frac{u}{s}\right)^{4+2\xi} + 8\left(\frac{u}{s}\right)^{5+2\xi} + 18\left(\frac{u}{s}\right)^{6+2\xi} + 56\left(\frac{u}{s}\right)^{7+2\xi} \\
&+ 40\left(\frac{u}{s}\right)^{8+2\xi} + 40\left(\frac{u}{s}\right)^{9+2\xi} + \frac{2\Gamma(2+\xi)\left(\frac{u}{s}\right)^{3+2\xi}}{\Gamma(1+\xi)} + \frac{2\Gamma(3+\xi)\left(\frac{u}{s}\right)^{5+2\xi}}{\Gamma(2+\xi)} + \frac{4\Gamma(4+\xi)\left(\frac{u}{s}\right)^{5+2\xi}}{\Gamma(3+\xi)} \\
&+ \frac{8\Gamma(5+\xi)\left(\frac{u}{s}\right)^{6+2\xi}}{\Gamma(4+\xi)} + \frac{18\Gamma(6+\xi)\left(\frac{u}{s}\right)^{7+2\xi}}{\Gamma(5+\xi)} + \frac{56\Gamma(7+\xi)\left(\frac{u}{s}\right)^{8+2\xi}}{\Gamma(6+\xi)} + \frac{40\Gamma(8+\xi)\left(\frac{u}{s}\right)^{9+2\xi}}{\Gamma(7+\xi)} \\
&+ \frac{40\Gamma(9+\xi)\left(\frac{u}{s}\right)^{10+2\xi}}{\Gamma(8+\xi)} + \frac{\Gamma(3+\xi)\left(\frac{u}{s}\right)^{4+2\xi}}{\Gamma(1+\xi)} + \frac{\Gamma(4+\xi)\left(\frac{u}{s}\right)^{5+2\xi}}{\Gamma(2+\xi)} + \frac{4\Gamma(5+\xi)\left(\frac{u}{s}\right)^{6+2\xi}}{\Gamma(3+\xi)} \\
&+ \frac{4\Gamma(6+\xi)\left(\frac{u}{s}\right)^{7+2\xi}}{2!\Gamma(4+\xi)} + \frac{9\Gamma(7+\xi)\left(\frac{u}{s}\right)^{8+2\xi}}{\Gamma(5+\xi)} + \frac{56\Gamma(8+\xi)\left(\frac{u}{s}\right)^{9+2\xi}}{\Gamma(6+\xi)} + \frac{20\Gamma(9+\xi)\left(\frac{u}{s}\right)^{10+2\xi}}{\Gamma(7+\xi)} + \frac{20\Gamma(10+\xi)\left(\frac{u}{s}\right)^{11+2\xi}}{\Gamma(8+\xi)} \\
&+ \frac{2\Gamma(4+\xi)\left(\frac{u}{s}\right)^{5+2\xi}}{3!\Gamma(1+\xi)} + \frac{2\Gamma(5+\xi)\left(\frac{u}{s}\right)^{6+2\xi}}{3!\Gamma(2+\xi)} + \frac{4\Gamma(6+\xi)\left(\frac{u}{s}\right)^{7+2\xi}}{3!\Gamma(3+\xi)} + \frac{8\Gamma(7+\xi)\left(\frac{u}{s}\right)^{8+2\xi}}{3!\Gamma(4+\xi)} \\
&+ \frac{18\Gamma(8+\xi)\left(\frac{u}{s}\right)^{9+2\xi}}{3!\Gamma(5+\xi)} + \frac{56\Gamma(9+\xi)\left(\frac{u}{s}\right)^{10+2\xi}}{3!\Gamma(6+\xi)} + \frac{40\Gamma(10+\xi)\left(\frac{u}{s}\right)^{11+2\xi}}{3!\Gamma(7+\xi)} + \frac{40\Gamma(11+\xi)\left(\frac{u}{s}\right)^{12+2\xi}}{3!\Gamma(8+\xi)}\end{aligned} \tag{104}$$

