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ON THE SECOND AND THIRD ORDER VANDERMONDE DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS IN THE LIMAÇON DOMAIN

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Abstract. A Limaçon curve is defined by $\partial\mathbb{L}(p) = \{a + ib \in \mathbb{C} : [(a-1)^2 + b^2 - p^4]^2 = 4p^2[(a-1+p^2)^2 + b^2]\}$ where $p \in [-1,1] \setminus \{0\}$. A Limaçon curve also known as Limaçon of Pascal has many applications in the field of mathematics, physics, engineering and fluid dynamics. Vandermonde determinants are used in linear algebra, optimization and frequency analysis. Motivated by this, in this paper we define a new subclass of analytic functions related to Limaçon domain. Let $\mathcal{T}\mathbb{L}_p(\alpha)$, $0 \le \alpha \le 1$, $0 < p \le \frac{1}{\sqrt{2}}$, denote the subclass of normalized analytic functions $f(z) = z + \sum_{r=2}^{\infty} a_r z^r$ in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the condition

$$\frac{2(\alpha z^2 f''(z) + zf'(z))}{\alpha z(f(z) - f(-z))' + (1-\alpha)(f(z) - f(-z))} \prec \mathbb{L}_p(z) \quad z \in \mathbb{U},$$

where $\mathbb{L}_p(z) = (1 + pz)^2$ is the Limaçon function and $\prec$ denotes the well known subordination of functions in geometric function theory. In this paper, we determine the sharp coefficient bounds for the second order Vandermonde determinants and upper bounds for the third order Vandermonde determinants for functions in the subclass $\mathcal{T}\mathbb{L}_p(\alpha)$. Further, we obtain as corollaries the results of already known classes.

1. Introduction

Let $H(\mathbb{U})$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the subclass of $H(\mathbb{U})$ consisting of functions f of the form

$$f(z) = z + \sum_{r=2}^{\infty} a_r z^r, \quad z \in \mathbb{U}. \tag{1}$$

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Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions and $\mathcal{W}$ denote, the class of Schwarz functions (analytic self-mappings) on $\mathbb{U}$ is given by

$$\mathcal{W} = \{w \in H(\mathbb{U}) : w(0) = 0, |w(z)| < 1, z \in \mathbb{U}\}.$$

Let g_1 and g_2 be two analytic functions in $H(\mathbb{U})$. Then the function g_1 is subordinate to g_2, ($g_1 \prec g_2$) if there exists a Schwarz function $w(z) \in \mathcal{W}$ such that $g_1(z) = g_2(w(z))$. Suppose g_2 is univalent in $\mathbb{U}$, then

$$g_1(z) \prec g_2(z) \Leftrightarrow\ g_1(0) = g_2(0) \text{ and } g_1(\mathbb{U}) \subset g_2(\mathbb{U}).$$

The following equation represents a Limaçon in polar coordinates.

$$r = v + u\cos\varphi \quad (u, v \in \mathbb{R},\ \varphi \in [0, 2\pi]).$$

The conditions for the Limaçon to be convex and to have an indentation bounded by two inflection points are $v \geq 2u$ and $2u > v > u$ respectively. For different values of v and u, we have different curves.

(i) If $v = u$, the Limaçon degenerates to a cardioid.
(ii) If $v < u$, the Limaçon has an inner loop and
(iii) when $v = u/2$, it is a trisectrix.

The image of unit disk $\mathbb{U}$ under the function

$$\mathbb{L}_p(z) = (1 + pz)^2, \quad (p \in [-1, 1] \setminus \{0\}), \tag{2}$$

is the region $\partial\mathbb{L}(p) = \{a + ib \in \mathbb{C} : [(a-1)^2 + b^2 - p^4]^2 = 4p^2[(a - 1 + p^2)^2 + b^2]\}$ bounded by the Limaçon [12].

In [12], Masih and Kanas established the following inclusion relation

$$\{w \in \mathbb{C} : |w - 1| < 1 - (1 - |p|)^2\} \subset \mathbb{L}_p(\mathbb{U}) \subset \{w \in \mathbb{C} : |w - 1| < (1 + |p|)^2 - 1\}.$$

Also, $\mathbb{L}_p(z) = (1 + pz)^2$, $(p \in [-1, 1] \setminus \{0\})$ is starlike in $\mathbb{U}$. For $0 < p \leq \frac{1}{\sqrt{2}}$, the function $\mathbb{L}_p(z)$ is a member of the class $\mathcal{M}$ of analytic univalent function ϕ in $\mathbb{U}$ with the properties

(i) $\Re(\phi) > 0$
(ii) $\phi(\mathbb{U})$ starlike with respect to $\phi(0) = 1$ and symmetric with respect to real axis, and
(iii) $\phi'(0) > 0$.