Taking the inverse Shehu transform of (104), we have

$$\begin{aligned} y_2(x) = &\frac{2x^{1+2\xi}}{\Gamma(2+2\xi)} + \frac{2x^{2+2\xi}}{\Gamma(3+2\xi)} + \frac{4x^{3+2\xi}}{\Gamma(4+2\xi)} + \frac{8x^{4+2\xi}}{\Gamma(5+2\xi)} + \frac{18x^{5+2\xi}}{\Gamma(6+2\xi)} \\ &\frac{56x^{6+2\xi}}{\Gamma(7+2\xi)} + \frac{40x^{7+2\xi}}{\Gamma(8+2\xi)} + \frac{40x^{8+2\xi}}{\Gamma(9+2\xi)} + \frac{2x^{2+2\xi}\Gamma(2+\xi)}{\Gamma(3+2\xi)\Gamma(1+\xi)} + \frac{2x^{3+2\xi}\Gamma(3+\xi)}{\Gamma(4+2\xi)\Gamma(2+\xi)} \\ &\frac{4x^{4+2\xi}}{\Gamma(4+2\xi)\Gamma(3+\xi)} + \frac{8x^{5+2\xi}\Gamma(5+\xi)}{\Gamma(6+2\xi)\Gamma(4+\xi)} + \frac{18x^{6+2\xi}\Gamma(6+2\xi)}{\Gamma(5+\xi)\Gamma(7+2\xi)} + \frac{56x^{7+2\xi}\Gamma(7+\xi)}{\Gamma(8+2\xi)\Gamma(6+\xi)} \\ &\frac{40x^{8+2\xi}\Gamma(8+\xi)}{\Gamma(7+2\xi)\Gamma(9+\xi)} + \frac{40x^{9+2\xi}\Gamma(9+\xi)}{\Gamma(8+\xi)\Gamma(10+2\xi)} + \frac{x^{3+2\xi}\Gamma(3+\xi)}{\Gamma(4+2\xi)\Gamma(1+2\xi)} + \frac{x^{4+2\xi}\Gamma(4+\xi)}{\Gamma(5+2\xi)\Gamma(2+2\xi)} \\ &+ \frac{2x^{5+\xi}\Gamma(5+2\xi)}{\Gamma(6+2\xi)\Gamma(3+\xi)} + \frac{4x^{6+2\xi}\Gamma(6+\xi)}{\Gamma(4+\xi)\Gamma(7+2\xi)} + \frac{9x^{7+2\xi}\Gamma(7+\xi)}{\Gamma(5+\xi)\Gamma(8+2\xi)} + \frac{28x^{8+2\xi}\Gamma(8+\xi)}{\Gamma(6+2\xi)\Gamma(9+2\xi)} \\ &\frac{20x^{9+2\xi}\Gamma(9+\xi)}{\Gamma(10+2\xi)\Gamma(7+\xi)} + \frac{20x^{10+2\xi}\Gamma(10+\xi)}{\Gamma(8+\xi)\Gamma(11+2\xi)} + \frac{2x^{4+2\xi}\Gamma(4+2\xi)}{3!\Gamma(1+\xi)\Gamma(5+2\xi)} + \frac{2x^{5+2\xi}\Gamma(5+\xi)}{3!\Gamma(2+\xi)\Gamma(6+2\xi)} \\ &+ \frac{4x^{6+2\xi}\Gamma(6+\xi)}{3!\Gamma(3+\xi)\Gamma(7+2\xi)} + \frac{8x^{7+2\xi}\Gamma(7+\xi)}{3!\Gamma(4+\xi)\Gamma(8+2\xi)} + \frac{18x^{8+2\xi}\Gamma(8+\xi)}{3!\Gamma(5+\xi)\Gamma(9+2\xi)} + \frac{56x^{9+2\xi}\Gamma(9+\xi)}{3!\Gamma(6+\xi)\Gamma(10+2\xi)} \\ &+ \frac{40x^{10+2\xi}\Gamma(10+\xi)}{3!\Gamma(7+\xi)\Gamma(11+2\xi)} + \frac{40x^{11+2\xi}\Gamma(11+\xi)}{3!\Gamma(8+\xi)\Gamma(12+2\xi)} \end{aligned} \tag{105}$$

Therefore, the general solution is given as

$$\begin{aligned} y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^\xi}{\Gamma(1+\xi)} + \frac{x^{1+\xi}}{\Gamma(2+\xi)} + \frac{2x^{2+\xi}}{\Gamma(3+\xi)} + \frac{4x^{3+\xi}}{\Gamma(4+\xi)} + \frac{9x^{4+\xi}}{\Gamma(5+\xi)} \\ + \frac{28x^{5+\xi}}{\Gamma(6+\xi)} + \frac{20x^{6+\xi}}{\Gamma(7+\xi)} + \frac{20x^{7+\xi}}{\Gamma(8+\xi)} + \frac{2x^{1+2\xi}}{\Gamma(2+2\xi)} + \frac{2x^{2+2\xi}}{\Gamma(3+2\xi)} + \frac{4x^{3+2\xi}}{\Gamma(4+2\xi)} + \dots \end{aligned}$$

Problem 4.2 [31]
Solve the Volterra integro-differential equation below using

$$D^\xi y(x) - \int_0^x [y(x)]^2 dt = -1, \tag{106}$$

subject to the initial condition $y(0) = 0$.
When the procedure described in Section 4.1 is followed, the solution to the problem is obtained as

$$y(x) = \frac{\Gamma(1+2\xi)x^{1+3\xi}}{\Gamma^2(1+\xi)\Gamma(2+3\xi)} - \frac{x^\xi}{\Gamma(1+\xi)} + \dots$$

5. Results and Discussion

In this section, the results obtained through our proposed method and the results from selected literatures are presented in tabular form for ease of comparison.

Table 1: Comparison between Exact solution and HAITM for Problem 3.1

x	Exact $u(x)$	Exact $v(x)$	HAITM $u(x)$	HAITM $v(x)$
0.0	0.000	0.000	0.000	0.000
0.1	0.001	-0.001	0.001	-0.001
0.2	0.008	-0.008	0.008	-0.008
0.3	0.027	-0.027	0.027	-0.027
0.4	0.064	-0.064	0.064	-0.064
0.5	0.125	-0.125	0.125	-0.125
0.6	0.216	-0.216	0.216	-0.216
0.7	0.343	-0.343	0.343	-0.343
0.8	0.512	-0.512	0.512	-0.512
0.9	0.729	-0.729	0.729	-0.729
1.0	1.000	-1.000	1.000	-1.000

Comparison between HAITM and the method in the literature PSM

Table 2: Comparison of Solution $y_1(t)$ and $y_2(t)$ at $\xi = 1$ by PSM and HAITM for Problem 3.2

t	Exact $y_1(t)$	Exact $y_2(t)$	HAITM $y_1(t)$	HAITM $y_2(t)$	PSM $y_1(t)$	PSM $y_2(t)$
0.0000	0.000	0.000	0.00000	0.000	0.000	0.000
0.1000	0.105	0.095	0.10500	0.0949915	0.105	0.095
0.2000	0.220	0.180	0.22000	0.179861	0.220	0.180
0.3000	0.345	0.255	0.34500	0.254285	0.345	0.255
0.4000	0.480	0.320	0.479999	0.317697	0.480	0.320
0.5000	0.625	0.375	0.624997	0.369274	0.625	0.375
0.6000	0.780	0.420	0.779988	0.407916	0.780	0.420
0.7000	0.945	0.455	0.944964	0.432226	0.945	0.455
0.8000	1.120	0.480	1.11991	0.440497	1.120	0.480
0.9000	1.305	0.495	1.30479	0.430695	1.305	0.495
1.000	1.500	0.500	1.49955	0.400446	1.500	0.5

Table 3: Comparison of Solution $y_1(t)$ and $y_2(t)$ at $\xi = 1$ by PSM and HAITM for Problem 3.3

t	Exact $y_1(t)$	Exact $y_2(t)$	HAITM $y_1(t)$	HAITM $y_2(t)$	PSM $y_1(t)$	PSM $y_2(t)$
0.0000	1.000	1.000	1.000	1.000	1.000	1.000
0.1000	1.001	0.999	1.001	0.999	1.001	0.999
0.2000	1.008	0.992	1.008	0.992	1.008	0.992
0.3000	1.027	0.973	1.027	0.973	1.027	0.973
0.4000	1.064	0.936	1.064	0.936	1.064	0.936
0.5000	1.125	0.875	1.125	0.875	1.125	0.875
0.6000	1.216	0.784	1.216	0.784	1.216	0.784
0.7000	1.343	0.657	1.343	0.657	1.343	0.657
0.8000	1.541	0.488	1.541	0.488	1.512	0.488
0.9000	1.748	0.271	1.748	0.271	1.729	0.271
1.0000	2.000	0.033	1.999	0.033	2.000	0.000