The class $\mathcal{M}$ was introduced by Ma and Minda [10].

Denote by $\mathcal{E}(\mathbb{L}_p)$ the class of function $s(z) = 1 + s_1 z + s_2 z^2 + ...$ analytic in $\mathbb{U}$ with $s(0) = 1$ and $s(z) \prec \mathbb{L}_p(z)$ for $0 < p \leq \frac{1}{\sqrt{2}}$. Clearly $\mathcal{E}(\mathbb{L}_p)$ is a subclass of Carathéodory class $\mathcal{E}$.

A Vandermonde matrix [2, 11] is a square matrix in which the terms of each row (or each column) is in geometric progression with the first element being one. The Vandermonde determinant is sometimes also known as a discriminant.

For $f \in \mathcal{A}$, the q^{th} order Vandermonde determinant $V_q(r)$ is defined as

$$V_q(r) = \begin{vmatrix} 1 & a_r & a_r^2 & \cdots & a_r^{q-1} \\ 1 & a_{r+1} & a_{r+1}^2 & \cdots & a_{r+1}^{q-1} \\ 1 & a_{r+2} & a_{r+2}^2 & \cdots & a_{r+2}^{q-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{r+q-1} & a_{r+q-1}^2 & \cdots & a_{r+q-1}^{q-1} \end{vmatrix}, \qquad (a_1 = 1,\ q, r \geq 1).$$

In particular,

$$V_2(1) = \begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = a_2 - a_1, \quad V_2(2) = \begin{vmatrix} 1 & a_2 \\ 1 & a_3 \end{vmatrix} = a_3 - a_2$$

and

$$V_3(1) = \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_3 - a_2)(a_3 - a_1)(a_2 - a_1).$$

For further details on Vandermonde determinants and their applications refer to articles [8, 9]. The second and third order Hankel determinants in geometric function theory have been extensively studied by many researchers [3, 4, 6, 13, 14, 15]. Recently, Vijayalakshmi et al. [16] introduced and studied the Vandermonde determinant of order two and three for Sakaguchi type of function in Limaçon domain. Further, Anand et al. [1] and Wahid et al. [17] have investigated on the upper bounds of Vandermonde determinants whose elements with coefficients of close-to-convex functions and logarithmic coefficients of bounded turning functions respectively.

Motivated by the works of Bucur et al. [4] and Vijayalakshmi et al. [16] , we define a new subclass $\mathcal{T}\mathbb{L}_p(\alpha)$ of analytic functions in Limaçon domain.

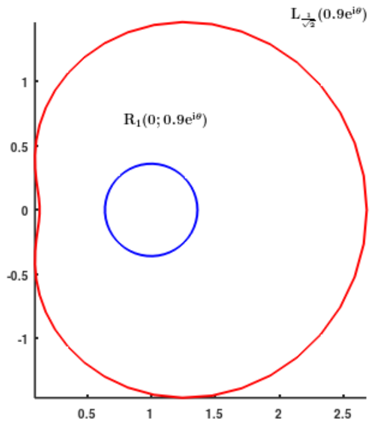
Definition 1.1. *Let $\mathcal{T}\mathbb{L}_p(\alpha)$ denote the subclass of $\mathcal{A}$ that consists of functions of the form (1) satisfying the condition*

$$\mathcal{T}\mathbb{L}_p(\alpha) = \left\{ f \in \mathcal{A} : \frac{2(\alpha z^2 f''(z) + z f'(z))}{\alpha z(f(z) - f(-z))' + (1-\alpha)(f(z) - f(-z))} \prec \mathbb{L}_p(z) \right\}, \quad (3)$$

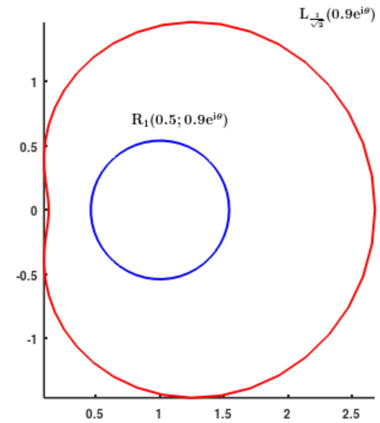
where $0 \leq \alpha \leq 1$, and $\mathbb{L}_p(z)$, $0 < p \leq \frac{1}{\sqrt{2}}$ is given by (2).

Remark 1.1. *The classes investigated in* [4, 14] *however do not apply subordination principle.*

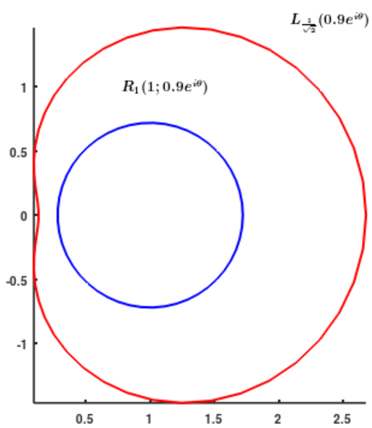
Remark 1.2. *i) Consider the function $\vartheta_1(z) = z + \frac{1}{5}z^2$ in $\mathcal{A}$. For $0 \leq \alpha \leq 1$, let $R_1(\alpha; z) = \dfrac{2(\alpha z^2 \vartheta_1''(z) + z\vartheta_1'(z))}{\alpha z(\vartheta_1(z) - \vartheta_1(-z))' + (1-\alpha)(\vartheta_1(z) - \vartheta_1(-z))}$. We observe that the images shown in Figures 1(a), 1(b) and 1(c) (in blue color) of $\mathbb{U}$ under transformations $R_1(\alpha; z) = 1 + \frac{2}{5}(\alpha+1)z$ at $z = 0.9e^{i\theta}, 0 \leq \theta \leq 2\pi$ for $\alpha = 0, 0.5, 1$ respectively lie in the images shown in Figures 1(a), 1(b) and 1(c) (in red color) of $\mathbb{U}$ under Limaçon function $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ drawn by Octave computer software.*



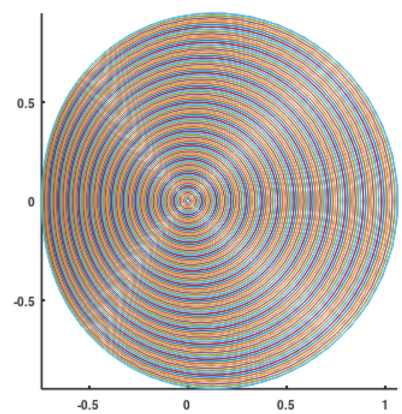
(a) *The images of* $R_1(0;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *(red color)*



(b) *The images of* $R_1(0.5;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *(red color)*



(c) *The images of* $R_1(1;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$*(red color)*



d) *The image of* $\vartheta_1(\mathbb{U})$

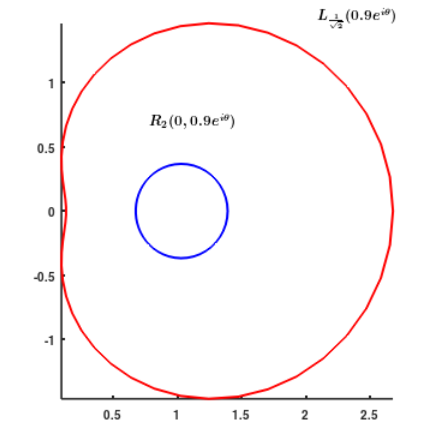
Figure 1 : *Figures for the Remark 1.2 (i)*

So we have $R_1(\alpha;0) = \mathbb{L}_{\frac{1}{\sqrt{2}}}(0)$ *and* $R_1(\alpha;\mathbb{U}) \subset \mathbb{L}_p(\mathbb{U})$. *By the univalence of* $\mathbb{L}_p(z)$, $0 < p \leq \frac{1}{\sqrt{2}}$, $R_1(\alpha;z) \prec \mathbb{L}_p(z)$ *for all* $\alpha \in [0,1]$. *Thus,* $\vartheta_1 \in \mathcal{T}\mathbb{L}_p(\alpha)$. *Also, the Figure 1(d) illustrates that the function* ϑ_1 *is univalent in* $\mathbb{U}$.