Table 4: Comparison between CAS solution and HAITM for Problem 4.1

x	CAS ($\xi = 3.25$)	CAS ($\xi = 3.75$)	HAITM ($\xi = 3.25$)	HAITM ($\xi = 3.75$)
0.000	1.0000	1.0000	1.0000	1.0000
0.1000	1.1053	1.1052	1.1052	1.1052
0.2000	1.2219	1.2216	1.2220	1.2215
0.3000	1.3523	1.3510	1.3521	1.3502
0.4000	1.4968	1.4941	1.4974	1.4928
0.5000	1.6635	1.8334	1.6600	1.6508
0.7000	2.0444	2.0293	2.0463	2.0203
0.8000	2.2776	2.2537	2.2748	2.2358
0.9000	2.5265	2.4949	2.5304	2.4748

Table 5: Comparison between BPM solution and HAITM for Problem 4.2				
x	BPM ($\xi = 1$)	BPM ($\xi = 0.9$)	HAITM ($\xi = 1$)	HAITM ($\xi = 0.9$)
0.0000	0.0000	0.0000	0.0000	0.0000
0.0625	-0.06250	-0.08576	-0.06250	-0.08576
0.1250	-0.12498	-0.15997	-0.12498	-0.15997
0.1825	-0.18740	-0.23025	-0.18740	-0.23025
0.2500	0.24968	-0.29791	0.24968	-0.29791
0.3125	-0.31171	-0.36344	-0.31171	-0.36344
0.3750	-0.37336	-0.42702	-0.37336	-0.42702
0.4375	-0.43446	-0.48866	-0.43446	-0.48866
0.5000	-0.49482	-0.54829	-0.49482	-0.54829
0.5625	-0.55423	-0.60576	-0.55423	-0.60576
0.6250	-0.61243	-0.66089	-0.61243	-0.66089
0.6875	-0.66917	-0.71347	-0.66917	-0.71347
0.7500	-0.72115	-0.76325	-0.72115	-0.76325
0.8125	-0.7709	-0.81007	-0.7709	-0.81007
0.8750	-0.82767	-0.85360	-0.82767	-0.85360
0.9375	-0.87557	-0.89363	-0.87557	-0.89363