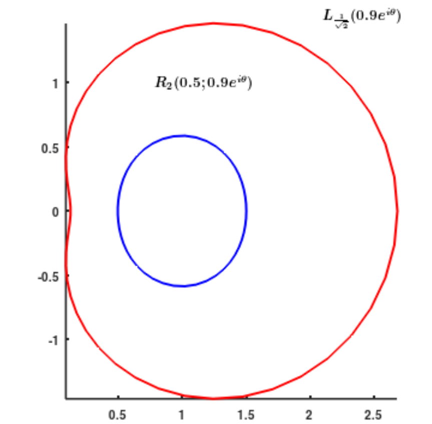
ii) Let $\vartheta_2(z) = ze^{\frac{z}{5}}$ *be in* $\mathcal{A}$. *For* $0 \leq \alpha \leq 1$, *We have*

$$\begin{aligned} R_2(\alpha;z) &= \frac{2(\alpha z^2 \vartheta_2''(z) + z\vartheta_2'(z))}{\alpha z(\vartheta_2(z)-\vartheta_2(-z))' + (1-\alpha)(\vartheta_2(z)-\vartheta_2(-z))} \\ &= \frac{e^{0.2z}[1+(2\alpha+1)(0.2)z+\alpha(0.04)z^2]}{\cosh(0.2z)+\alpha z\sinh(0.2z)}. \end{aligned}$$

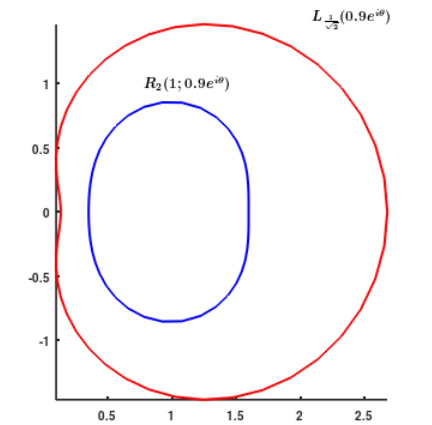
We observe that the images shown in Figures 2(a), 2(b) and 2(c) (in blue color) of $\mathbb{U}$ *under transformations* $R_2(\alpha;z)$ *at* $z = 0.9e^{i\theta}$, $0 \leq \theta \leq 2\pi$ *for* $\alpha = 0, 0.5, 1$ *respectively lie in the images shown in Figures 2(a), 2(b) and 2(c) (in red color) of* $\mathbb{U}$ *under Limaçon function* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *drawn by Octave computer software.*



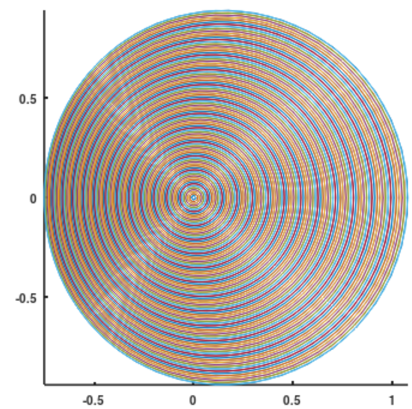
(a) *The images of* $R_2(0;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *(red color)*



(b) *The images of* $R_2(0.5;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *(red color)*



(c) *The images of* $R_2(1;0.9e^{i\theta})$ *(blue color) and* $\mathbb{L}_{\frac{1}{\sqrt{2}}}(0.9e^{i\theta})$ *(red color)*



d) *The image of* $\vartheta_2(\mathbb{U})$

Figure 2 : *Figures for the Remark 1.2 (ii)*

So we have $R_2(\alpha;0) = \mathbb{L}_{\frac{1}{\sqrt{2}}}(0)$ *and* $R_2(\alpha;\mathbb{U}) \subset \mathbb{L}_p(\mathbb{U})$. *By the univalence of* $\mathbb{L}_p(z)$, $0 < p \leq \frac{1}{\sqrt{2}}$, $R_2(\alpha;z) \prec \mathbb{L}_p(z)$ *for all* $\alpha \in [0,1]$. *Thus,* $\vartheta_2 \in \mathcal{T}\mathbb{L}_p(\alpha)$. *Also, the Figure 2(d) illustrates that the function* ϑ_2 *is univalent in* $\mathbb{U}$.

iii) From remarks 1.2 (i) and (ii), we conclude that the class $\mathcal{T}\mathbb{L}_p(\alpha)$ *is non-empty with the members* ϑ_1 *and* ϑ_2. *Further, we note that* ϑ_1 *and* ϑ_2 *are also in* $\mathcal{S}$. *This implies that* $\vartheta_1, \vartheta_2 \in \mathcal{T}\mathbb{L}_p(\alpha) \cap \mathcal{S}$. *So the subclass* $\mathcal{T}\mathbb{L}_p(\alpha) \cap \mathcal{S}$ *of* $\mathcal{T}\mathbb{L}_p(\alpha)$ *is also non-empty. The above examples and comments give the needed motivation to discuss the properties of functions belong to the class* $\mathcal{T}\mathbb{L}_p(\alpha)$ *in the next section.*

Remark 1.3.

i) If $\alpha = 0$, $\mathcal{T}\mathbb{L}_p(0) = S^*(\mathbb{L}_p)$, *where* $S^*(\mathbb{L}_p)$ *is the class considered by Vijayalakshmi et al.* [16].

ii) If $\alpha = 1$, $\mathcal{T}\mathbb{L}_p(1) = \mathcal{C}(\mathbb{L}_p)$, *where* $\mathcal{C}(\mathbb{L}_p)$ *is the class considered Vijayalakshmi et al.* [16].

We shall make use of the following lemmas to prove our main results in the next section.

Lemma 1.1. [7] *If the function $w(z) \in \mathcal{W}$ is of the form*

$$w(z) = \sum_{r=1}^{\infty} w_r z^r, \quad z \in \mathbb{U}, \tag{4}$$

then

$$w_2 = x(1 - w_1^2),$$
$$w_3 = (1 - w_1^2)(1 - |x|^2)t - w_1(1 - w_1^2)x^2,$$

for some x, t with $|x| \leq 1$ and $|t| \leq 1$.

Lemma 1.2. [5] *If the function $w \in \mathcal{W}$ is of the form (4), then $|w_r| \leq 1$ for all $r = 1, 2, 3, ...$*

In this paper, motivated by the works of [11] and [16], we obtain sharp bounds for the Vandermonde determinants $V_2(1), V_2(2)$ and upper bounds for the Vandermonde determinant $V_3(1)$ for functions in the subclass $\mathcal{T}\mathbb{L}_p(\alpha)$.

2. Main results

In the next Theorem 2.1, we determine sharp bounds for the second order Vandermonde determinant $V_2(2)$ of the class $\mathcal{T}\mathbb{L}_p(\alpha)$.

Theorem 2.1. *Let $0 < p \leq \frac{1}{\sqrt{2}}$, $0 \leq \alpha \leq 1$. If the function $f \in \mathcal{T}\mathbb{L}_p(\alpha)$ is defined by (1), then*

$$|V_2(2)| \leq \frac{p\left((2\alpha+1)^2 + 2(1+\alpha)^2(2-p)\right)}{2(1+\alpha)^2(2-p)(2\alpha+1)}. \tag{5}$$

The inequality is sharp.

Proof. For $0 \leq \alpha \leq 1$ and $0 < p \leq \dfrac{1}{\sqrt{2}}$, let $f \in \mathcal{T}\mathbb{L}_p(\alpha)$ is defined by (1). Then there exists a function $w \in \mathcal{W}$ of the form (4) such that

$$\frac{2(\alpha z^2 f''(z) + z f'(z))}{\alpha z(f(z) - f(-z))' + (1-\alpha)(f(z) - f(-z))} = (1 + p w(z))^2. \tag{6}$$

From (6), we have

$$\begin{aligned} z + 2(\alpha+1)a_2 z^2 &+ 3(2\alpha+1)a_3 z^3 + ... \\ &= z + 2pw_1 z^2 + (p^2 w_1^2 + 2pw_2)z^3 + (2\alpha+1)a_3 z^3 + ... \end{aligned}$$

Equating corresponding coefficients on both sides, we get

$$a_2 = \frac{pw_1}{\alpha+1}, \tag{7}$$

$$a_3 = \frac{pw_2}{2\alpha+1} + \frac{p^2 w_1^2}{2(2\alpha+1)}. \tag{8}$$

Using Lemma 1.1, for some x such that $|x| \leq 1$, we obtain

$$\begin{aligned}|V_2(2)| &= |a_3 - a_2| \\ &= \left| \left(\frac{pw_2}{2\alpha+1} + \frac{p^2 w_1^2}{2(2\alpha+1)} \right) - \frac{pw_1}{\alpha+1} \right| \\ &= \left| \left(\frac{px(1-w_1^2)}{2\alpha+1} + \frac{p^2 w_1^2}{2(2\alpha+1)} \right) - \frac{pw_1}{\alpha+1} \right|.\end{aligned}$$