FIGURE 1. Graph of $u(x)$ for Problem 3.1

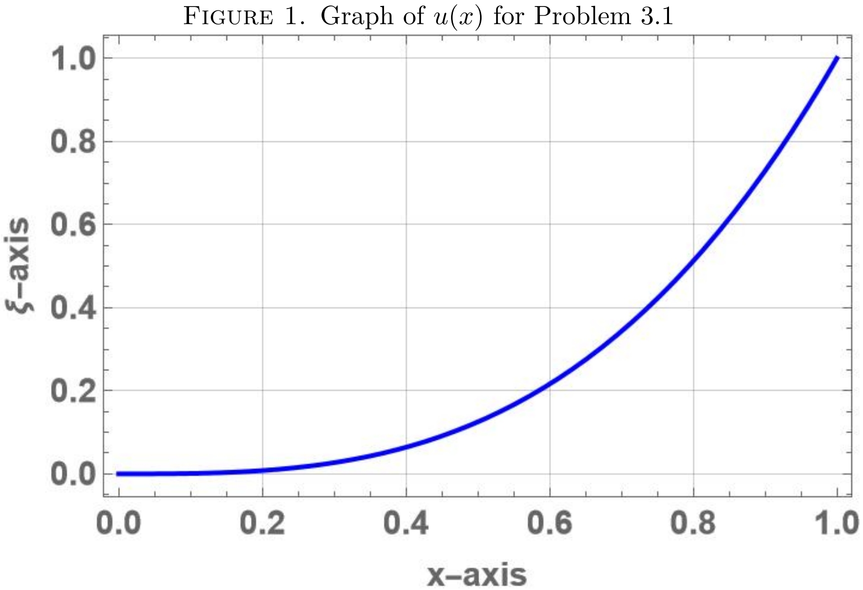


FIGURE 2. Graph of $v(x)$ for Problem 3.1

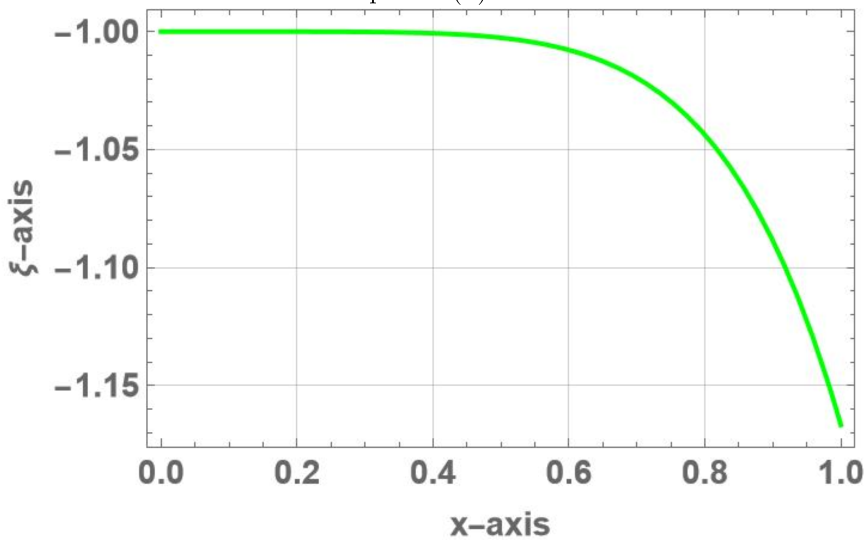


FIGURE 3. Graph of $y_1(t)$ for Problem 3.2

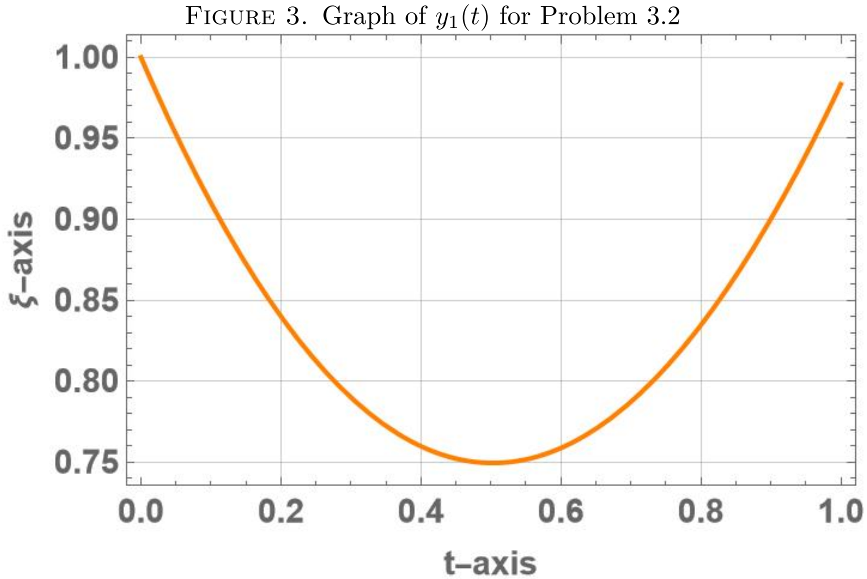


FIGURE 4. Graph of $y_2(t)$ for Problem 3.2

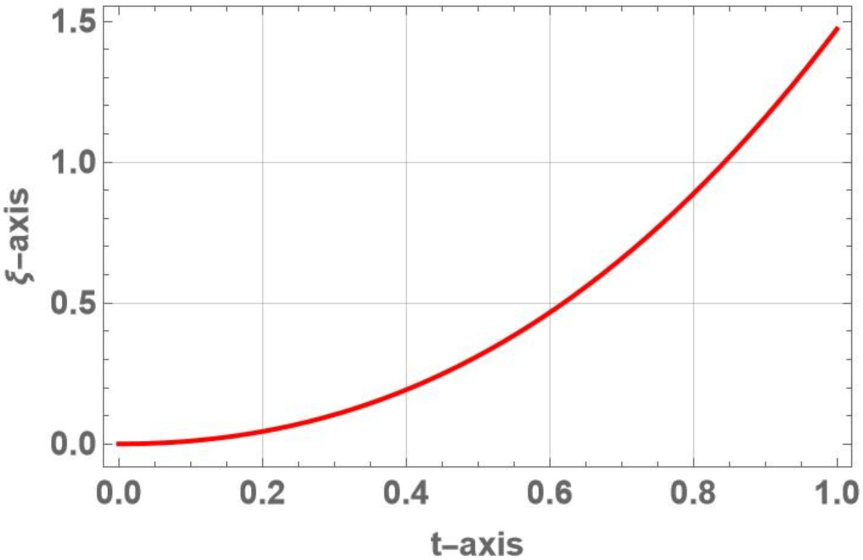


FIGURE 5. Graph of $y_1(t)$ for Problem 3.3

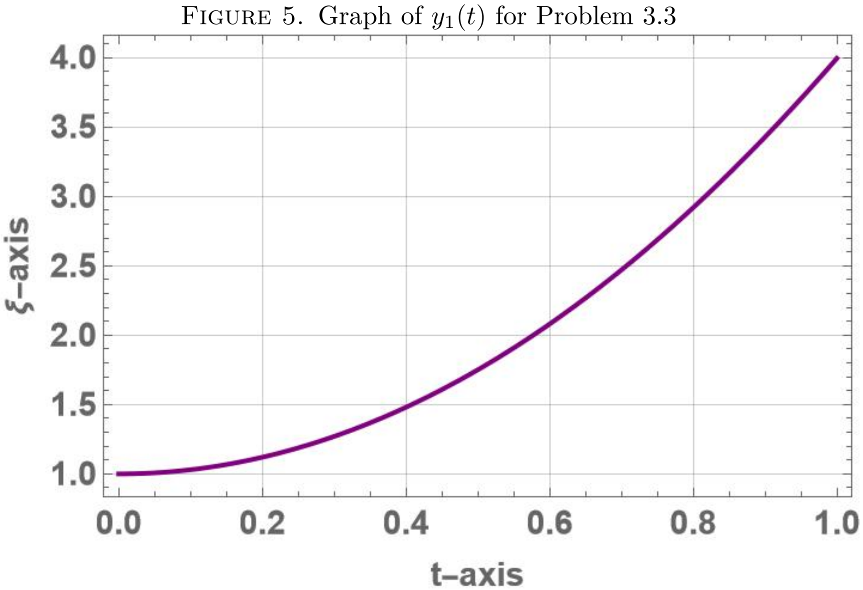


FIGURE 6. Graph of $y_2(t)$ for Problem 3.3

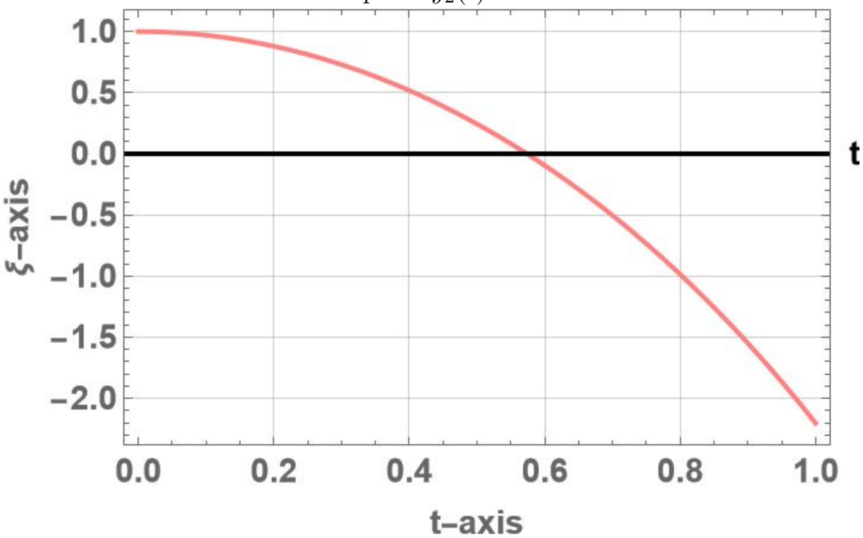


FIGURE 7. Graph of $y(t)$ for Problem 4.1

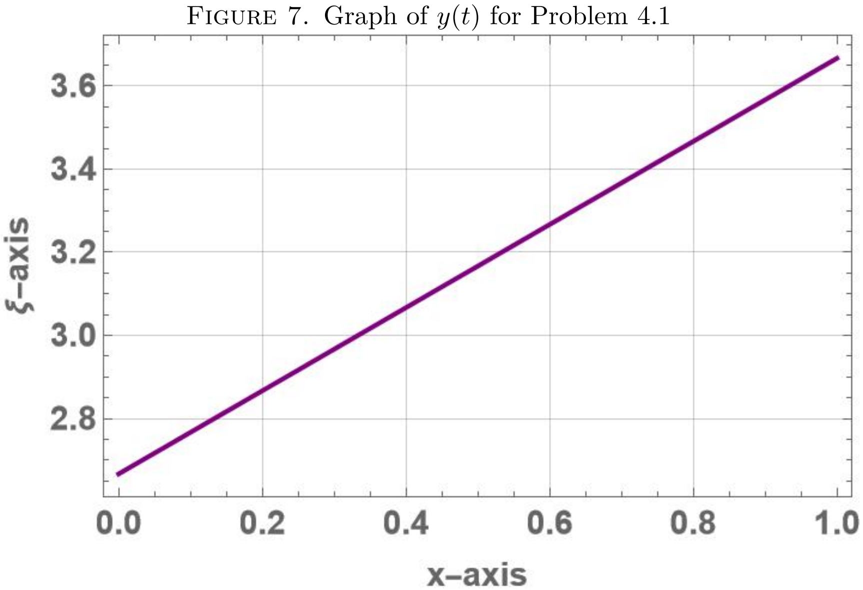
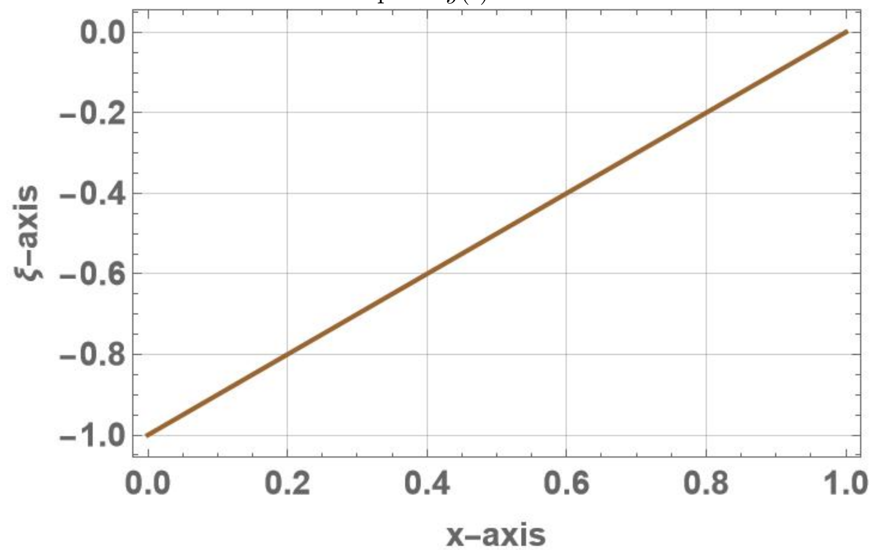


FIGURE 8. Graph of $y(t)$ for Problem 4.2