As $|w_1| \leq 1$, according to Lemma 1.2, we can assume without restriction that $w_1 = w$ with $0 \leq w \leq 1$. By implementation of the triangle inequality with $\tau = |x|$, we get

$$|V_2(2)| \leq \frac{p\tau(1-w^2)}{2\alpha+1} + \frac{p^2 w^2}{2(2\alpha+1)} + \frac{pw}{\alpha+1} =: G(w,\tau).$$

Since

$$\frac{\partial G}{\partial \tau} = \frac{p(1-w^2)}{2\alpha+1} \geq 0 \text{ for } 0 \leq \tau \leq 1,$$

$G(w,\tau)$ is an increasing function in closed region $R = \{(w,\tau) : 0 \leq w \leq 1 \text{ and } 0 \leq \tau \leq 1\}$. It has no maximum value in an interior of R and the maximum value of $G(w,\tau)$ occurs at $\tau = 1$.
For fixed $w \in [0,1]$,

$$\max_{0\leq\tau\leq1} G(w,\tau) = G(w,1) = -\frac{p(2-p)}{2(2\alpha+1)} w^2 + \frac{pw}{1+\alpha} + \frac{p}{2\alpha+1} =: h(w)$$

We can see that the maximum of $h(w)$ occurs at $w = w_0 = \frac{(2\alpha+1)}{(1+\alpha)(2-p)}$ in $[0,1]$. Hence, we have

$$|V_2(2)| \leq h(w_0) = \frac{p\left((2\alpha+1)^2 + 2(1+\alpha)^2(2-p)\right)}{2(1+\alpha)^2(2-p)(2\alpha+1)}.$$

This completes the proof. The equality holds for the function

$$f(z) = z - \frac{p}{(2-p)(2\alpha+1)} z^2 + \frac{p[4(2\alpha+1)(\alpha+1)+1] - 2p(1+\alpha)^2(p+1)}{2(1+\alpha)^2(2-p)(2\alpha+1)} z^3 - ...$$

□

In the following two Corollaries 2.1 and 2.2, we obtain sharp bounds for the second order Vandermonde determinant $V_2(2)$ of the classes $S^*(\mathbb{L}_p)$ and $\mathcal{C}(\mathbb{L}_p)$, when α takes the value 0 and 1 respectively.

Corollary 2.1. *When $\alpha = 0$ in (5), we get*

$$|V_2(2)| \leq \frac{p(5-2p)}{2(2-p)}.$$

This result coincides with Theorem 1 in [16].

Corollary 2.2. *When $\alpha = 1$ in (5), we get*

$$|V_2(2)| \leq \frac{p(25-8p)}{24(2-p)}.$$

This result coincides with Theorem 5 in [16].

We obtain sharp bounds for the second order Vandermonde determinant $V_2(1)$ of the class $\mathcal{T}\mathbb{L}_p(\alpha)$ in the following Theorem 2.2.

Theorem 2.2. *Let $0 < p \leq \frac{1}{\sqrt{2}}$, $0 \leq \alpha \leq 1$. If the function $f \in \mathcal{T}\mathbb{L}_p(\alpha)$ is defined by (1), then*

$$|V_2(1)| \leq \frac{p}{1+\alpha} + 1. \tag{9}$$

The inequality is sharp.

Proof. As per the proof of Theorem 2.1, we have

$$|V_2(1)| = \left| \frac{pw_1}{1+\alpha} - 1 \right|.$$

Using Lemma 1.1 and triangle inequality, we obtain

$$|V_2(1)| \leq \frac{pw}{1+\alpha} + 1 =: h(w).$$

But $h' = \frac{p}{1+\alpha} > 0$ implies that the maximum value of function $h(w)$ occurs at $w = 1$. Therefore, we get

$$|V_2(1)| \leq h(1) = \frac{p}{1+\alpha} + 1.$$

Hence, the proof of the theorem.
The inequality is sharp for the function

$$f(z) = z - \frac{p}{1+\alpha} z^2 + \dots.$$

□

In the next two Corollaries 2.3 and 2.4, we obtain sharp bounds for the second order Vandermonde determinant $V_2(1)$ of the classes $S^*(\mathbb{L}_p)$ and $\mathcal{C}(\mathbb{L}_p)$, when $\alpha = 0$ and $\alpha = 1$ respectively.

Corollary 2.3. *When $\alpha = 0$ in (9), we get*

$$|V_2(1)| \leq p + 1.$$

This result coincides with Theorem 2 in [16].

Corollary 2.4. *When $\alpha = 1$ in (9), we get*

$$|V_2(1)| \leq \frac{p+2}{2}.$$

This result coincides with Theorem 6 in [16].