5.1. **Discussion of Results.** The results obtained for systems of equations in Problems 3.1 and 3.3 in Section 3 using HAITM are similar to results in the cited literature where Optima Homotopy Analysis Method (OHAM) and Power series method were used [25] and [30] respectively as well as the exact solution, while the solutions obtained for Problem 3.2 for $y_2(t)$ have slight deviation from the exact solution, but $y_2(t)$ are the same when the required degree of approximation is maintained. Also, in Section 4, the proposed method produced similar results as obtained in [31] where Euler wavelet operational matrix method was used. In addition, the results for Problems 3.1 and 3.2 by HAITM are in total agreement with those in literature, [30] where Optimal Homotopy Analysis Method (OHAM) was used. For ease of visualization, the results obtained are presented in 2D graphs.

5.2. **Conclusion.** HAITM has been successfully applied to linear and nonlinear fractional order integro-differential equations. The results obtained for the selected problems from the literature agree perfectly with the solutions obtained through other methods in the literature, at reduced computational time and space. Conclusively, HAITM is an excellent mathematical tool for solving both linear and nonlinear integro-differential equations with difference kernel.

5.3. **Recommendation.** The HAITM proposed in this paper can as well be expanded in scope to cover the linear and nonlinear fractional order partial Volterra integro-differential equations.

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