In the following Theorem 2.3, we obtain sharp bounds for the functional $|a_3 - a_1|$ of the class $\mathcal{T}\mathbb{L}_p(\alpha)$.

Theorem 2.3. *Let $0 < p \leq \frac{1}{\sqrt{2}}$, $0 \leq \alpha \leq 1$. If the function $f \in \mathcal{T}\mathbb{L}_p(\alpha)$ is defined by (1), then*

$$|a_3 - a_1| \leq \frac{p}{2\alpha+1} + 1. \tag{10}$$

The inequality is sharp.

Proof. As per the proof of Theorem 2.1, we have

$$|a_3 - a_1| = \left| \frac{pw_1}{(2\alpha+1)} + \frac{p^2 w_1^2}{2(2\alpha+1)} - 1 \right|.$$

With the help of Lemma 1.1 for some x such that $|x| \le 1$, we obtain

$$|a_3 - a_1| = \left| \frac{px(1-w_1^2)}{(2\alpha+1)} + \frac{p^2 w_1^2}{2(2\alpha+1)} - 1 \right|.$$

As $w_1 \le 1$, according to Lemma 1.2, we can assume without restriction that $w_1 = w$ with $0 \le w \le 1$. Using the triangle inequality with $\tau = |x|$, we get

$$|a_3 - a_1| \le \frac{p\tau(1-w^2)}{2\alpha+1} + \frac{p^2 w^2}{2(2\alpha+1)} + 1 =: G(w,\tau).$$

Since

$$\frac{\partial G}{\partial \tau} = \frac{p(1-w^2)}{2(\alpha+1)} \ge 0, \text{ for } 0 \le \tau \le 1,$$

$G(w,\tau)$ is an increasing function in closed region $R = \{(w,\tau) : 0 \le w \le 1 \text{ and } 0 \le \tau \le 1\}$. It has no maximum value in an interior of R and the maximum value of $G(w,\tau)$ occurs at $\tau = 1$.

For fixed w with $0 \le w \le 1$,

$$\max_{0\le\tau\le1} G(w,\tau) = G(w,1) = \frac{p}{2\alpha+1} - \frac{c^2 p(2-p)}{2(2\alpha+1)} + 1 =: h(w).$$

The maximum of function $h(w)$ occurs at $w = 0$ in $[0,1]$. Hence, we have

$$|a_3 - a_1| \le h(0) = \frac{p}{2\alpha+1} + 1.$$

Thus we get the desired result.

The inequality is sharp for the function

$$f(z) = z - \frac{p}{2\alpha+1} z^3 + \ldots.$$

□

In the following two Corollaries 2.5 and 2.6, we determine sharp bounds for the functional $|a_3 - a_1|$ of the classes $S^*(\mathbb{L}_p)$ and $\mathcal{C}(\mathbb{L}_p)$, when α takes the value 0 and 1 respectively.

Corollary 2.5. *When $\alpha = 0$ in (10), we get*

$$|a_3 - a_1| \le p + 1.$$

This result coincides with Theorem 3 in [16].

Corollary 2.6. *When $\alpha = 1$ in (10), we get*

$$|a_3 - a_1| \le \frac{p+3}{3}.$$

This result coincides with Theorem 7 in [16].

We obtain upper bounds for the third order Vandermonde determinant $V_3(1)$ of the class $\mathcal{T}\mathbb{L}_p(\alpha)$ in the next Theorem 2.4.

Theorem 2.4. *Let $0 < p \le \frac{1}{\sqrt{2}}$, $0 \le \alpha \le 1$. If the function $f \in \mathcal{T}\mathbb{L}_p(\alpha)$ is defined by (1), then*

$$|V_3(1)| \le \left(\frac{p[(2\alpha+1)^2 + 2(1+\alpha)^2(2-p)]}{2(1+\alpha)^2(2-p)(2\alpha+1)} \right) \left(\frac{p}{1+\alpha} + 1 \right) \left(\frac{p}{2\alpha+1} + 1 \right). \quad (11)$$

Proof. The result follows upon using (5),(9) and (10). □

The upper bounds for the third order Vandermonde determinant $V_3(1)$ of the classes $S^*(\mathbb{L}_p)$ and $\mathcal{C}(\mathbb{L}_p)$ are determined in the next two Corollaries 2.7 and 2.8, when $\alpha = 0$ and $\alpha = 1$, respectively.

Corollary 2.7. *When $\alpha = 0$ in (11), we get*

$$|V_3(1)| \leq \frac{p(5-2p)(p+1)^2}{2(2-p)}.$$

This result coincides with Theorem 4 in [16].

Corollary 2.8. *When $\alpha = 1$ in (11), we get*

$$|V_3(1)| \leq \frac{p(25-8p)(p+3)(p+2)}{144(2-p)}.$$

This result coincides with Theorem 8 in [16].

3. Conclusions

Limaçon domain is applied in various branches of mathematics, statistics, fluid dynamics, engineering and science.

In this article, a new subclass $\mathcal{T}\mathbb{L}_p(\alpha)$ of analytic functions related to Limaçon domain is introduced. Sharp bounds for second order Vandermonde determinants and upper bounds for third order Vandermonde determinants in Limaçon domain are obtained for the subclass $\mathcal{T}\mathbb{L}_p(\alpha)$. The results of earlier well known work are also presented as corollaries from our results.

We believe that our results will motivate researchers in the area of geometric function theory to introduce and study new classes related to Limaçon domain.

Conflicts of Interest

All the authors declare that they have no conflict of interest.

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References

[1] S. Anand, P. Ravi and S. Kumar, *Close-to-convex functions associated with a rational function,* Math. Slovaca, 74, no. 2, 339-354, 2024.

[2] H. Anton, C. Rorres and A. Kaul, *Elementary Linear Algebra,* Applications Version, 12th Edition, John Wiley & Sons, 2019.

[3] K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions,* Inequal. Theory Appl. 6, 1-7, 2010.

[4] R. Bucur, D. Breaz and L. Georgescu, *Third Hankel determinant for a class of analytic functions with respect to symmetric points,* Acta Univ. Apulensis Math. Inform. 42, 79-86, 2015.

[5] P. L. Duren, *Univalent Functions,* Grundlehren Math. Wiss., 259, Springer-Verlag, New York, 1983.

[6] W. K. Hayman, *On the second Hankel determinant of mean univalent functions,* Proc. London Math. Soc. (3), 18, 77-94, 1968.

[7] S. Kanas, E. Analouei Adegani and A. Zireh, *An unified approach to second Hankel determinant of bi-subordinate functions,* Mediterr. J. Math. 14, no. 6, Paper No. 233, 12 pages, 2017.

[8] Y. Li and X. Dang, *Vandermonde Determinant and its Applications,* Journal of Education and Culture Studies, 7, no. 4, 16-24, 2023.

[9] K. Lundengård, J. Österberg and S. Silvestrov, *Optimization of the determinant of the Vandermonde matrix and related matrices,* Methodol. Comput. Appl. Probab. 20, no. 4, 1417-1428, 2018.

[10] W. Ma and D. Minda, *A unified treatment of some special classes of univalent functions,* Proceedings of the Conference on Complex Analysis (Tianjin), 157-169, 1992. Reprint: Conf. Proc. Lecture Notes Anal., I International Press, Cambridge, MA, ISBN:1-57146-017-9, 1994.

[11] N. Macon and A. Spitzbart, *Inverses of Vandermonde matrices,* Amer. Math. Monthly, 65, 95-100, 1958.

[12] V. S. Masih and S. Kanas, *Subclasses of starlike and convex functions associated with the Limaçon domain,* Symmetry, 12, no. 6, 1-11, 2020.

[13] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions,* Trans. Amer. Math. Soc. 223, 337-346, 1976.

[14] S. P. Vijayalakshmi and T. V. Sudharsan, *Second Hankel determinant for generalized Sakaguchi type functions,* Theory Appl. Math. Comput. Sci. 5, no. 1, 94-100, 2015.

[15] S. P. Vijayalakshmi, T. V. Sudharsan, D. Breaz and K. G. Subramanian, *Bounds on the third order Hankel determinant for certain subclasses of analytic functions,* An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. 25, no. 3, 199-214, 2017.

[16] S. P. Vijayalakshmi, S. Bulut and T. V. Sudharsan, *Vandermonde determinant for a certain Sakaguchi type function in Limaçon domain,* Asian-Eur. J. Math. 15, no. 12, Paper No. 2250212, 9 pages, 2022.

[17] N. H. A. A. Wahid, I. Q. Amirnuddin and N. I. M. Azmi, *Bounds for certain determinants of Logarithmic coefficients for the class of functions with bounded turning,* Eur. J. Pure Appl. Math. 17, no. 4, 2738-2752, 2024.

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